The index of a symmetric space

Carlos Olmos

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3. Applications of Simons holonomy theorem.

4. Reflective totally geodesic submanifolds.

5. Sufficient criteria for reflectivity.
   - Non-semisimple totally geodesic submanifolds.
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The existence and classification of totally geodesic submanifolds are two fundamental problems in submanifold geometry.

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The totally geodesic submanifolds in Riemannian symmetric spaces of rank one were classified by Joseph Wolf in 1963.

It is remarkable that the classification of totally geodesic submanifolds in Riemannian symmetric spaces of higher rank is a very complicated and essentially unsolved problem.

Élie Cartan already noticed an algebraic characterization of totally geodesic submanifolds in terms of Lie triple systems.

Although a Lie triple system is an elementary algebraic object, explicit calculations with them can be tremendously complicated.
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Using the Lie triple system approach, Sebastian Klein obtained between 2008-10, in a series of papers, the classification of totally geodesic submanifolds in irreducible Riemannian symmetric spaces of rank two.

No complete classifications are known for totally geodesic submanifolds in irreducible Riemannian symmetric spaces of rank greater than two.

A rather well-known result states that an irreducible Riemannian symmetric space which admits a totally geodesic hypersurface must be a space of constant curvature. As far as we know, the first proof of this fact was given by N. Iwahori in 1965.
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In 1980, A. L. Onishchik introduced the index $i(M)$ of a Riemannian symmetric space $M$ as the minimal codimension of a totally geodesic submanifold of $M$. Onishchik gave an alternative proof for Iwahori's result and also classified the irreducible Riemannian symmetric spaces with index 2.

In this lecture we would like to present a new approach to the index based, essentially, on geometric tools. By means of this approach we were able to determine the index of all symmetric spaces, with the exception of three families of classical type (on which we are still working).

Our point of view also allows us to determine the maximal totally geodesic submanifolds (of symmetric spaces) that are non-semisimple.
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Rank and index.

The starting point for dealing with the index of a symmetric space is the inequality in the following main result:

**Theorem (Berndt-O.)**

Let $M$ be an irreducible Riemannian symmetric space. Then

$$\text{rk}(M) \leq i(M).$$

Moreover, the equality holds if and only if, up to duality, $M = SL_{k+1}/SO_{k+1}$ or $M = G^*(\mathbb{R}^{n+k}) = SO^o_{k+n}/SO_k SO_n$.

We prove the inequality $\text{rank}(M) \leq i(M)$ by showing the following: if $\Sigma$ is a totally geodesic submanifold of a symmetric space $M$, then there exists a maximal flat $F$ of $M$ that intersects $\Sigma$ transversally.
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Let us now fix some notation.

\[ M = G/K \]

denotes a simply connected symmetric space with Cartan decomposition \( g = \mathfrak{k} \oplus \mathfrak{p} \) at \( p = [e] \), \( p \simeq T_p M \),

\[ \Sigma \subset M \]

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is the group of \textit{glide transformations} of \( \Sigma \), i.e.,

\[ \text{Lie}(G^\Sigma) = T_p \Sigma \oplus [T_p \Sigma, T_p \Sigma] \]

\[ \tilde{G}^\Sigma := \{ g \in G : g \Sigma = \Sigma \} \supset G^\Sigma \]

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\tilde{\rho}(g) = (dg)_{\nu_p\Sigma}
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The slice representation \( \rho : G^\Sigma_p \to O(\nu_p\Sigma) \) is

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If \( M \) is of the non-compact type then \( G^\Sigma_p \) is connected, since \( \Sigma \) is simply connected and so \( \rho : G^\Sigma_p \to SO(\nu_p\Sigma) \).

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An application of Simons holonomy theorem.

We now present a second auxiliary result which follows easily, in rank at least 2, from Simons theorem on holonomy systems.

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Let $\Sigma$ be a non-flat totally geodesic submanifold of an irreducible symmetric space $M$ which is not of constant curvature. Then the slice representation $\rho : (G^\Sigma_p) \circ \rightarrow SO(\nu_p \Sigma)$ is non-trivial.

The above theorem generalizes Iwahori’s result.

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Reflective totally geodesic submanifolds.

A totally geodesic submanifold $\Sigma$ of a symmetric space $M$ is called *reflective* if the exponential of the normal space $\exp_p(\nu_p\Sigma)$ is also a totally geodesic submanifold of $M$.

Equivalently, $\Sigma$ is reflective if $T_p\Sigma$ and $\nu_p\Sigma$ are both Lie triple systems.

One has that $\Sigma$ is reflective if and only if the reflection of $M$ in $\Sigma$ is an isometry.

Reflective submanifolds of symmetric spaces were classified by *D. S. P. Leung* in the 70’s.
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By making use of the Slice Lemma, we proved the following results:

Proposition (Berndt-O.; Berndt-O.-Rodríguez)

Let \( M = G/K \) be an irreducible Riemannian symmetric space with \( \text{rk}(M) \geq 2 \), where \( (G, K) \) is an effective Riemannian symmetric pair. Let \( \Sigma \) be a semisimple totally geodesic submanifold of \( M \) with \( p = [e] \in \Sigma \). Then \( \Sigma \) is reflective if and only if the kernel of the full slice representation \( \tilde{\rho} : \tilde{G}_p^\Sigma \rightarrow O(\nu_p\Sigma) \) is non-trivial.
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Corollary (Berndt-O.-Rodríguez)

Let $M = G/K$ be an irreducible simply connected Riemannian symmetric space with $\text{rk}(M) \geq 2$. Let $\Sigma_1, \Sigma_2$ be connected, complete, totally geodesic submanifolds of $M$ with $\Sigma_1 \subseteq \Sigma_2$. If $\Sigma_1$ is reflective, then $\Sigma_2$ is reflective.

We also have the following useful criterion:

Theorem (Berndt-O.-Rodríguez)

Let $\Sigma$ be a maximal semisimple totally geodesic submanifold of an irreducible symmetric space $M = G/K$. If $\dim(\tilde{G}\Sigma) > \dim(G\Sigma)$, then $\Sigma$ is reflective.
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Remark. Let us consider the (almost effective) symmetric pair \((G^Σ, G_p^Σ)\). The principal orbits, of the \(G_p^Σ\)-action on \(T_pΣ\), have dimension \(\dim(Σ) - \text{rk}(Σ)\). Then

\[
\dim(Σ) - \text{rk}(M) \leq \dim(G_p^Σ)
\]

On the other hand, the dimension of the image of the slice representation \(ρ(G_p^Σ) \subset \tilde{ρ}(\tilde{G}_p^Σ)\) is bounded from above by \(\dim(O(ν_p(Σ)) = \frac{1}{2}k(k - 1)\), where \(k\) is the codimension of \(Σ\).

Therefore, if \(\frac{1}{2}k(k - 1) < \dim(Σ) - \text{rk}(M)\), the full slice representation cannot be injective. Then

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*Let \(Σ^n\) be a totally geodesic submanifold of a symmetric space \(M^{n+k}\). If \(\frac{1}{2}k(k - 1) < n - \text{rk}(M)\), then \(Σ\) is reflective.*
Non-semisimple totally geodesic submanifolds.

If $\Sigma \subset M = G/K$ is totally geodesic and $p \in \Sigma$, then $T_p\Sigma$ is a Lie triple system of $p$. Assume that $\Sigma$ is not semisimple. Then there exists $0 \neq v \in T_p\Sigma$ such that $[v, T_p\Sigma] = 0$. Therefore $T_p\Sigma$ must be contained in the Lie triple system

$$C(v) = \{z \in p : [v, z] = 0\},$$

the centralizer of $v$ in $p$. One has that $C(v)$ coincides with the normal space $\nu_v(K.v)$ of the isotropy orbit. If $\Sigma$ is maximal, then the orbit $K.v$ must be most singular and hence

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The following result gives a nice characterization of the maximal non-semisimple totally geodesic submanifolds of a symmetric space.

**Theorem (Berndt-O.)**

Let $\Sigma$ be a non-semisimple totally geodesic submanifold of a symmetric space $M = G/K$, $p = [e] \in \Sigma$. Then $\Sigma$ is a maximal totally geodesic submanifold of $M$ if and only if $T_p \Sigma$ coincides with the normal space to an extrinsic symmetric orbit $K.v$ (and so it is reflective).
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Fixed vectors of the slice representation.

Theorem (Berndt-O.-Rodríguez)

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The index conjecture.

Let us define the *reflective index* of a symmetric space $M$ by

$$i_r = \min \ \{ \text{codim}(\Sigma) : \Sigma \subset M \text{ is totally geodesic and reflective} \}$$

The reflective index can be computed from Leung’s classification of reflective totally geodesic submanifolds of symmetric spaces. Clearly, $i(M) \leq i_r(M)$. Therefore

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In all known examples $i(M) = i_r(M)$, except for the space $M = G_2^2/\text{SO}_4$, or its compact dual. In this case the index is 3 but the reflective index is 4. We conjecture that this is the only exception. Namely,

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**Conjecture.** The index of an irreducible symmetric space $M$, which is different from $G_2/\text{SO}(4)$ and its symmetric dual, coincides with its reflective index $i_r(M)$. 
The index conjecture.

Let us define the \textit{reflective index} of a symmetric space \( M \) by

\[
i_r = \min \{ \text{codim}(\Sigma) : \Sigma \subset M \text{ is totally geodesic and reflective} \}
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The reflective index can be computed from Leung’s classification of reflective totally geodesic submanifolds of symmetric spaces. Clearly, \( i(M) \leq i_r(M) \). Therefore

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**Conjecture.** *The index of an irreducible symmetric space $M$, which is different from $G_2/\text{SO}(4)$ and its symmetric dual, coincides with its reflective index $i_r(M)$.***
We proved this conjecture for many classical families and for all symmetric spaces of group type. Moreover, we also determined the index, verifying the conjecture, for all exceptional symmetric spaces. The conjecture remains open only for the following three series of classical symmetric spaces and their symmetric dual spaces:

(i) $M = SO_{2k+2}/U_{k+1}$ for $k \geq 5$. Conjecture: $i(M) = 2k$.
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(iii) $M = Sp_{2k+1}/Sp_kSp_{k+1}$ for $l \geq 0$ and $k \geq \max\{3, l + 2\}$. Conjecture: $i(M) = 4k$.

We have some strategy, for dealing with these families, that hopefully would prove the index conjecture. But this is still a work in progress.
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References

Córdoba, November 2013

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Happy birthday Jürgen!!

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