Actions on positively curved manifolds and boundary in the orbit space

(Joint work with A. Kollross and B. Wilking)

Claudio Gorodski
University of São Paulo

Symmetry & Shape
Celebrating the 60th birthday of Prof. J. Berndt
Universidade de Santiago de Compostela, Spain
28-31 October 2019

- Let G be a compact Lie group acting by isometries on a complete Riemannian manifold M.

- Let G be a compact Lie group acting by isometries on a complete Riemannian manifold M.
- The orbit space $X = M/G$ is stratified by orbit types, and the boundary consists of the most important singular strata; here the *boundary* ∂X is defined as the closure of the union of all strata of codimension one of X.

Preliminaries

- Let G be a compact Lie group acting by isometries on a complete Riemannian manifold M.
- The orbit space $X = M/G$ is stratified by orbit types, and the boundary consists of the most important singular strata; here the *boundary* ∂X is defined as the closure of the union of all strata of codimension one of X.
- In case M is positively curved, this notion of boundary coincides with the boundary of X as an Alexandrov space and has a bearing on the geometry and topology of X.

Preliminaries

- Let G be a compact Lie group acting by isometries on a complete Riemannian manifold M.
- The orbit space $X = M/G$ is stratified by orbit types, and the boundary consists of the most important singular strata; here the *boundary* ∂X is defined as the closure of the union of all strata of codimension one of X.
- In case M is positively curved, this notion of boundary coincides with the boundary of X as an Alexandrov space and has a bearing on the geometry and topology of X.
- For instance, it is easy to see that ∂X is non-empty if and only if X is contractible.

- Let G be a compact Lie group acting by isometries on a complete Riemannian manifold M.
- The orbit space $X = M/G$ is stratified by orbit types, and the boundary consists of the most important singular strata; here the *boundary* ∂X is defined as the closure of the union of all strata of codimension one of X.
- In case M is positively curved, this notion of boundary coincides with the boundary of X as an Alexandrov space and has a bearing on the geometry and topology of X.
- For instance, it is easy to see that ∂X is non-empty if and only if X is contractible.
- The boundary often plays an important role in theorems regarding isometric actions.

- Let G be a compact Lie group acting by isometries on a complete Riemannian manifold M.
- The orbit space $X = M/G$ is stratified by orbit types, and the boundary consists of the most important singular strata; here the *boundary* ∂X is defined as the closure of the union of all strata of codimension one of X.
- In case M is positively curved, this notion of boundary coincides with the boundary of X as an Alexandrov space and has a bearing on the geometry and topology of X.
- For instance, it is easy to see that ∂X is non-empty if and only if X is contractible.
- The boundary often plays an important role in theorems regarding isometric actions.
- The existence of boundary is a *local condition*, in the sense that $X = M/G$ has non-empty boundary if and only if there exists a point $p \in M$ such that the slice representation of the isotropy group G_p on the normal space $\nu_p(Gp)$ to the orbit Gp has orbit space with non-empty boundary (slice theorem).

Relation to other classes

- In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:

- In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:
  (i) The principal isotropy group is non-trivial [Hsiang-Hsiang 1970].

Relation to other classes

- In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:
  (i) The principal isotropy group is non-trivial [Hsiang-Hsiang 1970].
  (ii) There exists a non-trivial *reduction*, that is, a representation of a group with smaller dimension and isometric orbit space [G.-Lytchak 2014].

- In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:
  - (i) The principal isotropy group is non-trivial [Hsiang-Hsiang 1970].
  - (ii) There exists a non-trivial *reduction*, that is, a representation of a group with smaller dimension and isometric orbit space [G.-Lytchak 2014].
  - (iii) The cohomogeneity, or codimension of the principal orbits is "low" [Hsiang-Lawson 1971].

- In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:
  (i) The principal isotropy group is non-trivial [Hsiang-Hsiang 1970].
  (ii) There exists a non-trivial *reduction*, that is, a representation of a group with smaller dimension and isometric orbit space [G.-Lytchak 2014].
  (iii) The cohomogeneity, or codimension of the principal orbits is “low” [Hsiang-Lawson 1971].
- (i) implies (ii) (take fix point set of principal isotropy group).

Relation to other classes

- In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:
  (i) The principal isotropy group is non-trivial [Hsiang-Hsiang 1970].
  (ii) There exists a non-trivial *reduction*, that is, a representation of a group with smaller dimension and isometric orbit space [G.-Lytchak 2014].
  (iii) The cohomogeneity, or codimension of the principal orbits is "low" [Hsiang-Lawson 1971].
- (i) implies (ii) (take fix point set of principal isotropy group).
- (ii) implies having non-empty boundary (apply Morse theory to sufficiently long geodesic contained in regular set).

- In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:
  - (i) The principal isotropy group is non-trivial [Hsiang-Hsiang 1970].
  - (ii) There exists a non-trivial *reduction*, that is, a representation of a group with smaller dimension and isometric orbit space [G.-Lytchak 2014].
  - (iii) The cohomogeneity, or codimension of the principal orbits is "low" [Hsiang-Lawson 1971].
- (i) implies (ii) (take fix point set of principal isotropy group).
- (ii) implies having non-empty boundary (apply Morse theory to sufficiently long geodesic contained in regular set).
- To some extent, (iii) is also related to non-empty boundary (as seen a posteriori).

Theorem

Let G be a compact connected simple Lie group acting effectively and isometrically on a connected complete orientable n-manifold M of positive sectional curvature. Assume that $X = M/G$ has non-empty boundary and $n \geq \ell_G$. Then G has a fixed point in M and $\dim M^G \geq \dim M - \ell_G$.

Theorem

Let G be a compact connected simple Lie group acting effectively and isometrically on a connected complete orientable n-manifold M of positive sectional curvature. Assume that $X = M/G$ has non-empty boundary and $n \geq \ell_G$. Then G has a fixed point in M and $\dim M^G \geq \dim M - \ell_G$.

(If $\dim M \geq \ell_G$ and $\partial X \neq \varnothing$, then $M^G \neq \varnothing$.)

Main Theorem (structural, “asymptotic” result)

Theorem

Let G be a compact connected Lie group acting effectively and isometrically on a connected complete orientable n-manifold M of positive sectional curvature. Assume that $X = M/G$ has non-empty boundary and

$$n > \alpha_G + \beta_G$$

where

$$\alpha_G = 2 \dim G_{ss} + 8 \operatorname{rk} G_{ss} + 4 \operatorname{nsf} G_{ss} \quad \text{and} \quad \beta_G = 2 \dim Z(G).$$

Then there exists a positive-dimensional normal subgroup N of G such that:

1. *The fixed point set M^N is non-empty (and G-invariant); let B be a component containing principal orbits of the G-action on M^N.*
2. *B/G has empty boundary and is contained in all faces of X.*
3. *In particular:*
    a. *N contains, up to conjugation, all isotropy groups of G corresponding to orbit types of strata of codimension one in X.*
    b. *At a generic point in B, the slice representation of N has orbit space with non-empty boundary.*

Main Theorem (structural, "asymptotic" result)

Theorem

Let G be a compact connected Lie group acting effectively and isometrically on a connected complete orientable n-manifold M of positive sectional curvature. Assume that $X = M/G$ has non-empty boundary and

$$n > \alpha_G + \beta_G$$

where

$$\alpha_G = 2\dim G_{ss} + 8\,\mathrm{rk}\,G_{ss} + 4\,\mathrm{nsf}\,G_{ss} \quad \text{and} \quad \beta_G = 2\dim Z(G).$$

Then there exists a positive-dimensional normal subgroup N of G such that:

1. *The fixed point set M^N is non-empty (and G-invariant); let B be a component containing principal orbits of the G-action on M^N.*
2. *B/G has empty boundary and is contained in all faces of X.*
3. *In particular:*
   a. *N contains, up to conjugation, all isotropy groups of G corresponding to orbit types of strata of codimension one in X.*
   b. *At a generic point in B, the slice representation of N has orbit space with non-empty boundary.*

Outline of proof

Outline of proof

- Basic idea of main theorem is to construct a normal subgroup containing all isotropy groups associated to codimension one strata of X for which we can prove its fixed point set is non-empty.

Outline of proof

- Basic idea of main theorem is to construct a normal subgroup containing all isotropy groups associated to codimension one strata of X for which we can prove its fixed point set is non-empty.
- Basic tool is **Frankel's theorem**:

$$\operatorname{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_\ell}) \leq \sum_{i=1}^{\ell} \operatorname{codim} M^{\sigma_i}.$$

Outline of proof

- Basic idea of main theorem is to construct a normal subgroup containing all isotropy groups associated to codimension one strata of X for which we can prove its fixed point set is non-empty.
- Basic tool is **Frankel's theorem**:

$$\operatorname{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_\ell}) \leq \sum_{i=1}^{\ell} \operatorname{codim} M^{\sigma_i}.$$

- Abelian case is easy.

Outline of proof

- Basic idea of main theorem is to construct a normal subgroup containing all isotropy groups associated to codimension one strata of X for which we can prove its fixed point set is non-empty.
- Basic tool is **Frankel's theorem**:

$$\operatorname{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_\ell}) \leq \sum_{i=1}^{\ell} \operatorname{codim} M^{\sigma_i}.$$

- Abelian case is easy.
- *Consider the special case G is simple*. We need to prove that G has a fixed point in M.

- Basic idea of main theorem is to construct a normal subgroup containing all isotropy groups associated to codimension one strata of X for which we can prove its fixed point set is non-empty.
- Basic tool is **Frankel's theorem**:

$$\mathrm{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_\ell}) \leq \sum_{i=1}^{\ell} \mathrm{codim}\, M^{\sigma_i}.$$

- Abelian case is easy.
- *Consider the special case G is simple*. We need to prove that G has a fixed point in M. We shall write M^G as a finite intersection of fixed points sets as in the LHS of Frankel's formula.

Outline of proof

- Basic idea of main theorem is to construct a normal subgroup containing all isotropy groups associated to codimension one strata of X for which we can prove its fixed point set is non-empty.
- Basic tool is **Frankel's theorem**:

$$\mathrm{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_\ell}) \leq \sum_{i=1}^{\ell} \mathrm{codim}\, M^{\sigma_i}.$$

- Abelian case is easy.
- *Consider the special case G is simple*. We need to prove that G has a fixed point in M. We shall write M^G as a finite intersection of fixed points sets as in the LHS of Frankel's formula. It suffices to find finitely many elements of G that generate a dense subgroup.

Outline, II

- Recall we are in case G is simple.

Outline, II

- Recall we are in case G is simple.
- An involutive inner automorphism σ of G defines a symmetric space of inner type G/K (here $K = G^\sigma$),

Outline, II

- Recall we are in case G is simple.
- An involutive inner automorphism σ of G defines a symmetric space of inner type G/K (here $K = G^\sigma$), and induces the geodesic symmetry of G/K at the basepoint.

Outline, II

- Recall we are in case G is simple.
- An involutive inner automorphism σ of G defines a symmetric space of inner type G/K (here $K = G^\sigma$), and induces the geodesic symmetry of G/K at the basepoint.

First remark

A finite number $\ell_{G/K}$ of generic conjugates of the involution generate a dense subgroup of G.

Outline, II

- Recall we are in case G is simple.
- An involutive inner automorphism σ of G defines a symmetric space of inner type G/K (here $K = G^\sigma$), and induces the geodesic symmetry of G/K at the basepoint.

First remark

A finite number $\ell_{G/K}$ of generic conjugates of the involution generate a dense subgroup of G. In fact, $\ell_{G/K}$ is the minimum number ℓ such that there exists $p_1, \ldots, p_\ell \in G/K$ "spanning" G/K in the sense that no proper connected closed totally geodesic submanifold of G/K contains those points.

Outline, II

- Recall we are in case G is simple.
- An involutive inner automorphism σ of G defines a symmetric space of inner type G/K (here $K = G^\sigma$), and induces the geodesic symmetry of G/K at the basepoint.

First remark

A finite number $\ell_{G/K}$ of generic conjugates of the involution generate a dense subgroup of G. In fact, $\ell_{G/K}$ is the minimum number ℓ such that there exists $p_1, \ldots, p_\ell \in G/K$ "spanning" G/K in the sense that no proper connected closed totally geodesic submanifold of G/K contains those points.

- For generic p_1, $p_2 \in G/K$, $\mathrm{span}\{p_1, p_2\}$ is a maximal flat torus.

- Recall we are in case G is simple.
- An involutive inner automorphism σ of G defines a symmetric space of inner type G/K (here $K = G^\sigma$), and induces the geodesic symmetry of G/K at the basepoint.

First remark

A finite number $\ell_{G/K}$ of generic conjugates of the involution generate a dense subgroup of G. In fact, $\ell_{G/K}$ is the minimum number ℓ such that there exists $p_1, \ldots, p_\ell \in G/K$ "spanning" G/K in the sense that no proper connected closed totally geodesic submanifold of G/K contains those points.

- For generic p_1, $p_2 \in G/K$, $\mathrm{span}\{p_1, p_2\}$ is a maximal flat torus.
- For generic $p_1, \ldots, p_k$ $(k \geq 2)$,

$$\mathrm{span}\{p_1, \ldots, p_k\} = L(p_1) = \cdots = L(p_k)$$

where L is the closure of the group generated by even products of the geodesic symmetries at $p_1, \ldots, p_k$.

Second remark

We can make the codimension in M of the fixed point set of the involution σ to be bounded by

$$4 + \dim G/K \qquad (1)$$

by suitably choosing σ to fix a regular point or an *important* point (i.e. a point projecting to a codimension one stratum of X) in M.

Second remark

We can make the codimension in M of the fixed point set of the involution σ to be bounded by

$$4 + \dim G/K \tag{1}$$

by suitably choosing σ to fix a regular point or an *important* point (i.e. a point projecting to a codimension one stratum of X) in M. In fact, we can find $\sigma \in G$ of order 2 in $\mathrm{Ad}(G) = G/Z(G)$ such that σ fixes a regular point (in case $\dim G_{princ} > 0$ or G is finite of even order) or an important point (in case G_{princ} is finite of odd order) in M.

Second remark

We can make the codimension in M of the fixed point set of the involution σ to be bounded by

$$4 + \dim G/K \tag{1}$$

by suitably choosing σ to fix a regular point or an *important* point (i.e. a point projecting to a codimension one stratum of X) in M. In fact, we can find $\sigma \in G$ of order 2 in $\mathrm{Ad}(G) = G/Z(G)$ such that σ fixes a regular point (in case $\dim G_{princ} > 0$ or G is finite of even order) or an important point (in case G_{princ} is finite of odd order) in M.

- We call an element $\sigma \in G$ of order 2 in $\mathrm{Ad}(G)$ satisfying estimate (1) a *nice involution*.

- Let

$$\ell_G := \max_K \{\ell_{G/K}(4 + \dim G/K)\},$$

where K runs through all symmetric subgroups of G with maximal rank.

Outline, IV

- Let
$$\ell_G := \max_K \{\ell_{G/K}(4 + \dim G/K)\},$$
where K runs through all symmetric subgroups of G with maximal rank.
- Now Frankel's theorem yields:

$$\operatorname{codim} M^G \quad = \quad \operatorname{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_{\ell_{G/K}}}) \quad (\sigma_i\text{'s: gen conj of } \sigma)$$

Outline, IV

- Let
$$\ell_G := \max_K \{\ell_{G/K}(4 + \dim G/K)\},$$
where K runs through all symmetric subgroups of G with maximal rank.
- Now Frankel's theorem yields:

$$\begin{aligned} \operatorname{codim} M^G &= \operatorname{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_{\ell_{G/K}}}) \quad (\sigma_i\text{'s: gen conj of } \sigma) \\ &\leq \sum_{i=1}^{\ell_{G/K}} \operatorname{codim} M^{\sigma_i} \quad \text{(Frankel)} \end{aligned}$$

- Let
$$\ell_G := \max_K\{\ell_{G/K}(4+\dim G/K)\},$$
where K runs through all symmetric subgroups of G with maximal rank.
- Now Frankel's theorem yields:

$$\begin{aligned}
\operatorname{codim} M^G &= \operatorname{codim}(M^{\sigma_1}\cap\cdots\cap M^{\sigma_{\ell_{G/K}}}) \quad (\sigma_i\text{'s: gen conj of } \sigma)\\
&\leq \sum_{i=1}^{\ell_{G/K}} \operatorname{codim} M^{\sigma_i} \quad \text{(Frankel)}\\
&\leq \ell_{G/K}(4+\dim G/K) \text{ (nice involututions)}
\end{aligned}$$

Outline, IV

- Let
$$\ell_G := \max_K\{\ell_{G/K}(4 + \dim G/K)\},$$
where K runs through all symmetric subgroups of G with maximal rank.
- Now Frankel's theorem yields:

$$\begin{aligned}
\operatorname{codim} M^G &= \operatorname{codim}(M^{\sigma_1} \cap \cdots \cap M^{\sigma_{\ell_{G/K}}}) \quad (\sigma_i\text{'s: gen conj of } \sigma) \\
&\leq \sum_{i=1}^{\ell_{G/K}} \operatorname{codim} M^{\sigma_i} \quad \text{(Frankel)} \\
&\leq \ell_{G/K}(4 + \dim G/K) \text{ (nice involututions)} \\
&\leq \ell_G.
\end{aligned}$$

- Let
$$\ell_G := \max_K\{\ell_{G/K}(4+\dim G/K)\},$$
where K runs through all symmetric subgroups of G with maximal rank.
- Now Frankel's theorem yields:

$$\begin{aligned}
\operatorname{codim} M^G &= \operatorname{codim}(M^{\sigma_1}\cap\cdots\cap M^{\sigma_{\ell_{G/K}}}) \quad (\sigma_i\text{'s: gen conj of } \sigma)\\
&\leq \sum_{i=1}^{\ell_{G/K}} \operatorname{codim} M^{\sigma_i} \quad \text{(Frankel)}\\
&\leq \ell_{G/K}(4+\dim G/K) \text{ (nice involututions)}\\
&\leq \ell_G.
\end{aligned}$$

- In the case of a general compact connected Lie group, the argument is more technical and one proceeds by induction using the simple factors and the center. (We skip the details.)

Application: representations of compact connected simple Lie groups with non-empty boundary in the orbit space

G	ker	V	Property	Effective p.i.g.
$\mathbf{SU}(2)$	1	$\mathbb{C}^2$	polar	1
$\mathbf{SO}(3)$	1	$\mathbb{R}^3$ $S_0^2\mathbb{R}^3 = \mathbb{R}^5$	polar	T^1 $\mathbb{Z}_2^2$
$\mathbf{SU}(n)$ $(n \geq 3)$	1 $\mathbb{Z}_n$ $\{\pm 1\}$ if n is even	$\mathbb{C}^n$ Ad $S^2\mathbb{C}^n$	polar toric	$\mathbf{SU}(n-1)$ T^{n-1} $\mathbb{Z}_2^{n-1}$
$\mathbf{SU}(n)$ $(n \geq 5)$	$\{\pm 1\}$ if n is even	$\Lambda^2\mathbb{C}^n$	polar if n is odd, toric otherwise	$\mathbf{SU}(2)^{\lfloor \frac{n}{2} \rfloor}/\ker$
$\mathbf{SU}(6)$	1	$\Lambda^3\mathbb{C}^6 = \mathbb{H}^{10}$	q-toric	T^2
$\mathbf{SU}(8)$	$\mathbb{Z}_4$	$[\Lambda^4\mathbb{C}^8]_{\mathbb{R}}$	polar	$\mathbb{Z}_2^7$
$\mathbf{SO}(n)$ $(n \geq 5)$	1 $\{\pm 1\}$ if n is even	$\mathbb{R}^n$ $\Lambda^2\mathbb{R}^n = \text{Ad}$ $S_0^2\mathbb{R}^n$	polar	$\mathbf{Spin}(n-1)$ $T^{\lfloor \frac{n}{2} \rfloor}$ $\mathbb{Z}_2^{n-1}$
$\mathbf{Spin}(7)$	1	$\mathbb{R}^8$ (spin)	polar	$\mathbf{G}_2$
$\mathbf{Spin}(8)$	$\mathbb{Z}_2$	$\mathbb{R}^8_\pm$ (half-spin)	polar	$\mathbf{Spin}(7)'$
$\mathbf{Spin}(9)$	1	$\mathbb{R}^{16}$ (spin)	polar	$\mathbf{Spin}(7)$
$\mathbf{Spin}(10)$	1	$\mathbb{C}^{16}_\pm$ (half-spin)	polar	$\mathbf{SU}(4)$
$\mathbf{Spin}(11)$	1	$\mathbb{H}^{16}$ (spin)	—	1
$\mathbf{Spin}(12)$	$\mathbb{Z}_2$	$\mathbb{H}^{16}_\pm$ (half-spin)	q-toric	$\mathbf{Sp}(1)^3$
$\mathbf{Spin}(16)$	$\mathbb{Z}_2$	$\mathbb{R}^{128}_\pm$ (half-spin)	polar	$\mathbb{Z}_2^8$
$\mathbf{Sp}(n)$ $(n \geq 3)$	1 ± 1	$\mathbb{C}^{2n} = \mathbb{H}^n$ $[S^2\mathbb{C}^{2n}]_{\mathbb{R}} = \text{Ad}$ $[\Lambda_0^2\mathbb{C}^{2n}]_{\mathbb{R}}$	polar	$\mathbf{Sp}(n-1)$ T^n $\mathbf{Sp}(1)^n/\{\pm 1\}$
$\mathbf{Sp}(3)$	1	$\Lambda_0^3\mathbb{C}^6 = \mathbb{H}^7$	q-toric	$\mathbb{Z}_2^2$
$\mathbf{Sp}(4)$	$\{\pm 1\}$	$[\Lambda_0^4\mathbb{C}^8]_{\mathbb{R}}$	polar	$\mathbb{Z}_2^6$

Representations, cont'd

G	ker	V	Property	Effective p.i.g.
$\mathbf{G}_2$	1	$\mathbb{R}^7$ Ad	polar	$\mathbf{SU}(3)$ T^2
$\mathbf{F}_4$	1	$\mathbb{R}^{26}$ Ad	polar	$\mathbf{Spin}(8)$ T^4
$\mathbf{E}_6$	1	$\mathbb{C}^{27}$	toric	$\mathbf{Spin}(8)$
$\mathbf{E}_6$	$\mathbb{Z}_3$	Ad	polar	T^6
$\mathbf{E}_7$	1	$\mathbb{H}^{28}$	q-toric	$\mathbf{Spin}(8)$
$\mathbf{E}_7$	$\mathbb{Z}_2$	Ad	polar	T^7
$\mathbf{E}_8$	1	Ad	polar	T^8

$\mathbf{SU}(n)$	$k\,\mathbb{C}^n$ $\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$	$2 \leq k \leq n-1$ $n \geq 4$
$\mathbf{SU}(4)$	$k\,\mathbb{R}^6 \oplus \ell\,\mathbb{C}^4$ $\mathbb{R}^6 \oplus \mathrm{Ad}$	$2 \leq k+\ell \leq 3$ –
$\mathbf{Spin}(n)$	$k\,\mathbb{R}^n$ $\mathbb{R}^n \oplus \mathrm{Ad}$	$2 \leq k \leq n-1$ $n \geq 4$
$\mathbf{Sp}(2)$	$\mathbb{H}^2 \oplus \mathbb{R}^5$	–
$\mathbf{Spin}(7)$	$k\,\mathbb{R}^7 \oplus \ell\,\mathbb{R}^8$	$2 \leq k+\ell \leq 4$
$\mathbf{Spin}(8)$	$k\,\mathbb{R}^8 \oplus \ell\,\mathbb{R}^8_+ \oplus m\,\mathbb{R}^8_-$	$2 \leq k+\ell+m \leq 5$
$\mathbf{Spin}(9)$	$k\,\mathbb{R}^{16}$ $\mathbb{R}^{16} \oplus k\,\mathbb{R}^9$ $2\mathbb{R}^{16} \oplus k\,\mathbb{R}^9$	$2 \leq k \leq 3$ $1 \leq k \leq 4$ $0 \leq k \leq 2$
$\mathbf{Spin}(10)$	$\mathbb{C}^{16} \oplus k\,\mathbb{R}^{10}$	$1 \leq k \leq 3$
$\mathbf{Spin}(12)$	$\mathbb{H}^{16} \oplus \mathbb{R}^{12}$	–
$\mathbf{Sp}(n)$	$k\,\mathbb{C}^{2n}$ $\mathbb{C}^{2n} \oplus [\Lambda^2_0\mathbb{C}^{2n}]_\mathbb{R}$	$2 \leq k \leq n$ $n \geq 3$
$\mathbf{Sp}(3)$	$2\,[\Lambda^2_0\mathbb{C}^6]_\mathbb{R}$	–
$\mathbf{G}_2$	$k\,\mathbb{R}^7$	$2 \leq k \leq 3$
$\mathbf{F}_4$	$2\,\mathbb{R}^{26}$	–

Theorem

Suppose G is a compact Lie group,

$$\rho : G \to \mathbf{O}(V)$$

is a **quaternionic** *representation of* **cohomogeneity at least two** *and*

$$\hat{\rho} : \hat{G} = G \times \mathbf{Sp}(1) \to \mathbf{O}(V)$$

is its natural extension. Then

$$\dim V/G = \dim V/\hat{G} + 3$$

.

Theorem

Suppose G is a compact Lie group,

$$\rho : G \to \mathbf{O}(V)$$

is a **quaternionic** *representation of* **cohomogeneity at least two** *and*

$$\hat{\rho} : \hat{G} = G \times \mathbf{Sp}(1) \to \mathbf{O}(V)$$

is its natural extension. Then

$$\dim V/G = \dim V/\hat{G} + 3$$

.

Proof.

Follows from previous classification by going to maximal connected groups. □

Application: Isometric actions of some simple groups

Let G be one of the following simple Lie groups:

$$\mathbf{SU}(2),\ \mathbf{SU}(n)/\mathbb{Z}_n\ (n\geq 3),\ \mathbf{SU}(8)/\mathbb{Z}_4,\ \mathbf{SO}(n)/\{\pm 1\}\ (n\geq 6 \text{ even}),$$
$$\mathbf{SO}'(16),\ \mathbf{Sp}(n)/\{\pm 1\}\ (n\geq 4),\ \mathbf{E}_6/\mathbb{Z}_3,\ \mathbf{E}_7/\mathbb{Z}_2,\ \mathbf{E}_8.$$

Let G be one of the following simple Lie groups:

$$\mathbf{SU}(2),\ \mathbf{SU}(n)/\mathbb{Z}_n\ (n \geq 3),\ \mathbf{SU}(8)/\mathbb{Z}_4,\ \mathbf{SO}(n)/\{\pm 1\}\ (n \geq 6 \text{ even}),$$
$$\mathbf{SO}'(16),\ \mathbf{Sp}(n)/\{\pm 1\}\ (n \geq 4),\ \mathbf{E}_6/\mathbb{Z}_3,\ \mathbf{E}_7/\mathbb{Z}_2,\ \mathbf{E}_8.$$

Theorem

An effective isometric action of G on a connected simply-connected compact positively curved manifold of dimension $n > \ell_G$ has non-empty boundary in the orbit space if and only if the action is polar (in this case, M is equivariantly diffeomorphic to a CROSS with a linearly induced action).

Let G be one of the following simple Lie groups:

$$\mathbf{SU}(2),\ \mathbf{SU}(n)/\mathbb{Z}_n\ (n\geq 3),\ \mathbf{SU}(8)/\mathbb{Z}_4,\ \mathbf{SO}(n)/\{\pm 1\}\ (n\geq 6 \text{ even}),$$
$$\mathbf{SO}'(16),\ \mathbf{Sp}(n)/\{\pm 1\}\ (n\geq 4),\ \mathbf{E}_6/\mathbb{Z}_3,\ \mathbf{E}_7/\mathbb{Z}_2,\ \mathbf{E}_8.$$

Theorem

An effective isometric action of G on a connected simply-connected compact positively curved manifold of dimension $n>\ell_G$ has non-empty boundary in the orbit space if and only if the action is polar (in this case, M is equivariantly diffeomorphic to a CROSS with a linearly induced action).

Proof.

Follows from classification above using deep results from Grove-Searle and Fang-Grove-Thorbergsson. □

Thank you!