Invariant Ricci-flat Kähler metrics on tangent bundles of compact symmetric spaces

José Carmelo González Dávila

Departamento de Matemáticas, Estadística e Investigación Operativa
University of La Laguna (Spain)

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Our goal

We give a new technique to determine explicitly all invariant Ricci-flat Kähler structures on the tangent bundle of compact symmetric spaces of any rank, not only for rank one. For rank one, we find new examples of Ricci-flat Kahler metrics.


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Polarizations and Kähler structures

Let $J$ be an almost complex structure on a $2n$-dimensional manifold $M$ ($J^2 = -Id$). The complex $\pm i$-eigenspaces of $J$ on $T^CM$ can be expressed as

\[ T^{(1,0)}M = \{z = u - iJu \mid u \in TM\}, \quad T^{(0,1)}M = \{z = u + iJu \mid u \in TM\}. \]

- $J$ defines a complex subbundle
  \[ F(J) = T^{(1,0)}M = \{z = u - iJu \mid u \in TM\} \subset T^CM \text{ s. t.} \]
  \[ T^CM = F(J) \oplus F(J). \]

The converse holds.

Existence of almost complex structures

Let $F$ be a complex subbundle of $T^CM$ such that $T^CM = F \oplus \overline{F}$. Then there exists a unique almost complex structure $J$ on $M$ s. t.

\[ F = F(J) = \{z = u - iJu \mid u \in TM\}. \]

Moreover, $F$ is involutive ($[F, F] \subset F$) if and only if $J$ is integrable.
Let $J$ be an almost complex structure on a $2n$-dimensional manifold $M$ ($J^2 = -\text{Id}$). The complex $\pm i$-eigenspaces of $J$ on $T^C M$ can be expressed as

$$T^{(1,0)} M = \{ z = u - iJu \mid u \in TM \}, \quad T^{(0,1)} M = \{ z = u + iJu \mid u \in TM \}.\$$

- $J$ defines a complex subbundle
  
  $F(J) = T^{(1,0)} M = \{ z = u - iJu \mid u \in TM \} \subset T^C M$ s. t. $T^C M = F(J) \oplus \overline{F(J)}$.

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On an almost Hermitian manifold \( (M, J, g) \) \((g(JX, JY) = g(X, Y))\), the fundamental 2-form \( \omega \) is given by

\[
\omega(X, Y) = -g(JX, Y), \quad X, Y \in \mathfrak{X}(M).
\]

Then, \( g(X, Y) = \omega(JX, Y) \).

- If \( d\omega = 0 \), \((M, J, g)\) is called almost Kähler.
- If, moreover \( J \) is integrable, it is called Kähler.

- \( F \subset T^\mathbb{C}M \) is said to be integrable if \( F \cap \overline{F} \) has constant rank and the subbundles \( F \) and \( F + \overline{F} \) are involutive. \( (F(J) \) is integrable if and only if it is involutive).
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Polarizations and Kähler structures

Fix a non-degenerate 2-form $\omega$ on a $2n$-dimensional manifold $M$:

- $F \subset T^\mathbb{C}M$ is said to be *Lagrangian* if $\omega(F, F) = 0$ and $\dim_{\mathbb{C}} F = n$.
- A *polarization* of $M$ is an integrable complex subbundle $F$ which is Lagrangian.
- A polarization $F$ is said to be *positive-definite* if the Hermitian form
  
  \[ h(u, v) = i\omega(u, \overline{v}), \quad u, v \in T^\mathbb{C}M, \]

  is positive-definite on $F$.

Equivalent Kähler condition

Let $(M, \omega)$ be a symplectic manifold and let $J$ be an almost complex structure on $M$. The pair $(J, g = \omega(J\cdot, \cdot))$ is a Kähler structure on $M$ if and only if the subbundle $F(J)$ is a positive-definite polarization.

Invariant Ricci-flat Kähler metrics
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The canonical complex structure on $T(G/K)$

The tangent bundle $T(G/K)$

Let $M = G/K$ where $G$ is a compact, connected Lie group and $K$ is closed subgroup of $G$. Then there exists a positive-definite $\text{Ad}(G)$-invariant form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.

- **Reductive decomposition**: $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ ($\text{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$).
- $(M = G/K, g)$ is a Riemannian homogeneous manifold determined by $\langle \cdot, \cdot \rangle_m$.

Consider $G \times \mathfrak{m}$ with two actions:

$$l_a: (g, w) \mapsto (ag, w), \quad r_k: (g, w) \mapsto (gk, \text{Ad}(k^{-1})(w)),$$

where $a, g \in G$ and $k \in K$.

- The projection $\pi: G \times \mathfrak{m} \to G \times_K \mathfrak{m}, (g, w) \mapsto [(g, w)]$, is $G$-equivariant.
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**Complexifications of Lie groups**

Any compact Lie group $G$ admits, up to isomorphisms, a unique complexification $G^\mathbb{C}$ which is given by $G^\mathbb{C} = G \exp(i\mathfrak{g})$.

- $G^\mathbb{C}/K^\mathbb{C}$ is a complex homogeneous manifold and $p_h : G^\mathbb{C} \to G^\mathbb{C}/K^\mathbb{C}$ is a holomorphic mapping. Moreover, $G^\mathbb{C} = G \exp(i\mathfrak{m}) \exp(i\mathfrak{k})$.
- The complex vector fields $\xi^x_h = \xi^x - i(l\xi)^x$, $\xi \in \mathfrak{g}$, $l\xi = i\xi$, determine a complex involutive subbundle of $T^\mathbb{C}G^\mathbb{C}$.
- The $G^\mathbb{C}$-invariant canonical complex structure $J^K_c$:
  $$(p_h)_*(\xi^x_h) = (p_h)_*(\xi^x) - i(p_h)_*(l\xi)^x$$
- A relevant fact: The mapping $f_K : G^\mathbb{C}/K^\mathbb{C} \to G \times_K m$, $g \exp(iw) \exp(i\zeta)K^\mathbb{C} \mapsto [(g, w)]$, $(g, w, \zeta) \in G \times m \times \mathfrak{k}$, is a $G$-equivariant diffeomorphism. Then it determines a $G$-invariant complex structure $J^K_c$ on $T(G/K)$. 

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Invariant Ricci-flat Kähler metrics
Let $G/K$ be a rank-$r$ symmetric space of compact type. Here, there exists an involutive automorphism $\sigma : g \to g$ and $\mathfrak{k} = \{\xi \in g : \sigma(\xi) = \xi\}$ and $\mathfrak{m} = \{\xi \in g : \sigma(\xi) = -\xi\}$.

Let $\mathfrak{a} \subset \mathfrak{m}$ be some Cartan subspace of $\mathfrak{m}$. Then $\dim \mathfrak{a} = r$ and there exists a Cartan subalgebra $\mathfrak{t} \sigma$-invariante de $g$ such that $\mathfrak{a} \subset \mathfrak{t}$.

- $\mathfrak{t}^C$ is a Cartan subalgebra of $g^C$.
- Root space decomposition

$$g^C = \mathfrak{t}^C \oplus \sum_{\alpha \in \Delta} g_\alpha, \quad g_\alpha = \{\xi \in g^C : [t, \xi] = \alpha(t)\xi, \ t \in \mathfrak{t}^C\}.$$  

- Restricted roots of $(g, \mathfrak{k}, \mathfrak{a})$

$$\Sigma = \{\lambda \in (\mathfrak{a}^C)^* : \lambda = \alpha|_{\mathfrak{a}^C}, \ \alpha \in \Delta \setminus \Delta_0\}.$$
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• $\lambda(a) \subset i\mathbb{R}$. Define $\lambda': a \to \mathbb{R}$, $\lambda \in \Sigma^+$, such that $i\lambda' = \lambda$.

• Weyl chamber $W^+$ in $\mathfrak{a}$:

$$W^+ = \{ w \in \mathfrak{a} : \lambda'(w) > 0 \text{ for all } \lambda \in \Sigma^+ \}.$$

• Regular points of $\mathfrak{m}$:

$$\mathfrak{m}^R = \text{Ad}(K)(W^+) \subset \mathfrak{m}.$$
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Fundamental property on our study

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- **Property of compact rank-one symmetric spaces:** The linear isotropy group $\text{Ad}(K)$ acts transitively on the unit sphere of $\mathfrak{m}$.
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- Let $H$ be the subgroup of $K$ given by $H = \{ k \in K : \text{Ad}_k u = u, \text{for all } u \in \mathfrak{a} \}$.

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Invariant Ricci-flat Kähler metrics on $T^+(G/K)$

The Ricci form

On a Riemannian Kähler manifold $(M, g, J)$,

$$\text{Ric}(g)(X, Y) = \text{Ric}(X, JY), \quad X, Y \in \mathfrak{X}(M),$$

is a 2-form, known as the **Ricci form** of $g$.

- Its complex extension can be expressed (locally) as $\text{Ric}(g) = -i\partial\bar{\partial} \ln \det(\omega_{js})$.
- If $g = \omega(J^K_c \cdot, \cdot)$ is a $G$-invariant Kähler metric on $T(G/K)$ then $\text{Ric}(g) = i\partial\bar{\partial} \ln S$, where $S : T(G/K) \to \mathbb{C}$ is a $G$-invariant function.
- Assume that the group $G$ is semisimple. If $g = \omega(J^K_c \cdot, \cdot)$ is a $G$-invariant Kähler metric on $G/H \times W^+$, then $\text{Ric}(g) = 0 \iff S = \text{const.}$
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- A two-form $\omega$ on $G/H \times W^+$ is a $G$-invariant symplectic structure if and only if $\tilde{\omega} = (\pi_H \times id)^* \omega$ satisfies the following three conditions:
  1. $\tilde{\omega}$ is closed;
  2. $\tilde{\omega}$ is left $G$-invariant and right $H$-invariant;
  3. $\text{Ker}(\tilde{\omega}) = \text{Ker}(\pi_H \times id)_*$.

- Consider $\mathcal{F} = (\pi_H \times id)_(-1)(F) \subset T^\mathbb{C}(G \times W^+)$. Then $\mathcal{F} = \text{Ker}(\tilde{\omega}) \oplus \tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is a $n$-dimensional $G$-invariant complex subbundle with $(\pi_H \times id)_* \tilde{\mathcal{F}} = F$ and there exists $G$-invariant complex vector fields $\{T_1, \ldots, T_n\}$ which generate $\tilde{\mathcal{F}}$. 
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• \( \omega \) satisfying (1) – (3), is a positive-definite polarization if and only if

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(4) \quad \tilde{\omega}(T_j, T_k) = 0, \ j, k = 1, \ldots, n;
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(5) \quad i\tilde{\omega}(T, \overline{T}) > 0 \text{ for each } T = \sum_{j=1}^{n} c_j T_j, \ (c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{0\}.
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• If moreover, \( G \) is semisimple and \( \tilde{\omega} \) satisfies

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Invariant Ricci-flat Kähler metrics on $T^+(G/K)$

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  (5) $i\tilde{\omega}(T, \overline{T}) > 0$ for each $T = \sum_{j=1}^{n} c_j T_j$, $(c_1, \ldots, c_n) \in \mathbb{C}^n \setminus \{0\}$.

- If moreover, $G$ is semisimple and $\tilde{\omega}$ satisfies

  (6) $\det(\tilde{\omega}(T_j, T_k)) = \text{const}$ on $G \times W^+$,

  then the corresponding Kähler metrics is Ricci-flat.

- The correspondence between $\omega$ and its pullback $\tilde{\omega}$ is one-to-one.
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and

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Invariant Ricci-flat Kähler metrics on $T^+(G/K)$

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**Theorem**

*(Main Theorem)* Let $G/K$ be a Riemannian symmetric space of compact type. Each $G$-invariant Kähler metric $g$, associated with the canonical complex structure $J^K_c$ on $G/H \times W^+ \cong T^+(G/K)$, is determined by the Kähler form $\omega(\cdot, \cdot) = -g(J^K_c \cdot, \cdot)$ on $G/H \times W^+$ given by $(\pi_H \times \text{id})^* \omega = d\tilde{\theta}^a$, where $a : W^+ \to g$ is a smooth vector-function which is unique for each $\omega$, satisfying certain conditions equivalent to the previous conditions (2)–(5) and $\tilde{\theta}^a$ is the $G$-invariant $1$-form on $G \times W^+$

$$\tilde{\theta}^a_{(g,x)}(\xi^l, w_x) = \langle a(x), \xi \rangle,$$

for all $(g, x) \in G \times W^+$, $\xi \in g$ and $w \in a$.

If, in addition, the corresponding condition (6) for $a$ holds, this metric $g$ is Ricci-flat.
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Other aspects which have been studied for these metrics

- Their differentiable \textit{extensions} to all the tangent bundle.
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Invariant Ricci-flat Kähler metrics on $T^+(G/K)$

A first application of the Main Theorem

Theorem

Let $G/K = SO(3)/SO(2) = S^2$. A 2-form $\omega$ on the punctured tangent bundle $G \times W^+ \cong T^+ S^2$ of $S^2$ defines a $G$-invariant Kähler structure, associated to the canonical complex structure $J^K_c$, and the corresponding metric $g = \omega(J^K_c \cdot, \cdot)$ is Ricci-flat, if and only if $\omega$ on $G \times W^+$ is expressed as $\omega = d\tilde{\theta}^a$, where the vector function $a(x) = f'(x)X + \frac{cz}{\cosh x}Z$, $cz$ being an arbitrary real number and

$$f'(x) = \sqrt{C \sinh^2 x + c^2_Z \sinh^2 x \cosh^{-2} x + C_1},$$

for some real constants $C > 0$ and $C_1 \geq 0$.

The corresponding $G$-invariant Ricci-flat Kähler metric on $T^+ S^2$ is uniquely extendable to a smooth complete metric on $T S^2$ if and only if $C_1 = 0$ (that is, $\lim_{x \to 0} f'(x) = 0$).
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A first application of the Main Theorem

**Theorem**

Let $G/K = SO(3)/SO(2) = \mathbb{S}^2$. A 2-form $\omega$ on the punctured tangent bundle $G \times W^+ \cong T^+\mathbb{S}^2$ of $\mathbb{S}^2$ defines a $G$-invariant Kähler structure, associated to the canonical complex structure $J^K_c$, and the corresponding metric $g = \omega(J^K_c \cdot, \cdot)$ is Ricci-flat, if and only if $\omega$ on $G \times W^+$ is expressed as $\omega = d\tilde{\theta}^a$, where the vector function $a(x) = f'(x)X + \frac{cZ}{\cosh x}Z$, $cZ$ being an arbitrary real number and

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All the metrics for $C_1 = 0$ and $c_Z \neq 0$ are new examples of complete Ricci-flat Kähler metrics on whole $T^+ S^2$.

Thanks for your attention
Invariant Ricci-flat Kähler metrics on $T^+(G/K)$

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