Homogeneous submanifolds in complex space forms

Symmetry and shape

Celebrating the 60th birthday of Prof. J. Berndt

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Homogeneous hypersurfaces

$\tilde{M}$ Riemannian manifold, $\tilde{\nabla}$ Levi-Civita connection

$M \subset \tilde{M}$ hypersurface, $\xi$ unit normal, $\nabla$ Levi-Civita connection

$M$ homogeneous

$M = G \cdot o$, with $o \in M, G \subset I(\tilde{M})$

$G$ is said to act with cohomogeneity one

Problem.
- Classify homogeneous hypersurfaces (up to isometric congruence)
- Characterize homogeneous hypersurfaces in terms of geometric data
### Complex space forms

#### Complex projective space

$$(\mathbb{C}^{n+1}, i) \langle v, w \rangle = \text{Re} \left( \sum_{i=0}^{n} \bar{v}_i w_i \right)$$

$S^{2n+1} = \{ z \in \mathbb{C}^{n+1} : \langle z, z \rangle = 1 \}$

$z \sim w \iff \exists \lambda \in \mathbb{C} : w = \lambda z$

$$\mathbb{C}P^n = S^{2n+1} / \sim$$

$\pi : S^{2n+1} \to \mathbb{C}P^n$ Hopf map

$\pi$ Riemannian submersion

**$\mathbb{C}P^n$ is a Kähler manifold with constant positive holomorphic sectional curvature**

#### Complex hyperbolic space

$$(\mathbb{C}^{1,n}, i) \langle v, w \rangle = \text{Re} \left( -\bar{v}_0 w_0 + \sum_{i=1}^{n} \bar{v}_i w_i \right)$$

$H_{1}^{2n+1} = \{ z \in \mathbb{C}^{1,n} : \langle z, z \rangle = -1 \}$

$z \sim w \iff \exists \lambda \in \mathbb{C} : w = \lambda z$

$$\mathbb{C}H^n = H_{1}^{2n+1} / \sim$$

$\pi : H_{1}^{2n+1} \to \mathbb{C}H^n$ Hopf map

$\pi$ semi-Riemannian submersion

**$\mathbb{C}H^n$ is a Kähler manifold with constant negative holomorphic sectional curvature**
Complex space forms

Complex projective space

\[(\mathbb{C}^{n+1}, i \cdot) \langle v, w \rangle = \text{Re} \left( \sum_{i=0}^{n} \bar{v}_i w_i \right)\]

\[S^{2n+1} = \{ z \in \mathbb{C}^{n+1} : \langle z, z \rangle = 1 \} \]

\[\mathbb{C}P^n = S^{2n+1} / \sim \]

\[SU(n+1) \text{ acts transitively on } \mathbb{C}P^n\]

\[\mathbb{C}P^n = \frac{SU(n+1)}{S(U(1)U(n))} \]

\[\mathbb{C}P^n \text{ is a symmetric space of rank one and compact type} \]

Complex hyperbolic space

\[(\mathbb{C}^{1,n}, i \cdot) \langle v, w \rangle = \text{Re} \left( -\bar{v}_0 w_0 + \sum_{i=1}^{n} \bar{v}_i w_i \right)\]

\[H^{2n+1}_1 = \{ z \in \mathbb{C}^{1,n} : \langle z, z \rangle = -1 \} \]

\[\mathbb{C}H^n = H^{2n+1}_1 / \sim \]

\[SU(1, n) \text{ acts transitively on } \mathbb{C}H^n\]

\[\mathbb{C}H^n = \frac{SU(1, n)}{S(U(1)U(n))} \]

\[\mathbb{C}H^n \text{ is a symmetric space of rank one and noncompact type} \]
The complex hyperbolic space

\[ CH^n = \frac{SU(1, n)}{S(U(1)U(n))} \]

Iwasawa decomposition

\[ I^0(CH^n) = KAN \]

acts simply transitively on \( CH^n \)

\[ CH^n \cong AN \]

with left-invariant metric

\[ n = g_\alpha \oplus g_{2\alpha} \]

Heisenberg algebra

\[ [U, V] = \langle JU, V \rangle Z \]

\[ [A, U] = \frac{1}{2} U \quad [A, Z] = Z \]
Homogeneous hypersurfaces in space forms

- Euclidean spaces $\mathbb{R}^n$ [Somigliana, Levi-Civita, Segre]:

- Real hyperbolic spaces $\mathbb{RH}^n$ [Cartan]:

- Spheres $S^n$ [Hsiang, Lawson]:
  Isotropy representations of symmetric spaces of rank 2
Homogeneous hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$

**Theorem.** [Takagi] A homogeneous hypersurface in $\mathbb{C}P^n$ is a principal orbit of the quotient of the isotropy representation of a Hermitian symmetric space of rank two.

**Theorem.** [Berndt, Tamaru] Homogeneous hypersurfaces in $\mathbb{C}H^n$:

- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, \ldots, n-1\}$
- tubes around totally geodesic $\mathbb{R}H^n$
- horospheres
- ruled homogeneous minimal Lohnherr hypersurfaces $W^{2n-1}$, or their equidistant hypersurfaces
- tubes around ruled homogeneous minimal Berndt-Brück submanifolds $W^{2n-k}_\varphi$, for $k \in \{2, \ldots, n-1\}$, $\varphi \in (0, \pi/2]$
  ($k$ even if $\varphi \neq \pi/2$)
Homogeneous hypersurfaces in $\mathbb{C}H^n$

**Hopf examples**

- Tubes around a totally geodesic $\mathbb{C}H^k$, $k \in \{0, \ldots, n-1\}$
  
  Group action: $S(U(1,k) \times U(n-k))$
  
  $g = 2$ if $k \in \{0, n-1\}$; $g = 3$ otherwise

- Tubes around a totally geodesic $\mathbb{R}H^n$
  
  Group action: $SO^0(1,n)$
  
  $g = 2$ if $r = \log(2 + \sqrt{3})$; $g = 3$ otherwise

- Horospheres
  
  Group action: $N$
  
  $g = 2$
Homogeneous hypersurfaces in $\mathbb{C}H^n$

**Non-Hopf examples**

$V \subset \mathbb{C}^n$ has constant Kähler angle $\varphi$

$\mathfrak{w} \subset \mathfrak{g}_\alpha$ such that $\mathfrak{w}^\perp$ is of constant Kähler angle $\varphi$, $k = \dim \mathfrak{w}^\perp$

$\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$

$S_\mathfrak{w}$ subgroup of $AN$ whose Lie algebra is $\mathfrak{s}_\mathfrak{w}$

**Theorem.** [Berndt, Brück] Tubes around $W^{2n-k}_\varphi := W_\mathfrak{w} = S_\mathfrak{w} \cdot o$ are homogeneous

- If $\mathfrak{w}$ is a hyperplane, $W_\mathfrak{w}$ is the Lohnherr hypersurface ($g = 3$)
- If $\varphi = \pi/2$, then $g = 3$ if $r = \log(2 + \sqrt{3})$, otherwise $g = 4$
- If $\varphi \neq \pi/2$, then $k$ is even; $g = 4$ if $k = 2$, otherwise $g = 5$
Characterization of homogeneous hypersurfaces

- $M$ homogeneous hypersurface
- $M$ has constant principal curvatures
- $M$ is isoparametric
Characterization in real space forms

**Theorem.** [Cartan] Isoparametric \( \Leftrightarrow \) constant principal curvatures

- Euclidean spaces \( \mathbb{R}^n \):

- Real hyperbolic spaces \( \mathbb{R}H^n \):

- Spheres \( S^n \):
  
  There are inhomogeneous examples
Constant principal curvatures

\[ M \text{ homogeneous hypersurface} \quad \Rightarrow \quad M \text{ has constant principal curvatures} \]

Shape operator: \[ SX = -\nabla_X \xi \]

\( S \) self-adjoint  
\( \Rightarrow S \) diagonalizable

principal curvatures:  
eigenvalues of \( S \)

\( g \): number of principal curvatures

\( J\xi \): Hopf vector field

\( h \): # of nontrivial projections of \( J\xi \) onto principal curvature spaces

\( M \) is Hopf  
\( \Leftrightarrow J\xi \) is an eigenvector of \( S \)  
\( \Leftrightarrow h = 1 \)
Constant principal curvatures

$M$ homogeneous hypersurface $\rightarrow$ $M$ has constant principal curvatures

The answer is **YES** if:

$g = 1$ [Tashiro, Tachibana] No umbilical hypersurfaces in $\mathbb{C}H^n$

$g = 2$ [Montiel]
- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, n-1\}$
- tubes of radius $r = \log(2 + \sqrt{3})$ around totally geodesic $\mathbb{R}H^n$
- horospheres

$g = 3$ [Berndt, Díaz-Ramos]
- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{1, \ldots, n-2\}$
- tubes of radii $r \neq \log(2 + \sqrt{3})$ around totally geodesic $\mathbb{R}H^n$
- ruled Lohnherr hypersurfaces $W_{\pi/2}^{2n-1}$, or their equidistant hypersurfaces
- tubes of radius $r = \log(2 + \sqrt{3})$ around Berndt-Brück submanifolds $W_{\pi/2}^{2n-k}$, for $k \in \{2, \ldots, n-1\}$
Constant principal curvatures

$M$ homogeneous hypersurface $\xrightarrow{?} M$ has constant principal curvatures

The answer is **YES** if:

$h = 1$ [Berndt]
- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, \ldots, n - 1\}$
- tubes around totally geodesic $\mathbb{R}H^n$
- horospheres

$h = 2$ [Díaz-Ramos, Domínguez-Vázquez]
- ruled Lohnherr hypersurfaces $W_{\pi/2}^{2n-1}$, or their equidistant hypersurfaces
- tubes around Berndt-Brück submanifolds $W_{\pi/2}^{2n-k}$, for $k \in \{2, \ldots, n - 1\}$
Isoparametric hypersurfaces

\( M \) homogeneous hypersurface \( \rightarrow \) \( M \) is isoparametric

\( f: \bar{M} \rightarrow \mathbb{R} \)

isoparametric function

\( M \subset \bar{M} \)

isoparametric hypersurface

\( \| \nabla f \|^2 \) and \( \Delta f \) constant along the level sets of \( f \)

level set of codimension 1 of isoparametric function

[Cartan]

\( M \) isoparametric hypersurface \( \leftrightarrow \) nearby parallel hypersurfaces of constant mean curvature

In real space forms: isoparametric \( \leftrightarrow \) constant principal curvatures
Isoparametric hypersurfaces in $\mathbb{C}H^n$

$M$ homogeneous hypersurface $\implies$ $M$ is isoparametric

**Inhomogeneous examples**

$\mathfrak{w} \subset \mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$, $k = \dim \mathfrak{w}^\perp$

$\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$

$S_\mathfrak{w}$ subgroup of $AN$ whose Lie algebra is $\mathfrak{s}_\mathfrak{w}$

**Theorem.** [Díaz-Ramos, Domínguez-Vázquez] Tubes around $W_\mathfrak{w} = S_\mathfrak{w} \cdot o$ are isoparametric

If $\mathfrak{w} \subset \mathfrak{g}_\alpha$, then $\mathfrak{w}^\perp = \bigoplus_{\varphi \in \Phi} \mathfrak{w}_\varphi^\perp$ is a sum of space of constant Kähler angle

[Díaz-Ramos, Domínguez-Vázquez, Kollross]

Thus, $W_\mathfrak{w}$ is homogeneous if and only if $\mathfrak{w}^\perp$ has constant Kähler angle
Theorem. [Díaz-Ramos, Domínguez-Vázquez, Sanmartín-López] Isoparametric hypersurfaces in $\mathbb{C}H^n$:

- tubes around totally geodesic $\mathbb{C}H^k$, $k \in \{0, \ldots, n-1\}$
- tubes around totally geodesic $\mathbb{R}H^n$
- horospheres
- ruled homogeneous minimal Lohnherr hypersurfaces $W^{2n-1}_{\pi/2}$, or their equidistant hypersurfaces
- tubes around a ruled homogeneous minimal Berndt-Brück submanifolds $W^{2n-k}_{\varphi}$, for $k \in \{2, \ldots, n-1\}$, $\varphi \in (0, \pi/2]$
  ($k$ even if $\varphi \neq \pi/2$)
- tubes around ruled homogeneous minimal submanifolds $W_\mathfrak{w}$, for some proper real subspace $\mathfrak{w}$ of $g_\alpha \cong \mathbb{C}^{n-1}$ such that $\mathfrak{w}^\perp$ has nonconstant Kähler angle
Characterization of homogeneous hypersurfaces

\( M \) homogeneous hypersurface \( \Rightarrow \)
- \( M \) has constant principal curvatures
- \( M \) is isoparametric

**Corollary.** If \( M \) is a connected complete hypersurface in \( CH^2 \) then the following statements are equivalent:
- \( M \) is homogeneous
- \( M \) has constant principal curvatures
- \( M \) is isoparametric

**Corollary.** If \( M \) is a connected complete hypersurface in \( CH^n \) then, \( M \) is homogeneous if and only if \( M \) is isoparametric and has constant principal curvatures
**Polar actions**

$G$ acts **polarly** if and only if there is a **section**

**Section**: submanifold that intersects all orbits of $G$ orthogonally.

A section is thought as a set of "canonical forms".

**Example.** $\mathfrak{sl}(n, \mathbb{R})/SO(n) \quad \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \{\text{symmetric matrices}\}$

$SO(n)$ acts on symmetric matrices by conjugation

$\{\text{diagonal matrices}\}$ is a section
Polar actions

$G$ acts polarly ⇔ there is a section

**Section:** submanifold that intersects all orbits of $G$ orthogonally

**Problem.**
- Classify polar actions (up to isometric congruence)
- Characterize orbits of polar actions in terms of geometric data
Polar actions on real space forms

- Spheres $S^n$:
  [Dadok] Isotropy representations of symmetric spaces

- Euclidean spaces $\mathbb{R}^n$
  Isotropy representations of symmetric spaces $\times$ translations

- Real hyperbolic spaces $\mathbb{R}H^n$
  [Wu] $SO(1, k) \times K$ or $N \times K$, where $K$ acts polarly on $\mathbb{R}^{n-k}$
Polar actions on $\mathbb{C}H^n$

**Theorem.** [Podestà, Thorbergsson] A polar action on $\mathbb{C}P^n$ is orbit equivalent to the a quotient of an isotropy representation of a Hermitian symmetric space.

**Theorem.** [ -, Domínguez-Vázquez, Kolbross] Polar actions on $\mathbb{C}H^n$:

- $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{so}(1, k), \ k \in \{0, \ldots, n\}$
  - $\mathfrak{q}$ subalgebra of $\mathfrak{u}(n - k)$
  - $Q$ acts polarly on $\mathbb{C}^{n-k}$ with totally real section

- $\mathfrak{h} = \mathfrak{q} \oplus \mathfrak{b} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$
  - $\mathfrak{b}$ linear subspace of $\mathfrak{a}$, $\mathfrak{w}$ real subspace of $\mathfrak{g}_\alpha$,
  - $\mathfrak{q}$ subalgebra of $\mathfrak{k}_0 = \mathfrak{n}_K(\mathfrak{a})$, $\mathfrak{q}$ normalizes $\mathfrak{w}$,
  - $Q$ acts polarly on $\mathfrak{g}_\alpha \oplus \mathfrak{w}$ with totally real section

Recall: $\mathfrak{w} = \bigoplus_{\varphi \in \Phi} \mathfrak{w}_\varphi$
Characterization of orbits of polar actions

[Heintze, Liu, Olmos]

\[ M \subset \tilde{M} \]

isoparametric

- normal bundle is flat
- parallel submanifolds have constant mean curvature in radial directions
- for any \( p \in M \) there exists a section \( \Sigma_p \) through \( p \)
  (totally geodesic submanifold s.t. \( T_p \Sigma_p = \nu_p M \))

Isoparametric in \( CH^2 \)

\[ CH^2 \]

principal orbit of polar action

[Terng]

\[ M \subset \tilde{M} \]

isoparametric

- normal bundle is flat
- eigenvalues of the shape operator with respect to any parallel normal vector field are constant

Terng-isoparametric in \( CH^2 \)

\[ CH^2 \]

- principal orbit of polar action
- Chen's surface
- circles