Totally geodesic submanifolds in the Riemannian symmetric spaces of rank 2

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31st October 2019
Totally geodesic submanifolds in Riemannian manifolds

- **Totally geodesic** submanifolds in **symmetric** spaces rank 2.
- **Reminder.** A submanifold $M'$ of a Riemannian manifold $M$ is called **totally geodesic**, if
  - the second fundamental form $h$ of $M' \hookrightarrow M$ vanishes,
  or equivalently, if
  - every geodesic of $M'$ also is a geodesic in $M$.
If $M'$ is totally geodesic, then $T_p M'$ is **curvature-invariant**, i.e. $R^M(T_p M', T_p M') T_p M' \subset T_p M'$.
- **Examples.**
  - $\mathbb{R}^k \subset \mathbb{R}^n$
  - $S^k \subset S^n$
  - $\mathbb{C}P^k \subset \mathbb{C}P^n$
  - $\mathbb{R}P^k \subset \mathbb{C}P^n$
  - $\mathbb{H}P^k \subset \mathbb{H}P^n$
  - $\mathbb{C}P^k \subset \mathbb{H}P^n$
  - $\mathbb{R}P^k \subset \mathbb{H}P^n$
- **Murphy** (2019): On a differentiable manifold $M$ with $\dim(M) \geq 4$, **generic** Riemannian metrics on $M$ do not admit any totally geodesic submanifolds of dimension $\geq 2$. 
Today we are interested in the following classification problem:

Given a Riemannian symmetric space \( M \), find all totally geodesic submanifolds of \( M \).

Clearly, totally geodesic submanifolds in \( M \) come in families by the action of \( I(M) \). In general, there exist several such families of totally geodesic submanifolds \( M' \).

Classify totally geodesic submanifolds in \( M \)?

- Up to congruence?
- Up to (local) isometry?
- Up to (local) homothety?
Known classification results for totally geodesic submanifolds

- **All** totally geodesic submanifolds are known in
  - **Rank 1 symmetric spaces.**
    Spheres, projective spaces, Cayley plane. **Wolf** 1963.
  - **Rank 2 symmetric spaces.** “We have to talk.”
  - No symmetric spaces of rank \( \geq 3 \).

- **Specific types** of totally geodesic submanifolds have been classified in all (irreducible) symmetric spaces, for example:
  - **Reflective submanifolds.** They are connected components of the fixed point set of involutive isometries of \( M \).
    **Leung** 1974/75.
  - **Complex submanifolds** (in Hermitian symmetric spaces).
    **Ihara** 1967.
  - **Maximal spheres.**
    **Makiko Sumi Tanaka** 1991.
  - **Subspaces of maximal rank.**
    **Ikawa/Tasaki** 2000, **Zhu/Liang** 2004.
Totally geodesic submanifolds in spaces of rank 2

- **Chen/Nagano 1978**: Classification to **local homothety**.
  - First application of \((M_+, M_-)\)-method (polars/meridians).
  - No information about the **position** of the submanifolds.
  - **Missed** some “skew” maximal totally geodesic submanifolds:
    \[
    S^2(\frac{1}{2}\sqrt{10}) \subset Q^3 = G_2^+(\mathbb{R}^5), \quad \mathbb{CP}^2 \subset G_2(\mathbb{C}^6), \quad \mathbb{HP}^2 \subset G_2(\mathbb{H}^7),
    \]
    \[
    S^3(\frac{1}{2}\sqrt{10}) \subset Sp(2), \quad S^2(\frac{2}{3}\sqrt{21}) \subset G_2/\text{SO}(4), \quad S^3(\frac{2}{3}\sqrt{21}) \subset G_2.
    \]

- **Kimura/Tanaka 2008**: Classification **global homothety**.
  - Refinement of the method by Chen/Nagano.
  - The above “skew” submanifolds are still missing.

- **K~ 2005–09**: Classification up to **congruence**.
  - Postdoctoral Fellowship at the University College Cork (2006–08), under the guidance of **Jürgen Berndt**.
  - Different methods: **Root systems**.
  - Description of the **position** of submanifolds (tangent spaces/totally geodesic embeddings).
  - The missing “skew” totally geodesic submanifolds were **found**.
Let $M = G/K$ be a Riemannian symmetric space with symmetric triple $(G, K, \sigma)$ and origin $p_0 := eK \in M$.

Every connected totally geodesic (t.g.) submanifold of $M$ is contained in a complete one, congruent to one through $p_0$.

Two connected, complete, t.g. submanifolds $M'$, $M''$ through $p_0$ with $T_{p_0}M' = T_{p_0}M''$ are identical: $M' = M''$.

A connected, complete submanifold $M'$ of $M$ with $p_0 \in M'$ is t.g. if and only if it is a symmetric subspace, i.e. if there exists a $\sigma$-invariant Lie subgroup $G'$ of $G$ so that $(G', G' \cap K, \sigma|G')$ is a symmetric triple for $M'$.

$U \subset T_{p_0}M$ a linear subspace. There exists a t.g. submanifold $M' \subset M$ with $p_0 \in M'$ and $T_{p_0}M' = U$ if and only if $U$ is curvature invariant (a Lie triple system), i.e. if $R_M(U, U)U \subset U$ (or $[[m', m'], m'] \subset m'$) holds.
The task that is set before us is to classify the Lie triple systems of \( M \), for every Riemannian symmetric space \( M \) of rank 2.

The simply connected, irreducible Riemannian symmetric spaces \( M \) of compact type are the following:

- The 2-Grassmannians
  \[ Q^m = G_2^+(\mathbb{R}^{m+2}), \ G_2(\mathbb{C}^{m+2}) \text{ and } G_2(\mathbb{H}^{m+2}). \]
- The classical quotient spaces
  \[ \text{SU}(3)/\text{SO}(3), \ \text{SU}(6)/\text{Sp}(3) \text{ and } \text{SO}(10)/\text{U}(5). \]
- The exceptional spaces
  \[ E_6/((\text{U}(1) \cdot \text{Spin}(10)), \ E_6/F_4 \text{ and } G_2/\text{SO}(4). \]
- The compact Lie groups \( \text{SU}(3), \text{Sp}(2) \) and \( G_2 \).
Roots and root spaces

- Suppose that \( M \cong (G, K, \sigma) \) is of compact type.
- Let \( g \) be the Lie algebra of \( G \), \( \sigma_L \) the linearisation of \( \sigma \). Then \( \mathfrak{k} = \text{Eig}(\sigma_L, 1) \), \( \mathfrak{m} = \text{Eig}(\sigma_L, -1) \cong T_{p_0}M \), \( g = \mathfrak{k} \oplus \mathfrak{m} \) and \(-[[u, v], w] \cong R_M(u, v)w \) for \( u, v, w \in \mathfrak{m} \cong T_{p_0}M \).
- Choose a Cartan subalgebra \( \mathfrak{a} \subset \mathfrak{m} \). For \( \lambda \in \mathfrak{a}^* \setminus \{0\} \) we let

\[
\mathfrak{m}_\lambda = \{X \in \mathfrak{m} \mid \forall H \in \mathfrak{a} : \text{ad}(H)^2 X = -\lambda(H)^2 \cdot X\}.
\]

If \( \mathfrak{m}_\lambda \neq \{0\} \), \( \lambda \) is called a root of \( M \), and \( \mathfrak{m}_\lambda \) is its root space. The set \( \Delta \subset \mathfrak{a}^* \setminus \{0\} \) of all roots is the root system.
- We have \(-\Delta = \Delta\). For \( H_0 \in \mathfrak{a} \) with \( \lambda(H_0) \neq 0 \) for all \( \lambda \in \Delta \), \( \Delta_+ := \{\lambda \in \Delta \mid \lambda(H_0) > 0\} \) is the set of positive roots with respect to \( H_0 \). We have

\[
\mathfrak{m} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta_+} \mathfrak{m}_\lambda.
\]
The complex quadric

$Q^m = G_2^+(\mathbb{R}^{m+2}) = SO(m+2)/SO(2) \times SO(m)$ is a Hermitian symmetric space of rank 2.

We can visualise the root system of $Q^m$ with respect to a Cartan algebra $\mathfrak{a}$ by plotting $\alpha^\# \in \mathfrak{a}$ for $\alpha \in \Delta \subset \mathfrak{a}^*$:

\[
\begin{array}{ccc}
& & m-2 \\
1\cdot & \cdot & \cdot 1 \\
& \cdot & \bigcirc & \cdot m-2 \\
& \cdot & \cdot & \cdot \\
\end{array}
\]
How to describe Lie triple systems in root theory

Let $m' \subset m$ be a Lie triple system, \( \mathfrak{k}' = [m', m'] \subset \mathfrak{k} \) and $g' = \mathfrak{k}' \oplus m'$.

There exists a Cartan subalgebra \( \mathfrak{a} \) of $m$ such that \( \mathfrak{a}' = \mathfrak{a} \cap m' \) is a Cartan subalgebra of $m'$. Let $\Delta' \subset (\mathfrak{a}')^*$ be the root system of $m'$ with respect to $\mathfrak{a}'$, and for $\alpha \in \Delta'$, let $m'_\alpha$ be the corresponding root space.

Then we have

\begin{align*}
\Delta' &\subset \{ \lambda|\mathfrak{a}' \mid \lambda \in \Delta, \lambda|\mathfrak{a}' \neq 0 \} \\
\forall \alpha \in \Delta' : \quad m'_\alpha &\subset \bigoplus_{\lambda \in \Delta, \lambda|\mathfrak{a}' = \alpha} m_\lambda \\
m' &\subset \mathfrak{a}' \oplus \bigoplus_{\alpha \in \Delta'_+} m'_\alpha.
\end{align*}

In particular for $\text{rk}(m') = \text{rk}(m)$:

\begin{align*}
\mathfrak{a}' &= \mathfrak{a} , \quad \Delta' \subset \Delta , \quad m'_\alpha \subset m_\alpha.
\end{align*}
Consider the case $\text{rk}(m') = \text{rk}(m) = 2$. Then $\alpha' = \alpha$, $\Delta' \subset \Delta$ and $m'_\alpha \subset m_\alpha$.

The possibilities for $m'$ are further restricted by:

- $\Delta'$ is invariant under its Weyl group.
- $[[m'_\alpha, m'_\beta], m'_\gamma] \subset \bigoplus_{\alpha \pm \beta \pm \gamma \in \Delta'} m'_{\alpha \pm \beta \pm \gamma}$

Need to evaluate the Lie bracket.

- If $G$ is a classical Lie group, do matrix calculations in $g$ (or something similar).
- If $G$ is an exceptional Lie group, consider the root system of $g^C$. Use $\dim_C(g^C_\lambda) = 1$ and $[X_\lambda, X_\mu] = c_{\lambda, \mu} \cdot X_{\lambda + \mu}$. The numbers $c_{\lambda, \mu}$ are determined up to sign from the root system, consistent choice of signs can be obtained. Computer algebra is useful. ⇔ http://satake.sourceforge.net.

In this way, one can classify the rank 2 Lie triple systems in every rank 2 symmetric space.
Lie triple systems of rank 2 in the complex quadric

• • •  \( M' = G_2^+ (\mathbb{R}^{k+2}) \)
• • •  \( 3 \leq k < m \)

• • •  \( M' = (S^k \times S^\ell)/\mathbb{Z}_2 \)
• • •  \( k, \ell \geq 2; \ k + \ell \leq m \)

• • •  \( M' = \mathbb{CP}^1 \times \mathbb{CP}^1 \cong G_2^+ (\mathbb{R}^4) \)

• • •  \( M' = (S^k \times S^1)/\mathbb{Z}_2 \)
• • •  \( 2 \leq k \leq m - 1 \)

• • •  \( M' = \mathbb{CP}^1 \times \mathbb{RP}^1 \)

• • •  \( M' = (S^1 \times S^1)/\mathbb{Z}_2 \)
• • •  (a maximal flat torus)
Lie triple systems of rank 1.

- Consider the case $rk(m') = 1$ and $rk(m) = 2$. Then $a'$ is a line in the plane $a$.
- Is every line $a' \subset a$ possible? Take $\alpha \in \Delta'$, then $a' = \mathbb{R}\alpha^\#$, and $\alpha = \lambda |a'$ for one or more $\lambda \in \Delta$.
  - We call $\alpha$ elementary, if there exists only one $\lambda \in \Delta$ with $\lambda |a' = \alpha$. In this case we have $\lambda |(a')^\perp = 0$, i.e. $\lambda^\# \in a'$.
  - We call $\alpha$ composite, if there exist (at least) two different $\lambda, \mu \in \Delta$ with $\lambda |a' = \alpha = \mu |a'$. Then $a \perp (\lambda^\# - \mu^\#)$.
- Therefore
  - either $a' = \mathbb{R}\lambda^\#$ for some $\lambda \in \Delta$,
  - or $a' = (\mathbb{R}(\lambda^\# - \mu^\#))^\perp$ for some $\lambda, \mu \in \Delta$, $\lambda \neq \mu$.
- It follows that for every space $M$, there exist only finitely many possible $a'$.
- Still have to evaluate $[[m'_{j\alpha}, m'_{k\alpha}], m'_{l\alpha}]$ (for $j, k, l \in \{\pm 1, \pm 2\}$) to determine the possibilities for $m'_{\alpha}$ and $m'_{2\alpha}$. 
Rank 1 Lie triple systems in the complex quadric

\[ M' = S^k(1), \quad 1 \leq k \leq m \]
\[ M' = G_2(\mathbb{R}^3) \cong S^2 \]

\[ M' = \mathbb{CP}^k, \quad 1 \leq k \leq \frac{n}{2} \]
\[ M' = \mathbb{RP}^k, \quad 1 \leq k \leq \frac{n}{2} \]

\[ M' = S^2\left(\frac{1}{2}\sqrt{10}\right) \]
in a special, “skew” position
The “skew” 2-sphere in $Q^3$

- We want to **embed** the 2-sphere $M = \text{SO}(3)/\text{SO}(2)$ in $Q^3 = \text{SO}(5)/\text{SO}(2) \times \text{SO}(3)$ as a **totally geodesic** submanifold (**symmetric subspace**).

- $V := \text{End}^0_+(\mathbb{R}^3)$: symmetric, trace-free real $(3 \times 3)$-matrices. The **Cartan representation** is the 5-dimensional irreducible, orthogonal, real representation

\[
\text{SO}(3) \times V \to V, \ (B, X) \mapsto BXB^t = BXB^{-1}.
\]

It acts on the complex quadric $Q^3 \cong G^+_2(V)$ via **isometries**.

- Let $Z_0 := \mathbb{R} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \mathbb{R} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in G^+_2(V)$.

- It turns out that the **orbit** $M$ of the action of $\text{SO}(3)$ on $G^+_2(V)$ through $Z_0$ is **totally geodesic**, and **isometric** to $S^2$. It is neither a complex nor a totally real submanifold of $Q(V, \beta)$, and is therefore the **totally geodesic 2-sphere** that we seek.