Symmetry and shape
Celebrating the 60th birthday of Prof. J. Berndt

A topological lower bound for the energy of a unit vector field on a closed Euclidean hypersurface

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Let $M^m$ be a compact oriented Riemannian manifold, $m \geq 2$, and let $\nabla$ denote its Levi-Civita connection. The energy of a unit vector field on $M$ is defined as the energy of the map $\vec{v} : M \to T^1M$, where $T^1M$ denotes the unit tangent bundle equipped with the Sasaki metric,

$$E(\vec{v}) = \frac{1}{2} \int_M \|\nabla \vec{v}\|^2 + \frac{m}{2} \text{vol}(M).$$ (1)

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An important question regarding these functionals is whether one can find unit vector fields Minimizing them. It is expected that these vector fields have nice properties.

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An important question regarding these functionals is whether one can find unit vector fields minimizing them. It is expected that these vector fields have nice properties.

**Theorem (Brito)**

*Hopf vector fields are the unique vector fields on $\mathbb{S}^3$ to minimize $E$.***

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Gluck and Ziller proved that Hopf flows are also the unit vector fields of minimum volume, with respect to the following definition of volume,

$$\text{vol}(\vec{v}) = \int_M \sqrt{\det(I + (\nabla \vec{v})(\nabla \vec{v})^*)},$$

where $I$ is the identity and $(\nabla \vec{v})^*$ represents adjoint operator.

**Theorem (Gluck and Ziller)**

The unit vector fields of minimum volume on $\mathbb{S}^3$ are precisely the Hopf vector fields, and no others.

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Construction of harmonic and minimal unit vector fields

Theorem (Berndt, Vanhecke and Verhóczki)

Let $M$ be a Riemannian symmetric space of compact or non-compact type, and let $F$ be a reflective submanifold of $M$ such that its codimension is greater than one and the rank of $F^\perp$ is equal to one. Then the radial unit vector field $\vec{v}$ associated to $F$ is harmonic and minimal.

- Harmonic = critical point of energy
- Minimal = critical point of volume

Reznikov compared the volume functional to the topology of an Euclidean hypersurface. Let $M^{n+1}$ be a smooth closed oriented immersed hypersurface in $\mathbb{R}^{n+2}$, endowed with the induced metric, and let $S = \sup_{x \in M} \|S_x\| = \sup_{x \in M} |\lambda_i(x)|$, where $S_x$ is the second fundamental operator in $T_xM$, and $\lambda_i(x)$ are the principal curvatures.

**Theorem (Reznikov)**

For any unit vector field $\vec{v}$ on $M$ we have

$$\text{vol}(\vec{v}) - \text{vol}(M) \geq \frac{\text{vol}(S^{n+1})}{S} |\text{deg}(\nu)|,$$

where $\text{deg}(\nu)$ is the degree of the Gauss map $\nu : M \to S^{n+1}$.

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A topological lower bound for the energy of a unit vector field on a closed Euclidean hypersurface

**Theorem A**

For a unit vector field on a closed oriented Euclidean hypersurface $M^{2n+1}$,

$$E(\vec{v}) \geq C(n) \frac{\text{deg}(\nu) \text{vol}(S^{2n+1})}{S^{[2n-1]}} + \frac{2n + 1}{2} \text{vol}(M^{2n+1})$$

where $S^{[2n-1]}$ and $C(n)$ are constants depending on the immersion of $M$ and on $n$.

- the energy of a given vector field depends on the topology of the immersion
Definition

If \( \{u_1, \ldots, u_{2n+1}\} \) is an orthonormal basis at \( x \in M \), then, for each \( 1 \leq A \leq 2n + 1 \),

\[
S[A] = \sup_{1 \leq i_1, \ldots, i_A \leq 2n+1; x \in M} \{ \| S(u_{i_1}) \wedge \cdots \wedge S(u_{i_A}) \|_{\infty} \},
\]

where \( \| \cdot \|_{\infty} \) denotes the maximum norm, naturally extended to \( \wedge^A(M) \).

\[
C(n) = \begin{cases} 
\frac{n}{2n-1}, & \text{if } M^{2n+1} = \mathbb{S}^{2n+1}(r), \\
\frac{1}{2}, & \text{otherwise}.
\end{cases}
\]
Theorem (Borrelli, Brito and Gil-Medrano)

The infimum of $E$ among all globally defined unit smooth vector fields of the sphere $S^{2n+1}$ $(n \geq 2)$ is

$$\left(\frac{2n + 1}{2} + \frac{n}{2n - 1}\right) \text{vol}(S^{2n+1}).$$

(2)

This value is not attained by any globally defined unit smooth vector field.

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Corollary

Let $S^{2n+1}(r)$ be the round sphere of radius $r$ in $\mathbb{R}^{2n+2}$. Then

$$E(\vec{v}) \geq \left( \frac{2n + 1}{2} r^{2n+1} + \frac{n}{2n - 1} r^{2n-1} \right) \text{vol}(S^{2n+1}).$$
Let \( \vec{v} \) be a unit vector field on a compact oriented Riemannian manifold \( M^m \). For every \( 1 \leq k \leq m - 1 \), define

\[
B_k(\vec{v}) = \int_M \left\| \nabla \vec{v} \wedge \cdots \wedge \nabla \vec{v} \right\|^2. 
\] (3)

If \( \sigma_{2n} \) denotes the \( 2n \)-th elementary symmetric function, and \( \mathcal{V} \) is the restriction of \( \nabla \vec{v} \) to \( V^\perp \) then our last theorem reads
Theorem B

Let $M^{2n+1}$ be a compact oriented Riemannian manifold, and let $\vec{v}$ be a unit vector field on $M$. Then

$$B_n(\vec{v}) \geq \binom{2n}{n} \int_M |\sigma_{2n}(V)|. \quad (4)$$

Furthermore, when $M^{2n+1}$ is a closed Euclidean hypersurface,

$$B_n(\vec{v}) \geq \frac{|\deg(\nu)|}{S} \binom{2n}{n} \text{vol}(S^{2n+1}), \quad (5)$$

where $S$ is the aforementioned constant.
Theorem B

Let $M^{2n+1}$ be a compact oriented Riemannian manifold, and let $\vec{v}$ be a unit vector field on $M$. Then

$$B_n(\vec{v}) \geq \binom{2n}{n} \int_M |\sigma_{2n}(\mathcal{V})|.$$  \hspace{1cm} (4)

Furthermore, when $M^{2n+1}$ is a closed Euclidean hypersurface,

$$B_n(\vec{v}) \geq \frac{\left|\deg(\nu)\right|}{S} \binom{2n}{n} \text{vol}(S^{2n+1}),$$ \hspace{1cm} (5)

where $S$ is the aforementioned constant.

Corollary

Hopf vector fields minimize $B_n$ on $S^{2n+1}$.

- What about uniqueness?
Thank you!