Cohomogeneity one actions on hyperbolic spaces

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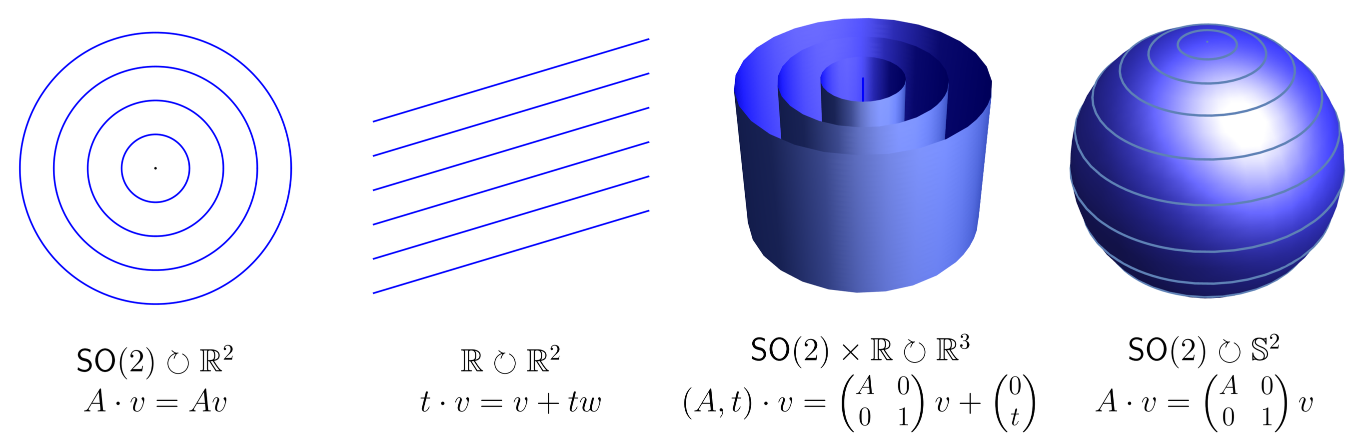
Introduction

A **cohomogeneity one action** on a Riemannian manifold $\bar{M}$ is an isometric action on $\bar{M}$ with codimension one orbits.

Properties:

- All the orbits, except at most two, are hypersurfaces.
- The orbit space is diffeomorphic to $\mathbb{S}^1$, $[0,1]$, $\mathbb{R}$ or $[0,1)$.



Let $\bar{M}$ be a hyperbolic space over over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We can construct a **solvable** Lie group AN equipped with a **left invariant** metric such that $AN \overset{\text{isom.}}{\cong} \bar{M} \cong G/K$. In particular, $\mathfrak{a} \simeq \mathbb{R}$, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and $K_0 := N_K(\mathfrak{a})$.

$\bar{M}$	$\mathbb{R}H^n$	$\mathbb{C}H^n$	$\mathbb{H}H^n$	$\mathbb{O}H^2$
	$\frac{\mathsf{SO}^0(1,n)}{\mathsf{SO}(n)}$	$\frac{\mathsf{SU}(1,n)}{\mathsf{S}(\mathsf{U}(1)\times\mathsf{U}(n))}$	$\frac{\mathsf{Sp}(1,n)}{\mathsf{Sp}(1)\mathsf{Sp}(n)}$	$\frac{\mathsf{F}_4^{-20}}{\mathsf{Spin}(9)}$
$\mathfrak{v}$	$\mathbb{R}^{n-1}$	$\mathbb{C}^{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{O}$
$\dim \mathfrak{z}$	0	1	3	7
K_0	$\mathsf{SO}(n-1)$	$\mathsf{U}(n-1)$	$\mathsf{Sp}(1)\mathsf{Sp}(n-1)$	$\mathsf{Spin}(7)$

Main problem:

Classify such actions on $\bar{M}$ up to orbit equivalence.

Known results

By [3] such an action is of one of these types:

(A) It has no singular orbits $\rightsquigarrow$ (Classified in [2]).

(B) It has 1 totally geodesic singular orbit $\rightsquigarrow$ (Classified in [1]).

(C) It has a non-totally geodesic singular orbit $S_{\mathfrak{w}} := \operatorname{Exp} \mathfrak{s}_{\mathfrak{w}}$, where $\mathfrak{s}_{\mathfrak{w}} := \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$ for $\mathfrak{w} \subset \mathfrak{v}$, with $N_{K_0}(\mathfrak{w})$ acting transitively on the unit sphere of $\mathfrak{w}^\perp$.

Theorem [1], [3], [4]

The moduli space of cohomogeneity one actions on $\mathbb{F}H^n$ is:

- For $\mathbb{R}H^n$: $\{1, \dots, n-1\} \cup \{N, K\}$.
- For $\mathbb{C}H^n$: $\{0, \pi/2\} \times \mathbb{Z}_{n-1} \cup (0, \pi/2) \times \mathbb{Z}_{\lceil (n-2)/2 \rceil} \cup \{N, K, \mathsf{SO}^0(1,n)\}$.
- For $\mathbb{H}H^2$: $\{N, K, \mathsf{SU}(1,2)\} \cup \{1,2,3,4\}$.
- For $\mathbb{O}H^2$: $\{K, N, \mathsf{Sp}(1,2) \times \mathsf{SU}(2)\} \cup \{1,2,3,6,7,8\} \cup (4 \times [0,1])$.

The problem in $\mathbb{H}H^n$

$V \subset \mathbb{H}^n$ is a **protohomogeneous** subspace if there is some $H \leq \mathsf{Sp}(1)\mathsf{Sp}(n)$ acting transitively on its unit sphere.

Any type (C) action is induced by a protohomogeneous V.

Let $\mathfrak{J} \subset \operatorname{End}_{\mathbb{R}}(\mathbb{H}^n)$ be a quaternionic structure of $\mathbb{H}^n$. Let $\{J_1, J_2, J_3\}$ span $\mathfrak{J}$ satisfying $J_i^2 = -\operatorname{Id}$ and $J_i J_{i+1} = J_{i+2}$.

$$L_v \colon \mathfrak{J} \times \mathfrak{J} \to \mathbb{R}, \quad L_v(J, J') := \langle \pi_V J v, \pi_V J' v \rangle$$

The **quaternionic Kähler angle** of $v \in V \setminus \{0\}$ with respect to V is $\Phi_V(v) = (\varphi_1, \varphi_2, \varphi_3)$, with $\varphi_1 \leq \varphi_2 \leq \varphi_3$, such that $\sigma(L_v) = \{\cos^2 \varphi_i \langle v, v \rangle\}_{i=1,2,3}$.

We say that V has **constant quaternionic Kähler angle** (c.q.k.a.) if $\Phi_V(v) = \Phi_V(w)$ $\forall v, w \in V \setminus \{0\}$.

V is protohomogeneous $\implies$ V has c.q.k.a.

Main Theorem

The moduli space of cohomogeneity one actions on $\mathbb{H}H^n$ is:

$$\{N, K, \mathsf{SU}(1, n+1)\} \sqcup \bigsqcup_{k=1}^{4n} \mathcal{M}_{k,n}.$$

$\mathcal{M}_{k,n}$	$k \leq n$	$n < k \leq \frac{4n}{3}$	$\frac{4n}{3} < k \leq 2n$	$k > 2n$
$k \equiv_4 0$	$(\mathfrak{R}_4^+ \setminus \mathfrak{R}_4^-) \sqcup (\mathfrak{R}_4^- \times \mathbb{Z}_2)$	$\mathfrak{S}$	$\{(0, \varphi, \varphi)\}_{\varphi \in [0, \frac{\pi}{2}]}$	$\{(0,0,0)\}$
$k \equiv_4 2$	$\{(\varphi, \frac{\pi}{2}, \frac{\pi}{2})\}_{\varphi \in [0, \frac{\pi}{2}]}$	$\{(0, \frac{\pi}{2}, \frac{\pi}{2})\}$	$\{(0, \frac{\pi}{2}, \frac{\pi}{2})\}$	$\emptyset$
$k \neq 3$ odd	$\{(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})\}$	$\emptyset$	$\emptyset$	$\emptyset$
$k = 3$	$(\mathfrak{R}_3^+ \setminus \mathfrak{R}_3^-) \sqcup (\mathfrak{R}_3^- \times \mathbb{Z}_2)$	$\emptyset$	$\{(\varphi, \varphi, \frac{\pi}{2})\}_{\varphi \in \{0, \frac{\pi}{3}\}}$	$\{(0, 0, \frac{\pi}{2})\}$

$$\begin{aligned}
\mathfrak{R}_3^+ &:= \{(\varphi, \varphi, \pi/2) \in \Lambda : \varphi \in [0, \pi/2]\}, \\
\mathfrak{R}_3^- &:= \{(\varphi, \varphi, \pi/2) \in \Lambda : \varphi \in [\pi/3, \pi/2)\}, \\
\mathfrak{R}_4^+ &:= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) - \cos(\varphi_3) \leq 1\}, \\
\mathfrak{R}_4^- &:= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) \leq 1, \ \varphi_3 \neq \pi/2\}, \\
\mathfrak{S} &:= \{(\varphi_1, \varphi_2, \varphi_3) \in \Lambda : \cos(\varphi_1) + \cos(\varphi_2) \pm \cos(\varphi_3) = 1\}.
\end{aligned}$$

Sketch of the proof

This is a brief sketch of our proof. Let $k := \dim_{\mathbb{R}}(V)$.

1. Using generalized Hairy ball theorem we classify c.q.k.a. subspaces with $k \neq 3, 4l$, $l \in \mathbb{N}$.
2. For $k \neq 3$ we prove that if V is protohomogeneous then $\{L_v\}_{v \in V}$ diagonalize simultaneously.
3. For $k \equiv_4 0$, we build a $\mathsf{Cl}(2)$ or $\mathsf{Cl}(3)$ structure in V, reducing the problem to $k = 3, 4$.
4. We classify c.q.k.a subspaces with $k = 3, 4$.

References

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