Left Invariant Structures on Statistical Manifolds

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- Information geometry, Fisher metric
- Manifolds of probability density functions-Amari, Lauritzen 80’s
- Statistical manifolds, Holomorphic statistical manifolds-Kurose, 2004
- Statistical manifolds-generalization of Hessian manifolds
- Holomorphic statistical manifolds-generalization of Kaehler manifolds
- H. Matsuzoe, Statistical manifolds and affine differential geometry, 2010
- H. Furuhata, Hypersurfaces in statistical manifolds, 2009
- B. Opozda, Bochner’s technique for statistical structures, 2015
FROM KÄHLER TO STATISTICAL MANIFOLDS
Kähler manifolds

\((M, g)\)-a Riemannian manifold
\(\nabla\)-the Levi-Civita connection on \(M\)
\((M, g)\) is called a complex manifold if it admits a complex structure \(J\).
g is called Hermitian if
\[
g(JX, JY) = g(X, Y).
\]
\(\Omega(X, Y) = g(JX, Y)\)-Kähler form
\((M, g)\) is called a Kähler manifold if
\[
d\Omega = 0 \iff \nabla J = 0.
\]
On statistical manifolds, we have a connection different than Levi-Civita one.

**Definition**
(Kurose) The triplet $(M, \nabla, g)$ is called a statistical manifold if:

1. $\nabla$ is torsion-free connection, and
2. $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$ (Codazzi equation).

**Definition**
$\nabla^*$-the dual connection of $\nabla$ wrt $g$ iff

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z).$$

$(M, \nabla^*, g)$-the dual statistical manifold of $(M, \nabla, g)$. 
**Definition**

(Kurose) A statistical manifold \((\overline{M}, \overline{\nabla}, \overline{g})\) is said to be of constant curvature \(c \in \mathbb{R}\) if

\[
R^{\overline{\nabla}}(X, Y)Z = c\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y\},
\]

\(R^{\overline{\nabla}}\)-the curvature tensor of \(\overline{\nabla}\).

A statistical structure \((\overline{\nabla}, \overline{g})\) of constant curvature 0 is called a **Hessian structure**.

Example

**Normal distributions.** We denote by \( l(x, \xi) = \log p(x, \xi) \)

\[
M = \{ p(x; \xi) | \xi = (\xi^1, \xi^2) = (\mu, \sigma), \\
p(x; \xi) = \frac{1}{\sqrt{2\pi(\xi^2)^2}} e^{-\frac{(x-\xi^1)^2}{2(\xi^2)^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \}
\]

We regard that \( M \) is a manifold with local coordinates \((\mu, \sigma)\).

\[
g_{ij} := \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial \xi^i} \log p(x; \xi) \right) \left( \frac{\partial}{\partial \xi^j} \log p(x; \xi) \right) p(x; \xi) dx \\
= E \left[ \frac{\partial l}{\partial \xi^i} \frac{\partial l}{\partial \xi^j} \right] \\
= -\frac{1}{\sigma^2} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\]

-the Fisher information
$$C_{ijk} = E \left[ \frac{\partial l}{\partial \xi^i} \frac{\partial l}{\partial \xi^j} \frac{\partial l}{\partial \xi^k} \right]$$

-the skewness or the cubic form

$$\Gamma_{ij,k} = E \left[ \frac{\partial^2 l}{\partial \xi^i \partial \xi^j} \frac{\partial l}{\partial \xi^k} - \frac{\partial l}{\partial \xi^i} \frac{\partial l}{\partial \xi^j} \frac{\partial l}{\partial \xi^k} \right]$$

$$= \Gamma^0_{ij,k} - \frac{1}{2} C_{ijk}$$

$\nabla^0$-the Levi-Civita connection wrt $g$

$$\Gamma^*_{ij,k} = \left[ \frac{\partial^2 l}{\partial \xi^i \partial \xi^j} \frac{\partial l}{\partial \xi^k} + \frac{\partial l}{\partial \xi^i} \frac{\partial l}{\partial \xi^j} \frac{\partial l}{\partial \xi^k} \right]$$

$$= \Gamma^0_{ij,k} + \frac{1}{2} C_{ijk}$$

$(M, \nabla, g)$ and $(M, \nabla^*, g)$ are statistical manifolds.
Definition

(Kurose) $(M, g, J)$- a Kaehler manifold, $\nabla$- an affine connection of $M$

$(M, \nabla, g, J)$ is called a holomorphic statistical manifold if

1. $(M, g, \nabla)$ is a statistical manifold, and
2. $\omega := g(\cdot, J)$ is a $\nabla$-parallel 2-form on $M$
HOW TO CONSTRUCT HOLOMORPHIC STATISTICAL MANIFOLDS?
If we define a connection $\nabla$ as $\nabla := \nabla^{LC} + K$, where $K$ is a $(1, 2)$ tensor field satisfying

$$K(X, Y) = -K(Y, X) \quad (1)$$

$$g(K(X, Y), Z) = g(Y, K(X, Z)) \quad (2)$$

and

$$K(X, JY) = -JK(X, Y) \quad (3)$$

then $(M, \nabla, g, J)$ is a holomorphic statistical manifold.
Kaehler manifolds:

\[ \nabla^{L_C} J = 0 \]  \hfill (4)

Holomorphic statistical manifolds:

\[ \nabla J = J \nabla^* \]  \hfill (5)
BACKGROUND
S. Amari, Differential-geometrical methods in statistics, 1985
S. L. Lauritzen, Statistical manifolds, 1987
- $\alpha$-connections
- first definition of statistical manifolds

$(M, g)$-a Riemannian manifold with Levi-Civita connection $\nabla^{LC}$. $C$-(0,3)-tensor symmetric in its first two arguments

$$\nabla := \nabla^{LC} + \overline{C},$$ (6)

$g(\overline{C}(X, Y), Z) = C(X, Y, Z)$.

$(M, g, C)$-statistical manifold

A conjugate symmetric manifold: $(M, g, \nabla, \nabla^{*})$ if for conjugate connections $R(X, Y)Z = R^{*}(X, Y)Z$
LEFT INVARIANT STRUCTURES
Some new results on Lie groups
G-Lie group, \( g \)-its Lie algebra (vector fields invariant under left translations)  

\( g \)-left invariant pseudo-Riemannian metric on \( G \) (left translations are isometries of \( (G, g) \)) 

\[
< X, Y > := g(X, Y) \tag{7}
\]

\( g(U, V) = \text{const.} \)

\[
0 = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{8}
\]

\[
\nabla^\Sigma := \frac{1}{2}(\nabla + \nabla^*) \tag{9}
\]

\[
\Phi(X, Y, Z) := g((\nabla^\Sigma_X J)Y, Z) + g((\nabla^\Sigma_Y J)Z, X) + g((\nabla^\Sigma_Z J)X, Y) \tag{10}
\]

\( \Phi \) is skew-symmetric tensor
(\nabla_X J)Y = -(\nabla^*_X J)Y \quad (11)

(\nabla_{JX} J)Y = -J(\nabla_Y J)X \quad (12)

D(X, Y) := \nabla_{JX} Y + J\nabla_Y X \quad (13)

\Theta(X, Y, Z) = \langle D(X, Y), Z \rangle + \langle D(Y, Z), X \rangle + \langle D(Z, X), Y \rangle \quad (14)
Equivalent conditions:

1. \((G, g, J)\) is holomorphic statistical manifold
2. \(\Theta(X, Y, Z) = -\Theta(X, Z, Y)\) and \(\Theta(JX, Y, Z) = \Theta(X, JY, Z)\)
THANK YOU!