Non-Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ and $\mathbb{C}H^n$

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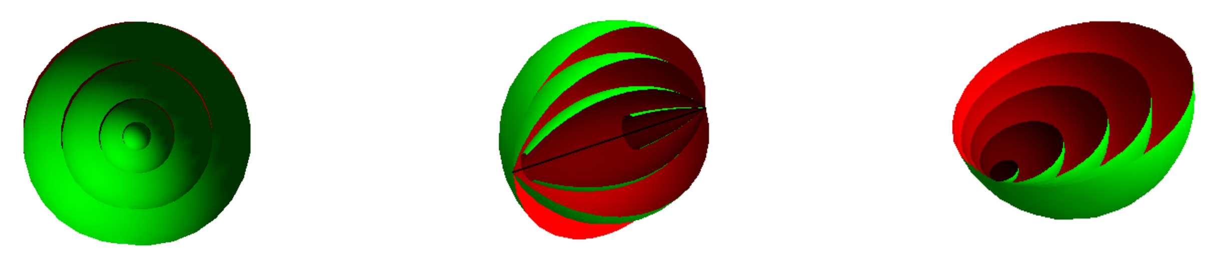
Introduction

Intuitively a hypersurface has constant principal curvatures if it looks the same at each point.
In 1938, É. Cartan classified the hypersurfaces with constant principal curvatures in the real hyperbolic space $\mathbb{R}H^n$ [2].



Geodesic spheres — Tubes or equidistant hypersurfaces to a totally geodesic $\mathbb{R}H^k$ — Horospheres

The problem in S^n is more involved. In fact the classification has not been completed yet, despite some recent advances [6].

Main problem:

Classification of real hypersurfaces in $\mathbb{C}H^n$ or $\mathbb{C}P^n$ with constant principal curvatures.

Preliminaries

$M \subset \overline{M}$ a real hypersurface, $\overline{M} = \mathbb{C}P^n$ or $\mathbb{C}H^n$.
ξ normal vector field and $S_\xi X = -\bar{\nabla}_X \xi$ shape operator.
J complex structure and $J\xi \in \mathfrak{X}(M)$ Hopf vector field
Principal curvatures: eigenvalues of S.
For any $M \subset \overline{M}$, we can write $J\xi = \sum_{i=1}^{h} b_i u_i$ with u_i a principal direction and $b_i > 0$.
Hopf hypersurface: $J\xi$ is a principal direction.

Construction of W^{2n-k}_φ

As a symmetric space, $\mathbb{C}H^n = G/K$, with $G := SU(1,n)$, $K := S(U(n)U(1))$, where K fixes some $o \in \mathbb{C}H^n$.
Iwasawa decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$:
$\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n) \oplus \mathfrak{u}(1))$, $\mathfrak{a} \simeq \mathbb{R}$, $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$, where $\mathfrak{n}$ is the Heisenberg algebra, $\mathfrak{g}_{2\alpha} \simeq \mathbb{R}$, $\mathfrak{g}_\alpha \simeq \mathbb{C}^{n-1}$.
Let $\mathfrak{w} \subset \mathfrak{g}_\alpha$ be a subspace such that $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$ has dimension k and constant Kähler angle $\varphi \in [0, \frac{\pi}{2}]$, that is, $\measuredangle(Jv, \mathfrak{w}^\perp) = \varphi, \forall v \in \mathfrak{w} \setminus \{0\}$.
We define $W^{2n-k}_\varphi := S \cdot o$, where $S = \operatorname{Exp}(\mathfrak{s}) \subset AN$,

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha} \subset \mathfrak{a} \oplus \mathfrak{n}.$$

Known results

Theorem [1], [3], [5]

Let $M \subset \overline{M}$, $n \geq 2$, with constant principal curvatures and whose Hopf vector field has at most $h \leq 2$ distinct non-trivial projections onto the principal curvature spaces. Then if $M \subset \mathbb{C}H^n$, it is an open part of:

1. a horosphere in $\mathbb{C}H^n$,
2. a tube around a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^n$, for some $k \in \{1, \dots, n-1\}$,
3. a tube around a totally geodesic $\mathbb{R}H^n$ in $\mathbb{C}H^n$,
4. $W^{2n-1}_{\pi/2}$ or an equidistant hypersurface to it,
5. a tube around a submanifold $W^{2n-k}_{\pi/2}$, for some $k \in \{2, \dots, n-1\}$.

If $M \subset \mathbb{C}P^n$, then $h = 1$ and M is homogeneous.

Non-Hopf hypersurfaces with $h = 3$

Main Theorem

Let $M \subset \overline{M}$, $n \geq 2$, with 4 constant principal curvatures and such that $J\xi$ has non-trivial projection onto 3 distinct curvature spaces of simple multiplicity.

- If $\overline{M} = \mathbb{C}H^n$, then M is locally congruent to a tube around W^{2n-k}_φ, with $\varphi \in (0, \pi/2)$.
- If $\overline{M} = \mathbb{C}P^n$, such M doesn't exist.

Conjecture:

Every hypersurface with constant principal curvatures in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ is homogeneous.

Sketch of the proof

We prove the Main Theorem after accomplishing the following steps:

1. Get polynomial relations between b_1, b_2 and b_3 using Gauss and Codazzi equations.
2. Prove that the b_i are constant using Gröbner basis theory.
3. Conclude using the isoparametric classification theorem in [4].

References

[1] J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, *J. Reine Angew. Math.* **395** (1989), 132-141.
[2] É. Cartan, Familles de surfaces isoparamétriques dans les espaces à courbure constante, *Ann. Mat. Pura Appl.* **395** (1938), no. 1, 177-191.
[3] J. C. Díaz-Ramos, M. Domínguez-Vázquez, Non-Hopf real hypersurfaces with constant principal curvatures in complex space forms, *Indiana Univ. Math. J.* **60** (2011) no. 3, 859-882.
[4] J. C. Díaz-Ramos, M. Domínguez-Vázquez, V. Sanmartín-López, Isoparametric hypersurfaces in complex hyperbolic spaces, *Adv. Math.* **314** (2017), 756-805.
[5] M. Kimura, Real hypersurfaces and complex submanifolds in complex projective space. *Trans. Amer. Math. Soc.* **296** (1986), 137-149.
[6] R. Miyaoka, Isoparametric hypersurfaces with (g,m) = (6,2), *Ann. of Math. (2)* **177** (2012), no. 1, 53-110.

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