Some homogeneous Lagrangian submanifolds in complex hyperbolic spaces

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Homogeneous (sub)manifolds: provide a manifold with several geometric structures and properties.
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Introduction

- **Homogeneous (sub)manifolds**: provide a manifold with several geometric structures and properties.
  - **Classifications** of cohomogeneity one actions in symmetric spaces: Hsiang-Lawson, Takagi, Iwata, Kollross, Berndt-Tamaru...

- **Lagrangian submanifolds**: an object in symplectic geometry.
  - A submfd $L$ in a symplectic mfd $(M, \omega)$ with $\omega|_L = 0$ & $\dim L = \frac{1}{2} \dim M$.
  - A widely-studied class of higher codimensional submfds by motivations related to Riemannian & Symplectic geometry.
  - **Homogeneous Lagrangian** submfds provide nice examples of Lag submfd.
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Problem: *Construct and classify homogeneous Lagrangian submanifolds in a specific Kähler manifold* (e.g. Hermitian symmetric spaces).
Introduction

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**Definition**

A submanifold \(L\) in \((M, \omega, J)\) is called *homogeneous* if \(L\) is obtained by an orbit \(H \cdot p\) of a connected Lie subgroup \(H\) of \(\text{Aut}(M, \omega, J)\). Furthermore, if we take \(H\) to be a compact subgroup, we say \(L = H \cdot p\) is *compact homogeneous*.
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We are interested in **homogeneous Lagrangian submfd**:

e.g. $T^n$-orbits in a toric Kähler manifold, real forms in cplx flag mfds, Gauss images in $\tilde{\mathcal{G}}r_2(\mathbb{R}^{n+2})$ of homog. hypersurfaces in a sphere... etc.
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**Classification results** (of actions admitting Lag orbits):

- \(M = \mathbb{C}P^n \& H\) is a cpt simple Lie group [Bedulli-Gori 08]. (Note that \(\exists\) 1-1 correspondence btw cpt homog Lag in \(\mathbb{C}P^n\) and the ones in \(\mathbb{C}^{n+1}\) via Hopf fibration).
- \(M = \tilde{Gr}_2(\mathbb{R}^{n+2}) \cong Q_n(\mathbb{C})\) [Ma-Ohnita 09]

Note: so far, we do not know any comprehensive method to classify homog Lag even for Hermitian symmetric spaces...
Consider the case when $M = \text{Hermitian symmetric space of non-compact type}$:

Theorem (cf. McDuff 88, Deltour 13)

Let $M = G/K$ be a Hermitian symmetric space of non-compact type, and $g = k + p$ the Cartan decomposition. Then, there exists a $K$-equivariant symplectic diffeomorphism $(M; !) \to (p; !_o)$. (Remark: this result is just an existence theorem, although they proved for more general setting [McDuff 88, Deltour 13])

g.e. $M = \mathbb{C}H^n \cong B_n \not\subset \mathbb{C}^n \cong \mathbb{C}^n, z \mapsto \sqrt{1 + jz}^{1/2}$

is a $K$-equivariant symplectic diffeomorphism (not holomorphic).

(Remark: [Di Scala-Loi 08] gives an explicit construction of $!$ for any Hermitian symmetric space of non-cpt type.)

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E.g. \( M = \mathbb{C}H^n \cong B^n \).

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\Phi : B^n \to \mathbb{C}^n \cong \mathfrak{p}, \quad z \mapsto \sqrt{\frac{1}{1 - |z|^2}} \cdot z
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(Remark: [Di Scala-Loi 08] gives an explicit construction of \( \Phi \) for any Hermitian symmetric space of non-cpt type.)
Cpt homg Lag in HSS of non-compact type

(continued) Since $\Phi : M \to p$ is $K$-equivariant and $K$ is a maximal compact subgroup of $G$, $\exists$ a map

$$\{\text{cpt homog Lag in } M = G/K\} \longrightarrow \{\text{cpt homog Lag in } p \cong \mathbb{C}^n\}.$$ 

In this sense, the classification problem of cpt homog Lag in $M$ is reduced to find an $H \subset \text{Ad}(K)$ admitting a Lag orbit in $p \cong \mathbb{C}^n$. 

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In this sense, the classification problem of cpt homog Lag in $M$ is reduced to find an $H \subset \text{Ad}(K)$ admitting a Lag orbit in $\mathfrak{p} \simeq \mathbb{C}^n$.

For example, if $M$ is rank 1, we see $\text{Ad}(K) = U(n)$, and it turns out that

**Theorem (Hashinaga-K. 17, Ohnita)**

*Suppose $M = \mathbb{C}H^n$ and let $L'$ be any cpt homog Lag in $\mathfrak{p} \simeq \mathbb{C}^n$. Then, $L := \Phi^{-1}(L')$ is a cpt homog Lag in $\mathbb{C}H^n$. In particular, any cpt homog Lag in $\mathbb{C}H^n$ (up to congruence) is obtained in this way.*

A geometric interpretation:

$$\begin{align*}
\mathbb{C}H^n, \omega & \xrightarrow{\Phi \text{ symp. diffeo.}} \mathbb{C}^n, \omega_0 \\
\bigcup & \quad \bigcup \\
C(K) = S^1 \curvearrowright L & \xrightarrow{\Phi \text{ diffeo.}} S^{2n-1}_r \\
S^1 & \xrightarrow{\text{diffeo.}} S^{2n-1}(\sinh r) \\
L/S^1 & \xrightarrow{\text{diffeo.}} \mathbb{C}P^{n-1}\left(\frac{4}{\sinh^2 r}\right) = \mathbb{C}P^{n-1}\left(\frac{4}{\sinh^2 r}\right)
\end{align*}$$
Non-cpt homg Lag in HSS of non-cpt type

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e.g. $M = \mathbb{C}H^1 \simeq B^1$

![Diagram of $L_0$-orbits (A-orbits) and $L_{\pi/2}$-orbits (N-orbits)]
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Since $\text{Aut}(M, \omega, J)$ of HSS of non-cpt type $M$ is non-cpt, there exist several types of non-cpt group actions:

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(Note: Since $\Phi : M \to p$ is a symplectic diffeo, we have a correspondence

$$\{\text{Lag submfd in HSS of non-cpt type } M\} \leftrightarrow \{\text{Lag submfd in } p \simeq \mathbb{C}^n\}. $$

Thus, a construction of (homog) Lag submfd in $M$ provides a way of constructing (new example of) a Lag submfd in $\mathbb{C}^n$.)

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Let $M = G/K$ be an irreducible HSS of non-cpt type.

- $g = \mathfrak{k} + \mathfrak{p}$: the Cartan decomposition.
- $\mathfrak{a} \subset \mathfrak{p}$: a maximal abelian subspace of $\mathfrak{p}$.
- $g = g_0 + \sum_{\lambda \in \Sigma} g_{\lambda}$: the restricted root decomposition w.r.t. $\mathfrak{a}$.
- Letting $n := \sum_{\lambda \in \Sigma_+} g_{\lambda}$, we obtain the Iwasawa decomposition

\[ g = \mathfrak{k} \oplus \mathfrak{a} \oplus n, \]

and $s := \mathfrak{a} + n$ is so called the solvable part of the Iwasawa decomposition.
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Let $M = G/K$ be an irreducible HSS of non-cpt type.

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and $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$ is so called the solvable part of the Iwasawa decomposition.

**Fact** Let $S$ be a connected subgroup of $G$ whose Lie algebra is $\mathfrak{s}$. Then, $S$ acts on $M$ simply transitively.

Hence, we obtain an identification $M \simeq S$ (as a Kähler mfd), and this is so called the solvable model of $M$. 

Let us consider a connected subgroup $S'$ of $S$ admitting a Lag orbit.
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Since $S$ acts on $M$ simply transitively, the classification of non-cpt homog Lag in $M$ obtained by a subgroup $S'$ of $S$ is reduced to classify Lagrangian subalgebras of $\mathfrak{s}$, that is, Lie subalgebra $\mathfrak{l}$ of $\mathfrak{s}$ satisfying Lagrangian condition i.e., $\omega|_{\mathfrak{l}} = 0$ and $\dim \mathfrak{l} = \frac{1}{2} \dim \mathfrak{s}$.

In [Hashinga-K. 17], we completely classify the Lagrangian subalgebra of $\mathfrak{s}$ when $M = \mathbb{C}H^n$, and give the details of Lagrangian orbits.
(The construction) Assume $M = \mathbb{C}H^n$. Then

$$ s = \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} = (\mathfrak{a} \oplus \mathfrak{g}_{2\alpha}) \oplus \mathfrak{g}_\alpha. $$

Both subspaces $\mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$ and $\mathfrak{g}_\alpha$ are symplectic (complex) subspaces of $\dim_{\mathbb{C}}(\mathfrak{a} \oplus \mathfrak{g}_{2\alpha}) = 1$ and $\dim_{\mathbb{C}}\mathfrak{g}_\alpha = n - 1$, hence, taking Lagrangian subspaces $l_1 \subset (\mathfrak{a} \oplus \mathfrak{g}_{2\alpha})$ and $l_2 \subset \mathfrak{g}_\alpha$,

$$ l := l_1 \oplus l_2 $$

is a Lagrangian subspace of $s$. 

(Remark: This construction is partially generalized to higher rank case [Hashinaga 18])

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For $X, Y \in l_1 \subset \mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$ and $U, V \in l_2 \subset \mathfrak{g}_\alpha$, the bracket relation of $\mathfrak{s}$ implies

$$[X + U, Y + V] = c_1 U + c_2 V + \{\omega_5(X, Y) + \omega_5(U, V)\} Z$$
$$= c_1 U + c_2 V \in l_2$$

for some $c_1, c_2$. Hence, $l$ is a subalgebra of $\mathfrak{s}$.

(Remark: This construction is partially generalized to higher rank case [Hashinaga 18])
(continued) Conversely, we proved the following:

**Lemma (H-K)**

Let $s'$ be any Lagrangian subalgebra of $s$. Then, $s'$ splits into a direct sum $s' = l_1 \oplus l_2$ of two Lagrangian subspaces $l_1 \subset \alpha \oplus g_{2\alpha}$ and $l_2 \subset g_{\alpha}$.
Non-cpt homg Lag in $\mathbb{C}H^n$

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Let $s'$ be any Lagrangian subalgebra of $s$. Then, $s'$ splits into a direct sum $s' = l_1 \oplus l_2$ of two Lagrangian subspaces $l_1 \subset a \oplus g_{2\alpha}$ and $l_2 \subset g_{\alpha}$.

Actually, $s' = l_1 \oplus l_2$ is isomorphic to the *canonical Lagrangian subalgebra* in $s$

$$l_\theta = \text{span}_\mathbb{R}\{\cos \theta A + \sin \theta Z\} \oplus \text{span}_\mathbb{R}\{X_1, \ldots, X_{n-1}\} \text{ for } \theta \in [0, \pi/2].$$

(Where $a = \text{span}_\mathbb{R}\{A\}$, $g_{2\alpha} = \text{span}_\mathbb{R}\{Z\}$ and $X_i \in g_{\alpha}$ s.t. $[X_i, JX_i] = Z$.)
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Denote the connected subgroup of $S$ whose Lie algebra $l_\theta$ by $L_\theta$. Lemma implies any Lag orbit $S' \cdot o$ for $S' \subset S$ is isometric to some $L_\theta \cdot o$. By computing the mean curvature, we see $L_\theta \cdot o$ is not isometric to $L_{\theta'} \cdot o$ if $\theta \neq \theta'$. Namely,
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**Theorem (Hashinaga-K. 17)**

The set $C(S)$ consisting of congruence classes of Lagrangian orbits obtained by connected subgroups of $S$ is parametrized by $\theta \in [0, \pi/2]$, and $L_\theta \cdot o$ represents each congruence class.
Non-cpt homg Lag in $\mathbb{C}H^n$

Furthermore, we determined the orbit equivalence class:

**Theorem (Hashinaga-K. 17)**

Let $S'$ be a connected Lie subgroup of $S \cong \mathbb{C}H^n$. If $S' \acts \mathbb{C}H^n$ admits a Lagrangian orbit, then the $S'$-action is orbit equivalent to either $L_0$ or $L_{\pi/2}$-action. Here,

- $L_0$-action yields a 1-parameter family of Lag orbit including all congruence classes in $\mathcal{C}(S)$ except $[L_{\pi/2} \cdot o]$ (exists unique totally geodesic orbit $L_0 \cdot o \cong \mathbb{R}H^n$).
- Every $L_{\pi/2}$-orbits is Lagrangian and congruent to each other (each orbit is contained in a horosphere).
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- Every $L_{\pi/2}$-orbits is Lagrangian and congruent to each other (each orbit is contained in a horosphere).

Note: The orbit space of Lagrangian orbits can be described by the moment map $\mu : \mathbb{C}H^n \rightarrow (s')^*$:

- (roughly speaking)
  $$\{\text{Lag } S'-\text{orbits}\} \ni \mu^{-1}(c) \iff c \in \mathfrak{z}((s')^*) = \{c \in (s')^* : \text{Ad}^*(g)c = c \ \forall g \in S'\}$$

- For example, if $S' = L_0$, then $\mathfrak{z}((s')^*) = \mathbb{R}A^*$. Thus, taking $\gamma(t) \in S \simeq \mathbb{C}H^n$ s.t. $\mu(\gamma(t)) = tA^*$, $\gamma(t)$ intersects to every Lag $L_0$-orbits.