Lagrangian submanifolds of the complex quadric

Joeri Van der Veken

SYMMETRY AND SHAPE
celebrating the 60th birthday of Prof. J. Berndt

Santiago de Compostela – 30/10/2019
1. How we started research on $Q^n$

2. The complex quadric $Q^n$

3. The Gauss map of a hypersurface of a sphere

4. Study of Lagrangian submanifolds of $Q^n$

5. Question
1 – Outline

1. How we started research on $Q^n$

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5. Question
1 – How we started research on $Q^n$

The homogeneous nearly Kähler $(S^3 \times S^3, g)$
The homogeneous nearly Kähler \((S^3 \times S^3, g)\) has

- an almost complex structure \(J\)
- and an almost product structure \(P\),
- which anti-commute,
- and the curvature tensor is given by

\[
R(X, Y)Z = \frac{5}{12} (g(Y, Z)X - g(X, Z)Y) \\
+ \frac{1}{12} (g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ) \\
+ \frac{1}{3} (g(PY, Z)PX - g(PX, Z)PY) \\
+ g(JPY, Z)JPX - g(JPX, Z)JPY).
\]
2 – Outline

1. How we started research on $Q^n$

2. The complex quadric $Q^n$

3. The Gauss map of a hypersurface of a sphere

4. Study of Lagrangian submanifolds of $Q^n$

5. Question
2 – The complex quadric $Q^n$

**Definition**

$$Q^n := \{ [(z_0, \ldots, z_{n+1})] \in \mathbb{C}P^{n+1}(4) \mid z_0^2 + \ldots + z_{n+1}^2 = 0 \}.$$
2 – The complex quadric $Q^n$

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$Q^n$ is a holomorphic submanifold of $\mathbb{C}P^{n+1}(4)$ and hence, equipped with the induced metric and almost complex structure, a Kähler manifold.
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What is the inverse image of $Q^n$ under the Hopf fibration

$$\pi : S^{2n+3}(1) \subseteq \mathbb{C}^{n+2} \to \mathbb{C}P^{n+1}(4) ?$$
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**Lemma**

$$\pi^{-1}Q^n = \left\{u + iv \mid u, v \in \mathbb{R}^{n+2}, \langle u, u \rangle = \langle v, v \rangle = \frac{1}{2}, \langle u, v \rangle = 0 \right\} \subseteq S^{2n+3}(1),$$  

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric on $\mathbb{R}^{n+2}$.  

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Remark. Alternative descriptions:

- $Q^n$ is the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$
- $Q^n = \frac{\text{SO}(n + 2)}{\text{SO}(n) \times \text{SO}(2)}$
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Lemma

$T^\perp_{[z]} Q^n = \text{span}\{(d\pi)_z(\bar{z}), (d\pi)_z(i\bar{z})\}$. 
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**Lemma**

$$T_{z}^\perp Q^n = \text{span \{ } (d\pi)_{z}(\bar{z}), (d\pi)_{z}(i\bar{z}) \text{\}}.$$ 

\[\text{Re}(z_0^2 + \ldots + z_{n+1}^2) = 0 \implies \bar{z} \perp z \implies \bar{z} \text{ is tangent to } S^{2n+3} (1)\]
\[\text{Im}(z_0^2 + \ldots + z_{n+1}^2) = 0 \implies \bar{z} \perp iz \implies \bar{z} \text{ is horizontal}\]
Lemma

Any shape operator $A$ of $Q^n$ in $\mathbb{CP}^{n+1}(4)$, associated to a unit normal vector field, has the following properties:

1. $A^2 = \text{id}$,
2. $g(AX, AY) = g(X, Y)$,
3. $AJ = -JA$. 
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Let $\mathcal{A}$ be the set of these operators. Choose $A_0 \in \mathcal{A}$, then

$$\mathcal{A} = \{\cos \varphi A_0 + \sin \varphi JA_0 \mid \varphi : Q^n \to \mathbb{R}\}.$$
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Lemma

For all $A \in \mathcal{A}$, there exists a non-zero one-form $s$ such that

$$\nabla_X A = s(X)JA.$$
2 – The complex quadric $Q^n$

From the equation of Gauss:

$$R^{Q^n}(X,Y)Z = g(Y, Z)X - g(X, Z)Y$$
$$+ g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ$$
$$+ g(AY, Z)AX - g(AX, Z)AY$$
$$+ g(JAY, Z)JAX - g(JAX, Z)JAY$$
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Remark. Although none of the $A \in \mathcal{A}$ are integrable, we have

$$Q^2 \cong S^2 \left( \frac{1}{2} \right) \times S^2 \left( \frac{1}{2} \right).$$
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Theorem (Jensen)

A Riemannian homogeneous Einstein four-manifold is symmetric and hence locally isometric to either a real space form $\mathbb{R}^4$, $S^4(c)$ or $H^4(c)$; a complex space form $\mathbb{C}P^2(4c)$ or $\mathbb{C}H^2(4c)$; or a product of surfaces $S^2(c) \times S^2(c)$ or $H^2(c) \times H^2(c)$. 
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3 – The Gauss map of a hypersurface of a sphere

Gauss map of a hypersurface of $\mathbb{R}^{n+1}$

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**Definition**

The map $G : M^n \to S^n(1) : p \mapsto b(p)$ is the **Gauss map** of $a$. 
Gauss map of a hypersurface of \( \mathbb{R}^{n+1} \)

A hypersurface with unit normal \( b \):

\[
a : M^n \rightarrow \mathbb{R}^{n+1}
\]

**Definition**

The map \( G : M^n \rightarrow S^n(1) : p \mapsto b(p) \) is the Gauss map of \( a \).

**Remark:**

- Any *parallel hypersurface* to \( a \), given by

\[
a_t : M^n \rightarrow \mathbb{R}^{n+1} : p \mapsto a(p) + t b(p)
\]

for some \( t \in \mathbb{R} \), has the same Gauss map.
3 – The Gauss map of a hypersurface of a sphere

Gauss map of a hypersurface of $S^{n+1}(1)$

$a : M^n \to S^{n+1}(1) \subseteq \mathbb{R}^{n+2}$ hypersurface with unit normal $b$. 
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**Remark:**

- $\langle \frac{a(p)}{\sqrt{2}}, \frac{a(p)}{\sqrt{2}} \rangle = \langle \frac{b(p)}{\sqrt{2}}, \frac{b(p)}{\sqrt{2}} \rangle = \frac{1}{2}, \langle \frac{a(p)}{\sqrt{2}}, \frac{b(p)}{\sqrt{2}} \rangle = 0 \Rightarrow \frac{a(p)}{\sqrt{2}} + i \frac{b(p)}{\sqrt{2}} \in \pi^{-1}Q^n$
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- A parallel hypersurface to $a$ is now given by

  $$a_t : M^n \rightarrow S^{n+1}(1) \subseteq \mathbb{R}^{n+2} : p \mapsto \cos t a(p) + \sin t b(p)$$

  for some $t \in \mathbb{R}$. Since $b_t = \cos t b - \sin t a$ is a unit normal to $a_t$, $a_t + ib_t = e^{-it}(a + ib)$ and $a_t$ has the same Gauss map as $a$. 
Proposition

The Gauss map $G : M^n \to Q^n$ of a hypersurface $a : M^n \to S^{n+1}(1)$ is a Lagrangian immersion.
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Proof. Diagonalize the shape operator $S$ of $a$: $Se_j = \lambda_j e_j$.

For the horizontal lift $\hat{G} : M^n \to \pi^{-1}Q^n : p \mapsto \frac{1}{\sqrt{2}}(a(p) + ib(p))$, one has

$$(d\hat{G})e_j = \frac{1}{\sqrt{2}}(e_j - iSe_j) = \frac{1}{\sqrt{2}}(1 - i\lambda_j)e_j.$$  

$(d\hat{G})e_1, \ldots, (d\hat{G})e_n$ are linearly independent $\Rightarrow G$ is an immersion.

$\forall j, k \in \{1, \ldots, n\} : \langle (d\hat{G})e_j, i(d\hat{G})e_k \rangle = 0$ $\Rightarrow G$ is Lagrangian.
Proposition

If the principal curvatures of a hypersurface $\alpha : M^n \to S^{n+1}(1)$ are constant, then its Gauss map $G : M^n \to Q^n$ is a minimal Lagrangian immersion.
3 – The Gauss map of a hypersurface of a sphere

**Proposition**

*If the principal curvatures of a hypersurface \( \alpha : M^n \to S^{n+1}(1) \) are constant, then its Gauss map \( G : M^n \to Q^n \) is a minimal Lagrangian immersion.*

The statement follows from the following formula by Palmer:

\[
g(JH, \cdot) = -\frac{1}{n} d \left( \text{Im} \left( \prod_{j=1}^{n} (1 + i\lambda_j) \right) \right).
\]
**Definition**

An *isoparametric hypersurface* is a hypersurface with constant principal curvatures.
3 – The Gauss map of a hypersurface of a sphere

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**Remark.** Originally defined as level sets of *isoparametric functions* $F$ on the ambient space: functions for which $\|\nabla F\| = \phi_1(F)$, $\Delta F = \phi_2(F)$.
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*Full classification of isoparametric hypersurfaces of* $\mathbb{R}^{n+1}$. 

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**Somigliana, Levi-Civita, Segre**

An isoparametric hypersurface of $\mathbb{R}^{n+1}$ is a part of a hyperplane $\mathbb{R}^n$, a hypersphere $S^n(r)$ or a product immersion $S^k(r) \times \mathbb{R}^{n-k}$. 

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Full classification of isoparametric hypersurfaces of $\mathbb{R}^{n+1}$.

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*Examples of isoparametric hypersurfaces of $S^{n+1}(1)$.***
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- A hypersphere ($0 < r \leq 1$)
  \[ a_1 : S^n(r) \to S^{n+1}(1) : p \mapsto (p, \sqrt{1 - r^2}) \]
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- A product of spheres $(0 < r_1, r_2 < 1$ with $r_1^2 + r_2^2 = 1)$
  
  \[ a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1) : (p_1, p_2) \mapsto (p_1, p_2) \]
3 – The Gauss map of a hypersurface of a sphere

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- A tube around the Veronese surface in $S^4(1)$ (*Cartan’s example*)
  
  $$a_3 : \mathbb{RP}^2 \times S^1(\varepsilon) \to S^4(1)$$
Principal curvatures of these examples:

- $a_1$: $\lambda_1 = \ldots = \lambda_n = \frac{\sqrt{1-r^2}}{r}$
- $a_2$: $\lambda_1 = \ldots = \lambda_k = \frac{r_2}{r_1}$, $\lambda_{k+1} = \ldots = \lambda_n = -\frac{r_1}{r_2}$
- $a_3$: $\lambda_1, \lambda_2, \lambda_3$ mutually different
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**Theorem (Münzner, 1981)**

Let \( g \) be the number of distinct constant principal curvatures of an isoparametric hypersurface of \( S^{n+1}(1) \), then \( g \in \{1, 2, 3, 4, 6\} \).
3 – The Gauss map of a hypersurface of a sphere

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Until today, the classification of isoparametric hypersurfaces of $S^{n+1}(1)$ is still not completely understood.
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5. Question
Let $f: M^n \rightarrow Q^n$ be a Lagrangian immersion. Assume $A \in A$ is fixed. If $X$ is tangent to $M^n$, the decomposition $AX = BX - JCX$ into a tangent and a normal part, defines two $(1,1)$-tensors on $M^n$.

Lemma

The $(1,1)$-tensors $B$ and $C$ on $M^n$ satisfy

1. $B$ and $C$ are symmetric,
2. $B^2 + C^2 = id$,
3. $[B,C] = 0$.

Hence, for every $p \in M^n$, there exists an ONB $\{e_1, \ldots, e_n\}$ of $T_p M^n$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$, determined up to an integer multiple of $\pi$, such that $A e_j = \cos(2\theta_j) e_j - \sin(2\theta_j) Je_j$. 


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$$Ae_j = \cos(2\theta_j)e_j - \sin(2\theta_j)Je_j.$$
In the neighborhood of a point, $\theta_1, \ldots, \theta_n$ define local angle functions, which we can change by changing $A \in \mathcal{A}$. 
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**Lemma**

Let $f : M^n \to Q^n$ be a Lagrangian immersion and $A_0, A \in \mathcal{A}$. Then there exists a function $\varphi : M^n \to \mathbb{R}$ such that $A = \cos \varphi A_0 + \sin \varphi JA_0$ along $M^n$ and the local angle functions $\theta_1, \ldots, \theta_n$ associated to $A$ are related to the local angle functions $\theta_0^1, \ldots, \theta_0^n$ associated to $A_0$ by

$$\theta_j = \theta_j^0 - \frac{\varphi}{2}. $$
4 – Study of Lagrangian submanifolds of $Q^n$

In the neighborhood of a point, $\theta_1, \ldots, \theta_n$ define local angle functions, which we can change by changing $A \in A$.

**Lemma**

Let $f : M^n \to Q^n$ be a Lagrangian immersion and $A_0, A \in A$. Then there exists a function $\varphi : M^n \to \mathbb{R}$ such that $A = \cos \varphi A_0 + \sin \varphi J A_0$ along $M^n$ and the local angle functions $\theta_1, \ldots, \theta_n$ associated to $A$ are related to the local angle functions $\theta_1^0, \ldots, \theta_n^0$ associated to $A_0$ by

$$\theta_j = \theta_j^0 - \frac{\varphi}{2}.$$ 

**Example.** One can choose $A \in A$ such that

$$\theta_1 + \ldots + \theta_n = 0 \mod \pi.$$
Equation of Gauss:

\[ g(R(X,Y)Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \]
\[ + g(BY,Z)g(BX,W) - g(BX,Z)g(BY,W) \]
\[ + g(CY,Z)g(CX,W) - g(CX,Z)g(CY,W) \]
\[ + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \]
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\[ + g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)) \]

Equation of Codazzi:

\[ (\nabla h)(X, Y, Z) - (\nabla h)(Y, X, Z) = g(CY, Z)JBX - g(CX, Z)JBY \]
\[ - g(BY, Z)JCX + g(BX, Z)JCY \]
Question:

Given a Lagrangian immersion \( f: M^n \to Q^n \), can we see it as the Gauss map of a hypersurface \( a: M^n \to S^n+1 \)?

Idea:

Take a horizontal lift \( \hat{f}: M^n \to \pi^{-1}Q^n \) and put

\[ a := \sqrt{2\text{Re} \hat{f}}, \quad b := \sqrt{2\text{Im} \hat{f}}. \]

Remark:

We have to work locally to guarantee that \( a \) is an immersion.

We expect a relation between the angle functions of \( f \) and the principal curvatures of \( a \).
**Question:** Given a Lagrangian immersion $f : M^n \rightarrow Q^n$, can we see it as the Gauss map of a hypersurface $a : M^n \rightarrow S^{n+1}(1)$?
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\[
\begin{align*}
a &:= \sqrt{2} \text{Re} \, \hat{f}, \\
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\end{align*}
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Remark:
- We have to work locally to guarantee that $a$ is an immersion
4 – Study of Lagrangian submanifolds of $Q^n$ – main results

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Take a horizontal lift $\hat{f} : M^n \to \pi^{-1}Q^n$ and put

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**Remark:**

- We have to work locally to guarantee that $a$ is an immersion
- We expect a relation between the angle functions of $f$ and the principal curvatures of $a$. 
**Theorem (VdV, Wijffels)**

**PART I**

Let \( a : M^n \to S^{n+1}(1) \) be a hypersurface with unit normal \( b \) and denote by \( G : M^n \to Q^n : p \mapsto [a(p) + ib(p)] \) its Gauss map. After a suitable choice of \( A \in \mathcal{A} \), the relation between the principal curvatures \( a \) and the angle functions of \( G \) is

\[
\lambda_j = \cot \theta_j
\]

for \( j = 1, \ldots, n \).
Theorem (VdV, Wijffels)

**PART I**

Let \( \alpha : M^n \to S^{n+1}(1) \) be a hypersurface with unit normal \( b \) and denote by \( G : M^n \to Q^n : p \mapsto [\alpha(p) + ib(p)] \) its Gauss map. After a suitable choice of \( A \in A \), the relation between the principal curvatures \( \alpha \) and the angle functions of \( G \) is

\[
\lambda_j = \cot \theta_j
\]

for \( j = 1, \ldots, n \).

**Remark.** The choice of \( A \) comes down to choosing

\[
\hat{G}(p) = \frac{1}{\sqrt{2}}(a(p) - ib(p))
\]

as a unit normal to \( \pi^{-1}Q^n \) in \( S^{2n+3}(1) \) along \( \hat{G} \).
Conversely, if \( f : M^n \to Q^n \) is a Lagrangian immersion, then for every point of \( M^n \) there exist an open neighborhood \( U \) of that point in \( M^n \) and an immersion \( \alpha : U \to S^{n+1}(1) \) with Gauss map \( f|_U \). This immersion is not unique, nor are its principal curvature functions. However, for any choice of the hypersurface \( \alpha \) and of the almost product structure \( A \in \mathcal{A} \), the principal curvature functions of \( \alpha \) are related to the corresponding angle functions of \( f \) by

\[
\cot(\theta_j - \theta_k) = \pm \frac{\lambda_j \lambda_k + 1}{\lambda_j - \lambda_k}
\]

for \( j, k = 1, \ldots, n \) in points where \( \lambda_j \neq \lambda_k \).
4 – Study of Lagrangian submanifolds of $Q^n$ – main results

Some classification theorems for \textit{minimal} Lagrangian immersions into $Q^n$
Some classification theorems for \textit{minimal} Lagrangian immersions into \( Q^n \)

\textbf{Theorem (Li, Ma, VdV, Vrancken, Wang)}

Let \( f : M^n \to Q^n, \ n \geq 2, \) be a minimal Lagrangian immersion with \textit{constant local angle functions}. If \( g \) is the number of different constant local angle functions modulo \( \pi \), then \( g \in \{1, 2, 3, 4, 6\} \). Moreover,

- if \( g = 1 \), then \( f \) is the Gauss map of a part of \( a_1 : S^n(r) \to S^{n+1}(1) \);

- if \( g = 2 \), then \( f \) is the Gauss map of a part of \( a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1) \);

- if \( g = 3 \), then \( f \) is the Gauss map of a part of \( a_3 : \mathbb{R}P^2 \times S^1(\varepsilon) \to S^4(1) \)
  or of tubes around standard embeddings \( \mathbb{C}P^2 \to S^7(1) \), \( \mathbb{H}P^2 \to S^{13}(1) \) or \( \mathbb{O}P^2 \to S^{25}(1) \).
### Theorem (Li, Ma, VdV, Vrancken, Wang)

Let \( f : M^n \to Q^n \), \( n \geq 2 \), be a **totally geodesic** Lagrangian immersion. Then \( f \) is the Gauss map of a part of \( a_1 : S^n(r) \to S^{n+1}(1) \) or of a part of \( a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1) \).
4 – Study of Lagrangian submanifolds of $Q^n$ – main results

Theorem (Li, Ma, VdV, Vrancken, Wang)

Let $f : M^n \to Q^n$, $n \geq 2$, be a totally geodesic Lagrangian immersion. Then $f$ is the Gauss map of a part of $a_1 : S^n(r) \to S^{n+1}(1)$ or of a part of $a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1)$.

Theorem (Li, Ma, VdV, Vrancken, Wang)

Let $f : M^n \to Q^n$, $n \geq 2$, be a minimal Lagrangian immersion, such that $M^n$ has constant sectional curvature $c$. Then either
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Theorem (Li, Ma, VdV, Vrancken, Wang)

Let \( f : M^n \to Q^n, \ n \geq 2, \) be a minimal Lagrangian immersion, such that \( M^n \) has constant sectional curvature \( c. \) Then either

- \( f \) is the Gauss map of a part of
  \[ a_1 : S^n(r) \to S^{n+1}(1); \]
- \( n = 2 \) and \( f \) is the Gauss map of a part of
  \[ a_2 : S^1(r_1) \times S^1(r_2) \to S^3(1); \]
4 – Study of Lagrangian submanifolds of $Q^n$ – main results

**Theorem (Li, Ma, VdV, Vrancken, Wang)**

Let $f : M^n \to Q^n$, $n \geq 2$, be a totally geodesic Lagrangian immersion. Then $f$ is the Gauss map of a part of $a_1 : S^n(r) \to S^{n+1}(1)$ or of a part of $a_2 : S^k(r_1) \times S^{n-k}(r_2) \to S^{n+1}(1)$.

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- $n = 2$ and $f$ is the Gauss map of a part of $a_2 : S^1(r_1) \times S^1(r_2) \to S^3(1)$;
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4 – Study of Lagrangian submanifolds of $Q^n$ – main results

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Theorem (Li, Ma, VdV, Vrancken, Wang)

Let $f : M^n \to Q^n$, $n \geq 2$, be a minimal Lagrangian immersion, such that $M^n$ has constant sectional curvature $c$. Then either

- $f$ is the Gauss map of a part of $a_1 : S^n(r) \to S^{n+1}(1)$; $c = 2$
- $n = 2$ and $f$ is the Gauss map of a part of $a_2 : S^1(r_1) \times S^1(r_2) \to S^3(1)$; $c = 0$
- $n = 3$ and $f$ is the Gauss map of a part of $a_3 : \mathbb{RP}^2 \times S^1(\varepsilon) \to S^4(1)$. $c = \frac{1}{8}$
Some steps in the proof of the last theorem
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Lemma

Let $f : M^n \to Q^n$, $n \geq 2$, be a Lagrangian immersion, such that $M^n$ has constant sectional curvature $c$. Then

$$
\sin(\theta_i - \theta_j) \sin(\theta_i + \theta_j - 2\theta_k)(\delta_{k\ell} h(e_i, e_j) + h^\ell_{ij} Je_k)
+ \sin(\theta_j - \theta_k) \sin(\theta_j + \theta_k - 2\theta_i)(\delta_{i\ell} h(e_j, e_k) + h^\ell_{jk} Je_i)
+ \sin(\theta_k - \theta_i) \sin(\theta_k + \theta_i - 2\theta_j)(\delta_{j\ell} h(e_i, e_k) + h^\ell_{ik} Je_j) = 0
$$

for all $i, j, k, \ell$. In particular, if $i, j, k$ are mutually different, then

$$
h^k_{ii} \sin(\theta_i - \theta_k) \sin(\theta_i + \theta_k - 2\theta_j) = h^k_{jj} \sin(\theta_j - \theta_k) \sin(\theta_j + \theta_k - 2\theta_i),
$$

$$
h^k_{ij} \sin(\theta_i - \theta_j) \sin(\theta_i + \theta_j - 2\theta_k) = 0
$$

and if $i, j, k, \ell$ are mutually different, then

$$
h^k_{ij} \sin(\theta_i - \theta_j) \sin(\theta_i + \theta_j - 2\theta_\ell) = 0.
$$
### Proposition

Let $f : M^n \to Q^n$, $n \geq 2$, be a minimal Lagrangian immersion such that $M^n$ has constant sectional curvature and choose $A \in A$ such that $\theta_1 + \ldots + \theta_n = 0 \mod \pi$. Then either

- all local angle functions are the same modulo $\pi$, or
- all local angle functions are mutually different modulo $\pi$.

In the former case, the immersion is the Gauss map of a part of $a_1 : S^n(r) \to S^{n+1}(1)$. 

Proposition

Let \( f : M^n \rightarrow Q^n, \ n \geq 2, \) be a minimal Lagrangian immersion such that \( M^n \) has constant sectional curvature and choose \( A \in \mathcal{A} \) such that \( \theta_1 + \ldots + \theta_n = 0 \mod \pi \). Then either

- all local angle functions are the same modulo \( \pi \), or
- all local angle functions are mutually different modulo \( \pi \).

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The conclusion follows by algebraic computations using all the obtained relations and the equations of Gauss and Codazzi.
Proposition

Let $f : M^n \to Q^n$, $n \geq 2$, be a minimal Lagrangian immersion such that $M^n$ has constant sectional curvature and choose $A \in \mathcal{A}$ such that $\theta_1 + \ldots + \theta_n = 0 \mod \pi$. Then either

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The conclusion follows by algebraic computations using all the obtained relations and the equations of Gauss and Codazzi.

Remark. For $M^2 \to Q^2 \cong S^2(\frac{1}{2}) \times S^2(\frac{1}{2})$, the classification was already obtained by Castro and Urbano.
5 – Outline

1. How we started research on $Q^n$
2. The complex quadric $Q^n$
3. The Gauss map of a hypersurface of a sphere
4. Study of Lagrangian submanifolds of $Q^n$
5. Question
Question:

Are there other Riemannian manifolds \((M, g)\) with anti-commuting almost complex structure \(J\) and almost product structure \(P\) such that

\[
R(X, Y)Z = a \left( g(Y, Z)X - g(X, Z)Y \right) \\
+ b \left( g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ \right) \\
+ c \left( g(PY, Z)PX - g(PX, Z)PY + g(JPY, Z)JPX - g(JPX, Z)JPY \right)
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+ b (g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ) \\
+ c (g(PY, Z)PX - g(PX, Z)PY + g(JPY, Z)JPX - g(JPX, Z)JPY) 
\]

Only examples that I know of so far:

- real space forms (no \(J\), no \(P\)), complex space forms (no \(P\))
- the homogeneous nearly Kähler \(S^3 \times S^3\)
- the complex quadric, the hyperbolic complex quadric

Remark. All such manifolds will be Einstein.
5 – References

- J. Van der Veken and A. Wijffels,

- H. Li, H. Ma, J. Van der Veken, L. Vrancken and X. Wang,

- I. Castro and F. Urbano,

- G. R. Jensen,

- H. F. Münzner,

- B. Palmer,
Thank you for your attention!