Geometry of homogeneous hypersurfaces in noncompact symmetric spaces

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Symmetry and shape
- Celebrating the 60th birthday of Prof. J. Berndt
(Universidade de Santiago de Compostela, Spain)
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Preface (1/3)

- In Mar/2000 - Feb/2001, I worked with Jürgen in University of Hull, and we have studied cohomogeneity one actions on noncompact symmetric spaces.

Research in Pairs, MFO, 2003

Preface (2/3)

A photo taken at Prof. Ohnita's house, around 2001/2002.

[in the talk I put here a photo]

Preface (3/3)

Note

- M : irreducible symmetric space of noncompact type.
- According to the results by Berndt-Brück and Berndt-T., homogeneous hypersurfaces in M can be divided into 3 types, namely, type (K), (A), (N).
- In this talk we summarize some recent works after that, and fill in the following table.

type	classification	extrinsic geometry	intrinsic geometry
(K)			
(A)			
(N)			

Sec. 1: Preliminaries (1/3)

- We need to mention what are type (K), (A), (N).

Def

Let $H \curvearrowright M$ be an isometric action. Then

- a **regular orbit** is a maximal dimensional orbit;
- other orbits are **singular**;
- it is of **cohomogeneity one** (**C1**) if $\mathrm{codim}(\text{regular orbit}) = 1$.

Note

- homogeneous hypersurfaces = regular orbits of C1-actions.

Sec. 1: Preliminaries (2/3)

Thm. (Berndt-Brück 2001, Berndt-T. 2003)

Let

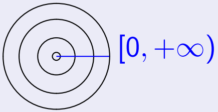
- M : irreducible symmetric space of noncompact type,
- $H \curvearrowright M$: C1-action with H being connected.

Then it satisfies one of:

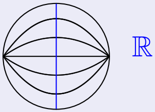
(K) $\exists 1$ singular orbit;

(A) $\nexists$ singular orbit, $\exists 1$ minimal orbit;

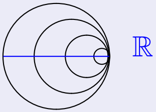
(N) $\nexists$ singular orbit, all orbits are congruent to each other.



type (K)



type (A)



type (N)

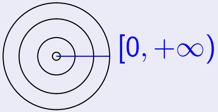
Sec. 1: Preliminaries (3/3)

Ex.

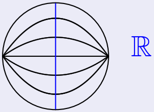
Let us consider

- $\mathbb{R}\mathrm{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$,
- $\mathrm{SL}(2,\mathbb{R}) = KAN$: an Iwasawa decomposition.

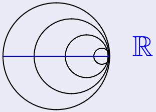
Then $K, A, N \curvearrowright \mathbb{R}\mathrm{H}^2$ look like as follows.



type (K)



type (A)



type (N)

Sec. 2: Type (A) (1/7)

Classification of type (A)-actions...

Thm. (Berndt-T. JDG 2003)

- $\#(\{\text{type } (A)\text{-actions on } M\}/\text{equiv.}) = \mathrm{rank}(M)$ (almost).

More Explicitly

- Take α_i (simple root) and $0 \neq \xi \in \mathfrak{g}_{\alpha_i}$.
- $\mathfrak{s}_\xi := \mathfrak{s} \ominus \xi = \mathfrak{a} \oplus (\mathfrak{n} \ominus \xi)$. Then $S_\xi \curvearrowright M$ is of type (A).
- Let $\xi \in \mathfrak{g}_{\alpha_i}$, $\eta \in \mathfrak{g}_{\alpha_j}$. Then $S_\xi \sim S_\eta$ (orbit equivalent) iff $\alpha_i \sim \alpha_j$ by Dynkin diagram auto.

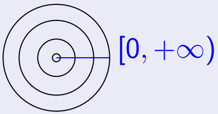
Sec. 2: Type (A) (2/7)

Note

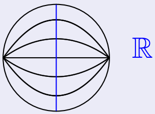
- If $\mathrm{rank}(M) = 1$, then $\exists 1$ type (A)-action.

Example

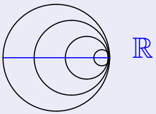
- For $M = \mathbb{R}\mathrm{H}^n$, the minimal orbit is a tot. geod. $\mathbb{R}\mathrm{H}^{n-1}$.
- For $M = \mathbb{C}\mathrm{H}^n$, the minimal orbit is "ruled".



type (K)



type (A)



type (N)

Sec. 2: Type (A) (3/7)

Extrinsic geometry of type (A)-actions...

Recall

(A) $\nexists$ singular orbit, $\exists 1$ minimal orbit.

Note

- We know principal curvatures of all orbits (Berndt-T.).

Thm.

The unique minimal orbit of type (A)-action is

- always austere (Berndt-T.);
- always **weakly reflective** (Hashinaga-Kubo-T., Gondo);
- totally geodesic iff $M = \mathbb{R}\mathrm{H}^n$ (orbit $= \mathbb{R}\mathrm{H}^{n-1}$).

Sec. 2: Type (A) (4/7)

Def. (Ikawa-Sakai-Tasaki 2009)

A submfd M in a Riem. mfd $(\overline{M}, g)$ is **weakly reflective** if

- $\forall p \in M$, $\forall \xi \in \nu_p(M)$, $\exists f \in \mathrm{Isom}(\overline{M}, g)$:

$$f(p) = p, \quad f(M) = M, \quad (df)_p(\xi) = -\xi.$$

Note

- reflective $\Rightarrow$ weakly reflective $\Rightarrow$ austere.

Note

- We can construct a "reflection" for the unique minimal orbit of type (A) action, in terms of the structure of AN.
- This shows that they are austere, without calculating the principal curvatures.

Sec. 2: Type (A) (5/7)

Intrinsic geometry of type (A)-actions...

Ex

For the type (A)-action on $M = \mathbb{R}\mathrm{H}^n$:

- all orbits are "$\mathbb{R}\mathrm{H}^{n-1}(c)$" (constant sectional curvature).

Thm. (Hashinaga-Kubo-T. Tohoku 2016)

For the type (A)-action on $M = \mathbb{C}\mathrm{H}^n$:

- an orbit is Ricci soliton iff it is minimal and $n = 2$.

Sec. 2: Type (A) (6/7)

Open Problem

- For type (A)-actions on other M, study whether orbits are Einstein/Ricci soliton.

Note

- By a result on $\mathbb{C}\mathrm{H}^n$, it would relate to the "multiplicities" of roots.
- By examples of Einstein solvmanifolds [T. 2011] and some general theory, Ricci soliton orbits seem to exist.

Sec. 2: Type (A) (7/7)

Summary of this section

type	classification	extrinsic geometry	intrinsic geometry
(K)			
(A)	**done**	**well-understood**	**many problems**
(N)			

Sec. 3: Type (N) (1/6)

Classification of type (N)-actions...

Thm. (Berndt-T. JDG 2003)

- $(\{\text{type } (N)\text{-action on } M\}/\text{equiv.}) \cong \mathbb{R}\mathrm{P}^{\mathrm{rank}(M)-1}$ (almost).

More Explicitly

- Take $0 \neq \xi \in \mathfrak{a}$.
- $\mathfrak{s}_\xi := \mathfrak{s} \ominus \xi = (\mathfrak{a} \ominus \xi) \oplus \mathfrak{n}$. Then $S_\xi \curvearrowright M$ is of type (N).
- All type (N)-actions can be constructed in this way.
- $S_\xi \sim S_\eta$ (orbit eq.) iff $\mathbb{R}\xi \sim \mathbb{R}\eta$ by Dynkin diagram auto.

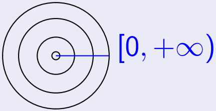
Sec. 3: Type (N) (2/6)

Note

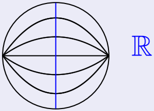
- If $\mathrm{rank}(M) = 1$, then $\exists 1$ type (N)-action.
- In this case, orbits are "horospheres".

Recall

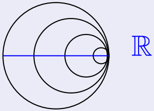
(N) $\not\exists$ singular orbit, all orbits are congruent to each other.



type (K)



type (A)



type (N)

Sec. 3: Type (N) (3/6)

Extrinsic geometry of type (N)-actions...

Note

- Since all orbits are congruent, we have only to study one orbit.

Results by Berndt-T.:

- One can calculate principal curvatures (at least theoretically).
- For some type (N)-actions, all orbits are minimal.

Recent Results by Gondo:

- For $\mathrm{rank}(M) \leq 3$, classify austere and weakly reflective orbits.
- $\exists$ austere orbits which are not weakly reflective.

Sec. 3: Type (N) (4/6)

Intrinsic geometry of type (N)-actions...

Thm. (observation)

- All orbits of all type (N)-actions are Ricci soliton.
- An orbit is Einstein **(& nonflat)** iff it is minimal.

Note

- This follows from theories by Heber (1998), Lauret (2013).
- But existence of Einstein hypersurfaces would have interest.

Sec. 3: Type (N) (5/6)

Thm. (Cho-Hashinaga-Kubo-Taketomi-T. 2018)

For $M = SO^0_{n,2}/S(O_n O_2)$,

- $\exists$ a particular action of type (N) such that the orbit is a contact $(0, 4)$-space & Ricci soliton.

Def. (Blair-Koufogiorgos-Papantoniou 1995)

A contact metric mfd $(M; \eta, \xi, \varphi, g)$ is **(κ, μ)-space** ($\kappa, \mu \in \mathbb{R}$) if

- $R(X, Y)\xi = (\kappa \, \mathrm{id} + (\mu/2)\mathcal{L}_\xi \varphi)(\eta(Y)X - \eta(X)Y)$.

Our result solves an open problem in contact geometry (a classification of Ricci soliton (κ, μ)-spaces).

Sec. 3: Type (N) (6/6)

Summary of this section

type	classification	extrinsic geometry	intrinsic geometry
(K)			
(A)	done	well-understood	many problems
(N)	**done**	$\exists$ **results & problems**	**many results**

Sec. 4: Type (K) (1/5)

- Recall: (K) $\exists 1$ singular orbit.

Known Classification

We have a classification of type (K)-actions on

- $\mathbb{R}\mathrm{H}^n$ (É. Cartan);
- $\mathbb{C}\mathrm{H}^n$, $\mathbb{H}\mathrm{H}^2$, $\mathbb{O}\mathrm{H}^2$ (Berndt-T. 2007);
- $SL_3\mathbb{R}/SO_3$, $G_2^*(\mathbb{R}^5)$, G_2^2/SO_4 (Berndt-T. 2013);
- $SL_3\mathbb{C}/SU_3$, $G_2^*(\mathbb{R}^n)$, $G_2^{\mathbb{C}}/G_2$ (Berndt-DominguezVazquez 2015);
- $\mathbb{H}\mathrm{H}^n$ (cf. Alberto RodriguezVazquez).

Sec. 4: Type (K) (2/5)

Summary for Classification

We have a classification of C1-actions on

- all rank one symmetric spaces;
- many rank two cases;
- but none of $\mathrm{rank}(M) \geq 3$.

Open Problem

- Classify C1-actions on the remaining M.
- Note: it is enough to classify type (K)-actions.
- Note: it is very hard for $\mathrm{rank}(M) \geq 3$.

Sec. 4: Type (K) (3/5)

Extrinsic geometry of type (K)-actions...

Question

- $\exists$ minimal regular orbits of type (K)-actions?

Note

- For the cases of $M = \mathbb{R}\mathrm{H}^n$, $\mathbb{C}\mathrm{H}^n$,
  $\nexists$ minimal regular orbits of type (K)-actions.

Sec. 4: Type (K) (4/5)

Intrinsic geometry of type (K)-actions...

Note

- $\exists$ several results for $\mathbb{R}\mathrm{H}^n$, $\mathbb{C}\mathrm{H}^n$;
- $\exists$ some results for $SO^0(n,2)/S(O_n O_2)$...

  (related to contact geometry; cf. Berndt-Suh, Hashinaga)

Question

- $\exists$ regular orbits of type (K)-actions which are Einstein or Ricci soliton?
- (I guess no; it would relate to Alekseevskii Conjecture...)

Sec. 4: Type (K) (5/5)

Summary of this section

type	classification	extrinsic geometry	intrinsic geometry
(K)	**only some**	**not yet**	**not yet**
(A)	done	well-understood	many problems
(N)	done	$\exists$ results & problems	many results

Summary (1/2)

Summary

- We are studying classification problem / extrinsic geometry / intrinsic geometry of homogeneous hypersurfaces in symmetric spaces of noncompact type.
- This would provide many interesting examples, and also gives applications to Einstein/Ricci soliton solvmanifolds, contact geometry, and so on.
- We still have so many open problems...

Summary (2/2)

Table

type	classification	extrinsic geometry	intrinsic geometry
(K)	only some	not yet	not yet
(A)	done	well-understood	many problems
(N)	done	$\exists$ results & problems	many results

Thank you very much!
&
Congulatulations and many thanks to Professor Jürgen Berndt.