Actions on positively curved manifolds and boundary in the orbit space

(Joint work with A. Kollross and B. Wilking)

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The orbit space $X = M/G$ is stratified by orbit types, and the boundary consists of the most important singular strata; here the boundary $\partial X$ is defined as the closure of the union of all strata of codimension one of $X$. In case $M$ is positively curved, this notion of boundary coincides with the boundary of $X$ as an Alexandrov space and has a bearing on the geometry and topology of $X$. For instance, it is easy to see that $\partial X$ is non-empty if and only if $X$ is contractible. The boundary often plays an important role in theorems regarding isometric actions. The existence of boundary is a local condition, in the sense that $X = M/G$ has non-empty boundary if and only if there exists a point $p \in M$ such that the slice representation of the isotropy group $G_p$ on the normal space $\nu_p(Gp)$ to the orbit $Gp$ has orbit space with non-empty boundary (slice theorem).
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In the case of orthogonal representations of compact Lie groups on vector spaces (or more generally, isometric actions on positively curved manifolds), the following criteria have been used to describe representations whose geometry is not too complicated, namely:

1. The principal isotropy group is non-trivial [Hsiang-Hsiang 1970].
2. There exists a non-trivial reduction, that is, a representation of a group with smaller dimension and isometric orbit space [G.-Lytchak 2014].
3. The cohomogeneity, or codimension of the principal orbits is "low" [Hsiang-Lawson 1971].

(i) implies (ii) (take fix point set of principal isotropy group).
(ii) implies having non-empty boundary (apply Morse theory to sufficiently long geodesic contained in regular set).
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Theorem

Let $G$ be a compact connected simple Lie group acting effectively and isometrically on a connected complete orientable $n$-manifold $M$ of positive sectional curvature. Assume that $X = M/G$ has non-empty boundary and $n \geq \ell_G$. Then $G$ has a fixed point in $M$ and $\dim M^G \geq \dim M - \ell_G$. (If $\dim M \geq \ell_G$ and $\partial X \neq \emptyset$, then $M^G \neq \emptyset$.)
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$$n > \alpha_G + \beta_G$$

where

$$\alpha_G = 2 \dim G_{ss} + 8 \rk G_{ss} + 4 \nnsf G_{ss} \quad \text{and} \quad \beta_G = 2 \dim Z(G).$$

Then there exists a positive-dimensional normal subgroup $N$ of $G$ such that:

1. The fixed point set $M^N$ is non-empty (and $G$-invariant); let $B$ be a component containing principal orbits of the $G$-action on $M^N$.
2. $B/G$ has empty boundary and is contained in all faces of $X$.
3. In particular:
   a. $N$ contains, up to conjugation, all isotropy groups of $G$ corresponding to orbit types of strata of codimension one in $X$.
   b. At a generic point in $B$, the slice representation of $N$ has orbit space with non-empty boundary.
Main Theorem (structural, “asymptotic” result)

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Outline of proof

Basic idea of main theorem is to construct a normal subgroup containing all isotropy groups associated to codimension one strata of $X$ for which we can prove its fixed point set is non-empty.

Basic tool is Frankel's theorem:
\[ \text{codim}(M_{\sigma_1} \cap \cdots \cap M_{\sigma_\ell}) \leq \ell \sum_{i=1}^{\ell} \text{codim}(M_{\sigma_i}). \]

Abelian case is easy.

Consider the special case $G$ is simple. We need to prove that $G$ has a fixed point in $M$.

We shall write $M_G$ as a finite intersection of fixed points as in the LHS of Frankel's formula. It suffices to find finitely many elements of $G$ that generate a dense subgroup.
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A finite number $\ell_{G/K}$ of generic conjugates of the involution generate a dense subgroup of $G$. In fact, $\ell_{G/K}$ is the minimum number $\ell$ such that there exists $p_1, \ldots, p_\ell \in G/K$ “spanning” $G/K$ in the sense that no proper connected closed totally geodesic submanifold of $G/K$ contains those points.
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where $L$ is the closure of the group generated by even products of the geodesic symmetries at $p_1, \ldots, p_k$. 
Second remark

We can make the codimension in $M$ of the fixed point set of the involution $\sigma$ to be bounded by

$$4 + \dim G/K$$

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by suitably choosing $\sigma$ to fix a regular point or an important point (i.e. a point projecting to a codimension one stratum of $X$) in $M$. 

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- We call an element $\sigma \in G$ of order 2 in $\text{Ad}(G)$ satisfying estimate (1) a *nice involution*. 
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Now Frankel’s theorem yields:
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\text{codim } M^G = \text{codim} (M^{\sigma_1} \cap \cdots \cap M^{\sigma_{\ell_G/K}}) \quad (\sigma_i's: \text{gen conj of } \sigma)
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In the case of a general compact connected Lie group, the argument is more technical and one proceeds by induction using the simple factors and the center. (We skip the details.)
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### Application: representations of compact connected simple Lie groups with non-empty boundary in the orbit space

<table>
<thead>
<tr>
<th>$G$</th>
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<tr>
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<td>1</td>
<td>$\mathbb{R}^{26}$</td>
<td>polar</td>
<td>$Spin(8)$ $T^4$</td>
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<td></td>
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<td>$\mathbb{C}^{27}$</td>
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<td>$Spin(8)$</td>
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<td>1</td>
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<td>$Spin(8)$</td>
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<td>$E_8$</td>
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<td>$Ad$</td>
<td>polar</td>
<td>$T^8$</td>
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</tbody>
</table>

For $SU(n)$:

- $k \mathbb{C}^n \oplus \Lambda^2 \mathbb{C}^n$
- $2 \leq k \leq n - 1$
- $n \geq 4$

For $SU(4)$:

- $k \mathbb{R}^6 \oplus \ell \mathbb{C}^4$
- $2 \leq k + \ell \leq 3$
- $-$

For $Spin(n)$:

- $k \mathbb{R}^n \oplus \ell \mathbb{R}^8$
- $2 \leq k + \ell \leq 4$
- $-$

For $Spin(8)$:

- $k \mathbb{R}^8 \oplus \ell \mathbb{R}^8 \oplus m \mathbb{R}^8$
- $2 \leq k + \ell + m \leq 5$

For $Spin(9)$:

- $k \mathbb{R}^16$
- $1 \leq k \leq 3$
- $1 \leq k \leq 4$
- $0 \leq k \leq 2$

For $Spin(10)$:

- $k \mathbb{C}^{16} \oplus \mathbb{R}^{10}$
- $1 \leq k \leq 3$

For $Spin(12)$:

- $k \mathbb{H}^{16} \oplus \mathbb{R}^{12}$
- $-$

For $Sp(n)$:

- $k \mathbb{C}^{2n}$
- $2 \leq k \leq n$
- $n \geq 3$

For $Sp(3)$:

- $2 \mathbb{C}^6$
- $-$

For $G_2$:

- $k \mathbb{R}^7$
- $2 \leq k \leq 3$

For $F_4$:

- $2 \mathbb{R}^{26}$
- $-$
Theorem

Suppose $G$ is a compact Lie group,

$$\rho : G \to O(V)$$

is a quaternionic representation of cohomogeneity at least two and

$$\hat{\rho} : \hat{G} = G \times Sp(1) \to O(V)$$

is its natural extension. Then

$$\dim V / G = \dim V / \hat{G} + 3$$
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$$\dim V / G = \dim V / \hat{G} + 3$$

Proof.

Follows from previous classification by going to maximal connected groups.
Let $G$ be one of the following simple Lie groups:

\[
\begin{align*}
\text{SU}(2), & \quad \text{SU}(n)/\mathbb{Z}_n \ (n \geq 3), \quad \text{SU}(8)/\mathbb{Z}_4, \quad \text{SO}(n)/\{\pm 1\} \ (n \geq 6 \text{ even}), \\
\text{SO}'(16), & \quad \text{Sp}(n)/\{\pm 1\} \ (n \geq 4), \quad \text{E}_6/\mathbb{Z}_3, \quad \text{E}_7/\mathbb{Z}_2, \quad \text{E}_8.
\end{align*}
\]
Let $G$ be one of the following simple Lie groups:

- $\text{SU}(2)$
- $\text{SU}(n)/\mathbb{Z}_n$ ($n \geq 3$)
- $\text{SU}(8)/\mathbb{Z}_4$
- $\text{SO}(n)/\{\pm 1\}$ ($n \geq 6$ even)
- $\text{SO}'(16)$
- $\text{Sp}(n)/\{\pm 1\}$ ($n \geq 4$)
- $E_6/\mathbb{Z}_3$
- $E_7/\mathbb{Z}_2$
- $E_8$

**Theorem**

An effective isometric action of $G$ on a connected simply-connected compact positively curved manifold of dimension $n > \ell_G$ has non-empty boundary in the orbit space if and only if the action is polar (in this case, $M$ is equivariantly diffeomorphic to a CROSS with a linearly induced action).
Let $G$ be one of the following simple Lie groups:

\[ \text{SU}(2), \text{SU}(n)/\mathbb{Z}_n \ (n \geq 3), \text{SU}(8)/\mathbb{Z}_4, \text{SO}(n)/\{\pm 1\} \ (n \geq 6 \text{ even}), \]
\[ \text{SO}'(16), \text{Sp}(n)/\{\pm 1\} \ (n \geq 4), \text{E}_6/\mathbb{Z}_3, \text{E}_7/\mathbb{Z}_2, \text{E}_8. \]

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**Proof.**

Follows from classification above using deep results from Grove-Searle and Fang-Grove-Thorbergsson.
Thank you!