REAL HYPERSURFACES WITH CONSTANT PRINCIPAL CURVATURES IN COMPLEX HYPERBOLIC SPACES

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ABSTRACT. We present the classification of all real hypersurfaces in complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures.

1. Introduction

The aim of submanifold geometry is to understand geometric invariants of submanifolds and to classify submanifolds according to given geometric data. In Riemannian geometry, the structure of a submanifold is encoded in the second fundamental form and its geometry is controlled by the equations of Gauß, Codazzi and Ricci. The situation simplifies for hypersurfaces, as the Ricci equation is trivial and the second fundamental form can be written in terms of a self-adjoint tensor field, the shape operator. The eigenvalues of the shape operator, the so-called principal curvatures, are the simplest geometric invariants of a hypersurface. Two basic problems in submanifold geometry are to understand the geometry of hypersurfaces for which the principal curvatures are constant, and to classify them. Élie Cartan [7] proved that in spaces of constant curvature a hypersurface has constant principal curvatures if and only if it is isoparametric. The classification of isoparametric hypersurfaces has a long history and over the years many surprising features have been discovered, see [10] for a survey.

Using the Gauß-Codazzi equations, Élie Cartan [7] also proved that the number g of distinct principal curvatures of an isoparametric hypersurface in the real hyperbolic space $\mathbb{R}H^n$ is either 1 or 2. This easily leads to a complete classification: geodesic hyperspheres, horospheres, totally geodesic hyperplanes and its equidistant hypersurfaces, tubes around totally geodesic subspaces of dimension ≥ 1. As a consequence, all hypersurfaces in real hyperbolic spaces with constant principal curvatures are open parts of homogeneous hypersurfaces.

In this paper we deal with the classification problem of real hypersurfaces with constant principal curvatures in complex hyperbolic spaces. We briefly describe the current state of the problem. Obviously, any homogeneous real hypersurface has constant principal curvatures. The first author and Tamaru [5] derived recently the complete classification of

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homogeneous real hypersurfaces in $\mathbb{C}H^n$. The number g of distinct principal curvatures of all these homogeneous real hypersurfaces is either 2,3,4 or 5. No examples are known of real hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ which are not an open part of a homogeneous real hypersurface. It is also not known whether for any real hypersurface with constant principal curvatures in $\mathbb{C}H^n$ the number g of distinct principal curvatures must necessarily be 2,3,4 or 5.

From the Codazzi equation one can easily deduce that $g > 1$ (see Corollary 2.3). It follows from work by Montiel [8] that every real hypersurface with two distinct constant principal curvatures in complex hyperbolic space $\mathbb{C}H^n$, $n \geq 3$, is an open part of a geodesic hypersphere, of a horosphere, of a tube around a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$, or of a tube with radius $\ln(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. For $n = 2$ this problem appears to be still open. In Corollary 2.4 we present a proof for this classification which includes this low-dimensional case as well. All these real hypersurfaces are homogeneous Hopf hypersurfaces. If ξ is a (local) unit normal field of a real hypersurface M in a Hermitian manifold $\bar{M}$, and J denotes the complex structure of $\bar{M}$, then the Hopf vector field $J\xi$ is tangent to M everywhere. The hypersurface M is said to be a Hopf hypersurface if the integral curves of $J\xi$ are geodesics in M. If $\bar{M}$ is a Kähler manifold this is equivalent to the condition that $J\xi$ is a principal curvature vector of M everywhere.

The first author obtained in [1] the classification of all Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$. Any such hypersurface is an open part of a horosphere, of a tube around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for some $k \in \{0, \ldots, n-1\}$, or to a tube around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. All these tubes and the horospheres are homogeneous hypersurfaces and satisfy $g \in \{2, 3\}$. But not all homogeneous real hypersurfaces in $\mathbb{C}H^n$ are necessarily Hopf hypersurfaces, see [2] for the construction of the following examples.

Let KAN be an Iwasawa decomposition of $SU(1, n)$, the connected component of the isometry group of $\mathbb{C}H^n$. The solvable Lie group AN acts simply transitively on $\mathbb{C}H^n$. The Riemannian metric on $\mathbb{C}H^n$ therefore induces in a natural way an inner product on the Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ of AN. The nilpotent Lie group N is isomorphic to the $(2n-1)$-dimensional Heisenberg group, and the orbits of the action of N on $\mathbb{C}H^n$ give a foliation by horospheres. The Lie algebra $\mathfrak{n}$ of N is a Heisenberg algebra and has a natural orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the one-dimensional center of $\mathfrak{n}$. Let $\mathfrak{w}$ be a linear hyperplane of $\mathfrak{v}$. Then $\mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ of codimension one. The corresponding connected Lie subgroup of AN therefore induces a foliation on $\mathbb{C}H^n$ by homogeneous hypersurfaces. None of these homogeneous hypersurfaces is a Hopf hypersurface. Exactly one of the orbits is minimal and has a simple geometric description. Consider a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^2 \subset \mathbb{C}H^n$ and pick a horocycle γ in $\mathbb{R}H^2$. At each point $p \in \gamma$ we attach the totally geodesic complex hyperbolic hyperplane which is tangent to the orthogonal complement of the complex span of the tangent line to γ at p. In this way we obtain a ruled real hypersurface W^{2n-1} in $\mathbb{C}H^n$. This hypersurface W^{2n-1} is congruent to the unique minimal orbit in the above foliation. As can be seen from the construction, the other homogeneous hypersurfaces in the foliation are geometrically the equidistant hypersurfaces to W^{2n-1}.

It was shown in [2] that each of these homogeneous hypersurfaces has three distinct constant principal curvatures. Saito claims in [9] that every real hypersurface with three distinct constant principal curvatures in $\mathbb{C}H^n$ is a Hopf hypersurface, and hence the assumption in [1] on the Hopf hypersurface would be redundant. The above examples show that this is not true.

The construction of W^{2n-1} can be generalized in the following way. Consider a totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$, $1 \leq k \leq n-1$, and fix a horosphere H in $\mathbb{R}H^{k+1}$. At each point $p \in H$ we attach the totally geodesic $\mathbb{C}H^{n-k}$ which is tangent to the orthogonal complement of the complex span of the tangent space to H at p. In this way we obtain a $(2n-k)$-dimensional ruled minimal submanifold W^{2n-k} in $\mathbb{C}H^n$ with totally real normal bundle of rank k. In terms of the above Iwasawa decomposition, denote by $o \in \mathbb{C}H^n$ the fixed point of the action of the compact group K on $\mathbb{C}H^n$. Then W^{2n-k} is holomorphically congruent to the orbit through o of the closed subgroup of AN with Lie algebra $\mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$, where $\mathfrak{w}$ is the orthogonal complement in $\mathfrak{v}$ of a real subspace of $\mathfrak{v}$. For $k = 1$ we just obtain the above ruled real hypersurface. For $k > 1$ the tubes around W^{2n-k} are homogeneous hypersurfaces (see [3]) and hence have constant principal curvatures. The number of distinct principal curvatures is four except for the radius $r = \ln(2 + \sqrt{3})$, where there are just three distinct principal curvatures.

In this paper we obtain the classification of all real hypersurfaces in $\mathbb{C}H^n$ with three distinct constant principal curvatures .

Theorem 1.1. *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures. Then M is holomorphically congruent to an open part of one of the following real hypersurfaces:*

(a) *the tube of radius $r > 0$ around the totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for some $k \in \{1, \dots, n-2\}$;*
(b) *the tube of radius $r > 0$, $r \neq \ln(2 + \sqrt{3})$, around the totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$;*
(c) *the ruled minimal real hypersurface $W^{2n-1} \subset \mathbb{C}H^n$, or to one of the equidistant hypersurfaces to W^{2n-1};*
(d) *the tube of radius $r = \ln(2 + \sqrt{3})$ around the ruled minimal submanifold $W^{2n-k} \subset \mathbb{C}H^n$ for some $k \in \{2, \dots, n-1\}$.*

For $n = 2$ the problem remains open. The hypersurfaces in (a) and (b) are Hopf hypersurfaces, the hypersurfaces in (c) and (d) are not Hopf hypersurfaces, and all of them are homogeneous. For the proof, we first derive some rigidity results of the ruled minimal submanifolds W^{2n-k} in terms of certain geometric data. In view of the known classification of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ (see [1]), we may assume that M is not a Hopf hypersurface. Using the Gauß-Codazzi equations and Jacobi field theory we then show that one of the focal sets or equidistant hypersurfaces of M has these geometric data.

We briefly describe the contents of this paper. In Section 2 we derive from the Gauß-Codazzi equations some basic formulae for real hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures, and settle the cases $g \leq 2$. The above mentioned rigidity results for the ruled minimal submanifolds are proved in Section 3. In Section 4 we determine the principal curvatures and some other geometric data for real hypersurfaces in $\mathbb{C}H^n$ with three distinct constant principal curvatures. Using Jacobi field theory we then proof the classification result in Section 5.

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2. Preliminaries

We denote by $\mathbb{C}H^n$ the n-dimensional complex hyperbolic space equipped with the Fubini Study metric $\langle \cdot , \cdot \rangle$ of constant holomorphic sectional curvature -1. We assume $n \geq 2$ and denote by $\bar{\nabla}$ and $\bar{R}$ the Levi Civita covariant derivative and the Riemannian curvature tensor of $\mathbb{C}H^n$, respectively, using the sign convention $\bar{R}_{XY} = [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}$. Then

$$\bar{R}_{XY}Z = -\frac{1}{4}\Big(\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY - 2\langle JX, Y\rangle JZ \Big),$$

where J is the complex structure of $\mathbb{C}H^n$. We also write $\bar{R}_{XYZW} = \langle \bar{R}_{XY}Z, W\rangle$.

Let M be a connected submanifold of $\mathbb{C}H^n$. We denote by ∇ and R the Levi Civita covariant derivative and the Riemannian curvature tensor of M, respectively. By TM and νM we denote the tangent bundle and the normal bundle of M, respectively. By $\Gamma(TM)$ and $\Gamma(\nu M)$ we denote the module of all vector fields tangent and normal to M, respectively. Let $X, Y, Z, W \in \Gamma(TM)$ and $\xi \in \Gamma(\nu M)$.

The Levi Civita covariant derivatives of M and $\mathbb{C}H^n$ are related by the Gauß formula

$$\bar{\nabla}_X Y = \nabla_X Y + I\!I(X, Y),$$

where $I\!I$ is the second fundamental form of M. The Weingarten formula is

$$\bar{\nabla}_X \xi = -S_\xi X + \nabla^\perp_X \xi,$$

where S_ξ denotes the shape operator of M with respect to ξ and $\nabla^\perp$ is the induced covariant derivative on νM. The second fundamental form and shape operator are related by $\langle S_\xi X, Y\rangle = \langle I\!I(X, Y), \xi\rangle$. If M is a real hypersurface and ξ is a unit normal vector field on M, we often write S instead of S_ξ. The fundamental equations of second order of interest to us are the Gauß equation

$$\bar{R}_{XYZW} = R_{XYZW} - \langle I\!I(Y, Z), I\!I(X, W)\rangle + \langle I\!I(X, Z), I\!I(Y, W)\rangle$$

and the Codazzi equation

$$\bar{R}_{XYZ\xi} = \langle (\nabla^\perp_X I\!I)(Y, Z) - (\nabla^\perp_Y I\!I)(X, Z), \xi\rangle,$$

where the covariant derivative of the second fundamental form is given by

$$(\nabla^\perp_X I\!I)(Y, Z) = \bar{\nabla}^\perp_X I\!I(Y, Z) - I\!I(\nabla_X Y, Z) - I\!I(Y, \nabla_X Z).$$

If M is a connected real hypersurface of $\mathbb{C}H^n$ and ξ is a global unit normal vector field on M, the equations simplify to

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + \langle SX, Y\rangle\xi,\\ \bar{\nabla}_X \xi &= -SX,\\ \bar{R}_{XYZW} &= R_{XYZW} - \langle SY, Z\rangle\langle SX, W\rangle + \langle SX, Z\rangle\langle SY, W\rangle,\\ \bar{R}_{XYZ\xi} &= \langle(\nabla_X S)Y - (\nabla_Y S)X, Z\rangle.\end{aligned}$$

We assume from now on that M is a connected real hypersurface of $\mathbb{C}H^n$ with constant principal curvatures. For each principal curvature λ of M we denote by T_λ the distribution on M formed by the principal curvature spaces of λ. By $\Gamma(T_\lambda)$ we denote the set of all sections in T_λ, that is, all vector fields on M satisfying $SX = \lambda X$.

The Codazzi equation readily implies

Lemma 2.1. *For all $X \in \Gamma(T_{\lambda_i})$, $Y \in \Gamma(T_{\lambda_j})$ and $Z \in \Gamma(T_{\lambda_k})$ we have*

$$\bar{R}_{XYZ\xi} = (\lambda_j - \lambda_k)\langle\nabla_X Y, Z\rangle - (\lambda_i - \lambda_k)\langle\nabla_Y X, Z\rangle.$$

Assume that $\lambda_i = \lambda_j = \lambda_k$ in the previous lemma. Then $\bar{R}_{XYZ\xi} = 0$ for all $X, Y, Z \in \Gamma(T_{\lambda_i})$. Choosing $Z = X$ we get $0 = \langle JX, Y\rangle\langle X, J\xi\rangle$ for all $X, Y \in \Gamma(T_{\lambda_i})$, which implies $0 = 4\langle X, J\xi\rangle\,\bar{R}_{XYZ\xi} = \langle JY, Z\rangle\,\langle X, J\xi\rangle^2$ for all $X, Y, Z \in \Gamma(T_{\lambda_i})$. Thus we have proved the following

Lemma 2.2. *If the orthogonal projection of $J\xi_p$ onto $T_{\lambda_i}(p)$ is nonzero at $p \in M$, then $T_{\lambda_i}(p)$ is a real subspace of $T_p\mathbb{C}H^n$, that is, $JT_{\lambda_i}(p) \subset T^\perp_{\lambda_i}(p)$, where $T^\perp_{\lambda_i}(p)$ is the orthogonal complement of $T_{\lambda_i}(p)$ in $T_p\mathbb{C}H^n$.*

This immediately implies

Corollary 2.3. *The number g of distinct principal curvatures of M satisfies $g > 1$.*

Corollary 2.4. (Montiel [8] for $n \geq 3$) *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 2$, with two distinct constant principal curvatures. Then M is holomorphically congruent to an open part of a horosphere in $\mathbb{C}H^n$, or of a geodesic hypersphere in $\mathbb{C}H^n$, or of a tube around a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$, or of the tube with radius $r = \ln(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$.*

Proof. We just need to prove that M is a Hopf hypersurface. The result then follows from the classification of real Hopf hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures (see [1]). Let λ_1, λ_2 be the two principal curvatures of M, and assume that there exists a point $p \in M$ such that $J\xi_p = \langle J\xi_p, u_1\rangle u_1 + \langle J\xi_p, u_2\rangle u_2$ with some unit vectors $u_i \in T_{\lambda_i}(p)$ and $0 \neq \langle J\xi_p, u_i\rangle$. According to Lemma 2.2 both $T_{\lambda_1}(p)$ and $T_{\lambda_2}(p)$ are real, which implies $JT_{\lambda_1}(p) \subset T_{\lambda_2}(p) \oplus \mathbb{R}\xi_p$ and $JT_{\lambda_2}(p) \subset T_{\lambda_1}(p) \oplus \mathbb{R}\xi_p$. Since $n \geq 2$ we can assume $\dim T_{\lambda_1}(p) \geq 2$. Then we have $J(T_{\lambda_1}(p) \ominus \mathbb{R}u_1) \subset T_{\lambda_2}(p)$, which implies $\dim T_{\lambda_2}(p) \geq \dim T_{\lambda_1}(p) - 1$. But $u_2 \notin J(T_{\lambda_1}(p) \ominus \mathbb{R}u_1)$ because of $\langle u_2, J\xi_p\rangle \neq 0$, and thus we have $\dim T_{\lambda_2}(p) \geq \dim T_{\lambda_1}(p)$. The previous equality implies $\dim T_{\lambda_2}(p) \geq 2$, and an analogous argument yields $\dim T_{\lambda_1}(p) \geq \dim T_{\lambda_2}(p)$. Therefore, $\dim T_{\lambda_1}(p) = \dim T_{\lambda_2}(p)$. This implies that $\dim T_pM = \dim T_{\lambda_1}(p) + \dim T_{\lambda_2}(p)$ is even, which contradicts $\dim M = 2n-1$. $\square$

Putting $\lambda_i = \lambda_k$ in Lemma 2.1 and then interchanging Y and Z yields

Lemma 2.5. *For all $X, Y \in \Gamma(T_{\lambda_i})$ and $Z \in \Gamma(T_{\lambda_j})$ with $\lambda_i \neq \lambda_j$ we have*

$$4(\lambda_j - \lambda_i)\langle \nabla_X Y, Z\rangle = \langle JY, Z\rangle\langle X, J\xi\rangle + \langle JX, Y\rangle\langle Z, J\xi\rangle + 2\langle JX, Z\rangle\langle Y, J\xi\rangle.$$

Corollary 2.6. *For all $X \in \Gamma(T_{\lambda_i})$ with $\langle X, J\xi\rangle = 0$ we have $\nabla_X X \in \Gamma(T_{\lambda_i})$.*

The following equation is a consequence of the Gauß and Codazzi equations and will be used later to obtain some relations among the principal curvatures.

Lemma 2.7. *For all unit vector fields $X \in \Gamma(T_{\lambda_i})$ and $Y \in \Gamma(T_{\lambda_j})$ with $\lambda_i \neq \lambda_j$ we have*

$$\begin{aligned} 0 \;=\;& (\lambda_j - \lambda_i)\big(1 - 4\lambda_i\,\lambda_j + 2\langle JX, Y\rangle^2 + 8\langle \nabla_X Y, \nabla_Y X\rangle - 4\langle \nabla_X X, \nabla_Y Y\rangle\big)\\ &+4\langle JX, Y\rangle\big(X\langle Y, J\xi\rangle + Y\langle X, J\xi\rangle\big)\\ &+\langle X, J\xi\rangle\big(3Y\langle JX, Y\rangle + \langle \nabla_Y X, JY\rangle - 2\langle \nabla_X Y, JY\rangle\big)\\ &+\langle Y, J\xi\rangle\big(3X\langle JX, Y\rangle - \langle \nabla_X Y, JX\rangle + 2\langle \nabla_Y X, JX\rangle\big).\end{aligned}$$

Proof. The Gauß equation implies

$$4R_{XYYX} = (4\lambda_i\lambda_j - 1) - 3\langle JX, Y\rangle^2.$$

On the other hand, the definition of R yields

$$\begin{aligned} R_{XYYX} &= \langle \nabla_X\nabla_Y Y - \nabla_Y\nabla_X Y - \nabla_{[X,Y]}Y, X\rangle\\ &= X\langle \nabla_Y Y, X\rangle - \langle \nabla_X X, \nabla_Y Y\rangle - Y\langle \nabla_X Y, X\rangle\\ &\quad +\langle \nabla_X Y, \nabla_Y X\rangle - \langle \nabla_{[X,Y]}Y, X\rangle.\end{aligned}$$

From Lemma 2.5 we get

$$\begin{aligned} 4(\lambda_j - \lambda_i)X\langle \nabla_Y Y, X\rangle &= 3\langle Y, J\xi\rangle X\langle JX, Y\rangle + 3\langle JX, Y\rangle X\langle Y, J\xi\rangle,\\ 4(\lambda_i - \lambda_j)Y\langle \nabla_X Y, X\rangle &= 3\langle X, J\xi\rangle Y\langle JX, Y\rangle + 3\langle JX, Y\rangle Y\langle X, J\xi\rangle.\end{aligned}$$

Next, using the Codazzi equation and the algebraic Bianchi identity, we get

$$\begin{aligned} &(\lambda_j - \lambda_i)\langle \nabla_{[X,Y]}Y, X\rangle\\ =\;& \langle \big(\nabla_{[X,Y]}S\big)\,Y, X\rangle\\ =\;& \langle (\nabla_Y S)\,[X,Y], X\rangle + \bar{R}_{[X,Y]YX\xi}\\ =\;& \langle (\nabla_Y S)\,X, \nabla_X Y\rangle - \langle (\nabla_Y S)\,X, \nabla_Y X\rangle + \bar{R}_{[X,Y]YX\xi}\\ =\;& \langle (\nabla_Y S)\,X, \nabla_X Y\rangle - \langle (\nabla_X S)\,Y, \nabla_Y X\rangle - \bar{R}_{YX\nabla_Y X\xi} + \bar{R}_{[X,Y]YX\xi}\\ =\;& (\lambda_i - \lambda_j)\langle \nabla_X Y, \nabla_Y X\rangle + \bar{R}_{\nabla_X YYX\xi} + \bar{R}_{X\nabla_Y XY\xi}\\ =\;& (\lambda_i - \lambda_j)\langle \nabla_X Y, \nabla_Y X\rangle\\ &-\frac{1}{4}\Big((\lambda_i - \lambda_j)\langle JX, Y\rangle^2 + \langle JX, Y\rangle\big(X\langle Y, J\xi\rangle + Y\langle X, J\xi\rangle\big)\\ &\qquad + \langle X, J\xi\rangle\big(\langle JY, \nabla_Y X\rangle - 2\langle JY, \nabla_X Y\rangle\big)\\ &\qquad - \langle Y, J\xi\rangle\big(\langle JX, \nabla_X Y\rangle - 2\langle JX, \nabla_Y X\rangle\big)\Big)\end{aligned}$$

Altogether this implies the lemma. □

3. The ruled minimal submanifolds W^{2n-k}

In this section we present a characterization of the ruled minimal submanifolds W^{2n-k}, $k \in \{1, \dots, n-1\}$. Let KAN be an Iwasawa decomposition of $SU(1,n)$ and $o \in \mathbb{C}H^n$ the fixed point of the action of K on $\mathbb{C}H^n$. Then AN acts simply transitively on $\mathbb{C}H^n$ and we can identify $\mathbb{C}H^n$ with the solvable Lie group AN equipped with a suitable left-invariant metric. This induces an inner product on the Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ of AN. There is a natural decomposition of the Lie algebra $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ of N, where $\mathfrak{z}$ is the one-dimensional center of $\mathfrak{n}$ and $\mathfrak{v}$ is the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. The Kähler structure on $\mathbb{C}H^n$ induces a complex structure i on the vector space $\mathfrak{v}$, so that $\mathfrak{v}$ becomes isomorphic to the complex vector space $\mathbb{C}^{n-1}$. Let $\mathfrak{w}$ be a linear subspace of $\mathfrak{v}$ such that the orthogonal complement $\mathfrak{w}^\perp$ of $\mathfrak{w}$ in $\mathfrak{v}$ is a real subspace of dimension k. Then $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and the orbit through o of the closed subgroup S of AN with Lie algebra $\mathfrak{s}$ is holomorphically congruent to the ruled minimal submanifold W^{2n-k}.

Let $\mathfrak{w}_{\mathbb{C}}$ be the maximal complex subspace of $\mathfrak{w}$, that is, the orthogonal complement in $\mathfrak{w}$ of $i\mathfrak{w}^\perp$. Then we have an orthogonal decomposition $\mathfrak{w} = \mathfrak{w}_{\mathbb{C}} \oplus i\mathfrak{w}^\perp$. The subspace $\mathfrak{a} \oplus \mathfrak{z} \oplus \mathfrak{w}_{\mathbb{C}}$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and the corresponding Lie subgroup of AN induces a foliation of W^{2n-k} by totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$. The subspace $\mathfrak{a} \oplus i\mathfrak{w}^\perp$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and the corresponding Lie subgroup of AN induces a foliation of W^{2n-k} by totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$. Moreover, the subspace $i\mathfrak{w}^\perp$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and the corresponding Lie subgroup of AN induces a foliation of W^{2n-k} by Euclidean spaces $\mathbb{R}^k$ which are embedded in the real hyperbolic spaces $\mathbb{R}H^{k+1}$ as horospheres in the usual way as totally umbilical submanifolds with parallel mean curvature vector.

The procedure for the computation of the Levi Civita connection for a left-invariant Riemannian metric on a Lie group is well-known and allows us to calculate the second fundamental form $I\!I$ of W^{2n-k} in an elementary way via the Gauß formula. Using for instance the expression for the Levi Civita connection of AN given in [6], p. 84, shows that $I\!I$ is determined by

$$\forall\, \xi \in \mathfrak{w}^\perp : 2I\!I(Z, i\xi) = \xi \ ,$$

where $Z \in \mathfrak{z}$ is a unit vector with a suitable orientation, and $Z, \xi, i\xi$ are viewed as left-invariant vector fields on AN. In other words, let $\xi \in \mathfrak{w}^\perp$ be a unit normal vector field of W^{2n-k}. Then the principal curvatures of W^{2n-k} with respect to ξ are $0, \frac{1}{2}, -\frac{1}{2}$ with multiplicities $2n-k-2, 1, 1$, respectively, and the principal curvature spaces with respect to $\pm\frac{1}{2}$ are spanned by $Z \pm i\xi$. This clearly shows that W^{2n-k} is a minimal submanifold of $\mathbb{C}H^n$.

We will now show that this second fundamental form characterizes W^{2n-k} among all $(2n-k)$-dimensional submanifolds of $\mathbb{C}H^n$ with totally real normal bundle.

Theorem 3.1. *Let M be a $(2n-k)$-dimensional connected submanifold in $\mathbb{C}H^n$, $n \geq 3$, with totally real normal bundle $\nu M \subset T\mathbb{C}H^n$. Assume that there exists a unit vector field Z tangent to the maximal holomorphic subbundle of $TM \subset T\mathbb{C}H^n$ such that the second fundamental form $I\!I$ of M is given by the trivial bilinear extension of $2I\!I(Z, J\xi) = \xi$ for* *all* $\xi \in \Gamma(\nu M)$. *Then* M *is holomorphically congruent to an open part of the ruled minimal submanifold* W^{2n-k}.

Proof. We will show the following:

(i) The maximal holomorphic subbundle $\mathfrak{D}$ of TM is integrable and each integral manifold is an open part of a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$;

(ii) The totally real subbundle $\mathbb{R}JZ \oplus J(\nu M)$ of TM is integrable and each integral manifold is an open part of a totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$;

(iii) The totally real subbundle $J(\nu M)$ of TM is integrable and each integral manifold is an open part of a horosphere in a totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$.

The rigidity of totally geodesic submanifolds of Riemannian manifolds (see e.g. [4], p. 230), and of horospheres in real hyperbolic spaces (see e.g. [4], pp. 24-26), then implies the assertion.

Ad (i): For $U, V \in \Gamma(\mathfrak{D})$ and $\xi \in \Gamma(\nu M)$ we have

$$\langle \nabla_U V, J\xi \rangle = \langle \bar{\nabla}_U V, J\xi \rangle = -\langle J\bar{\nabla}_U V, \xi \rangle = -\langle \bar{\nabla}_U JV, \xi \rangle = -\langle II(U, JV), \xi \rangle = 0$$

and

$$\langle \bar{\nabla}_U V, \xi \rangle = \langle II(U, V), \xi \rangle = 0.$$

This shows that $\mathfrak{D}$ is an autoparallel subbundle of TM and each integral manifold is a totally geodesic submanifold of $\mathbb{C}H^n$. As $\mathfrak{D}$ is a complex subbundle of complex rank $n-k$, each of these integral manifolds must be an open part of a totally geodesic $\mathbb{C}H^{n-k} \subset \mathbb{C}H^n$.

Ad (ii): Let $X \in \Gamma(\mathfrak{D} \ominus \mathbb{R}JZ)$ and $\zeta \in \Gamma(\nu M)$ be a local unit normal vector field of M. Using the Gauß formula, $\bar{\nabla} J = 0$, the Codazzi equation, the assumption on II, and the explicit expression for $\bar{R}$ we get

$$\begin{aligned} \langle \nabla_{JZ} JZ, X \rangle &= \langle \bar{\nabla}_{JZ} JZ, X \rangle = \langle \bar{\nabla}_{JZ} JX, Z \rangle = \langle \nabla_{JZ} JX, Z \rangle \\ &= 2\langle II(\nabla_{JZ} JX, J\zeta), \zeta \rangle \rangle = -2\langle (\nabla_{JZ} II)(JX, J\zeta), \zeta \rangle \\ &= -2\langle (\nabla_{J\zeta} II)(JZ, JX), \zeta \rangle - 2\bar{R}_{JZ\, J\zeta\, JX\, \zeta} = 0. \end{aligned}$$

For all $\xi, \eta \in \Gamma(\nu M)$ we get

$$\begin{aligned} \langle \nabla_{JZ} J\xi, X \rangle &= \langle \bar{\nabla}_{JZ} J\xi, X \rangle = \langle \bar{\nabla}_{JZ} JX, \xi \rangle = \langle II(JZ, JX), \xi \rangle = 0, \\ \langle \nabla_{J\xi} JZ, X \rangle &= \langle \bar{\nabla}_{J\xi} JZ, X \rangle = \langle \bar{\nabla}_{J\xi} JX, Z \rangle = \langle \nabla_{J\xi} JX, Z \rangle \\ &= 2\langle II(\nabla_{J\xi} JX, J\zeta), \zeta \rangle \rangle = -2\langle (\nabla_{J\xi} II)(JX, J\zeta), \zeta \rangle \\ &= -2\langle (\nabla_{JX} II)(J\xi, J\zeta), \zeta \rangle - 2\bar{R}_{J\xi\, JX\, J\zeta\, \zeta} = 0, \\ \langle \nabla_{J\xi} J\eta, X \rangle &= \langle \bar{\nabla}_{J\xi} J\eta, X \rangle = \langle \bar{\nabla}_{J\xi} JX, \eta \rangle = \langle II(J\xi, JX), \eta \rangle = 0. \end{aligned}$$

Finally, for all $U, V \in \Gamma(\mathbb{R}JZ \oplus J(\nu M))$ we obviously have

$$\langle \bar{\nabla}_U V, \zeta \rangle = \langle II(U, V), \zeta \rangle = 0.$$

Altogether this shows that $\mathbb{R}JZ \oplus J(\nu M)$ is integrable and each integral manifold is a totally geodesic submanifold of $\mathbb{C}H^n$. As $\mathbb{R}JZ \oplus J(\nu M)$ is a totally real subbundle of rank $k+1$, each of these totally geodesic submanifolds must be an open part of a totally geodesic $\mathbb{R}H^{k+1} \subset \mathbb{C}H^n$.

Ad (iii): For all $\xi, \eta \in \Gamma(\nu M)$ we get

$$\begin{aligned}\langle \nabla_{J\xi} J\eta, JZ\rangle &= \langle \bar{\nabla}_{J\xi} J\eta, JZ\rangle = \langle \bar{\nabla}_{J\xi} \eta, Z\rangle = -\langle \bar{\nabla}_{J\xi} Z, \eta\rangle \\ &= -\langle I\!I(Z, J\xi), \eta\rangle = -\frac{1}{2}\langle \xi, \eta\rangle = -\frac{1}{2}\langle J\xi, J\eta\rangle.\end{aligned}$$

It follows that $\langle [J\xi, J\eta], JZ\rangle = 0$ for all $\xi, \eta \in \Gamma(\nu M)$. Together with (ii) this implies that $J(\nu M)$ is integrable and the second fundamental form $\tilde{I\!I}$ of an integral manifold is given by

$$\tilde{I\!I}(J\xi, J\eta) = -\frac{1}{2}\langle J\xi, J\eta\rangle JZ.$$

Thus each integral manifold is a totally umbilical submanifold with constant mean curvature $1/2$ in a real hyperbolic space $\mathbb{R}H^{k+1}$ of constant sectional curvature $-1/4$. If $k \geq 2$, the classification of totally umbilical submanifolds in real hyperbolic spaces shows that each integral manifold is an open part of a horosphere in $\mathbb{R}H^{k+1}$. If $k = 1$, we have $2\bar{\nabla}_{J\xi} J\xi = 2J\bar{\nabla}_{J\xi}\xi = -2JSJ\xi = -JZ$ and hence $4\bar{\nabla}_{J\xi}\bar{\nabla}_{J\xi} J\xi = -2\bar{\nabla}_{J\xi} JZ = -J\xi$. Thus the integral curves of $J\xi$ satisfy the differential equation for a horocycle in $\mathbb{R}H^2$, which implies that the integral manifolds of the distribution $J(\nu M)$ are open parts of horocycles in $\mathbb{R}H^2$. $\square$

For $k = 1$ we have the following improvement:

Theorem 3.2. *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct principal curvatures 0, $+1/2$ and $-1/2$ and corresponding multiplicities $2n-3$, 1 and 1, respectively. Then M is holomorphically congruent to an open part of the ruled real hypersurface W^{2n-1}.*

Proof. Let $p \in M$ and suppose that the orthogonal projection of $J\xi_p$ onto $T_0(p)$ is nonzero. Then $T_0(p)$ is a real subspace of $T_p\mathbb{C}H^n$ by Lemma 2.2. Since $\dim T_0(p) = 2n-3$, this is impossible for $n > 3$ and we must have $n = 3$. Since $\xi_p \in T_0^\perp(p)$ it follows that $J\xi_p \in T_0(p)$. Since orthogonal projection onto subbundles is a continuous mapping, this must hold on an open neighborhood U of p in M. Therefore, U is a Hopf hypersurface in $\mathbb{C}H^3$ with three distinct constant principal curvatures 0, $+1/2$ and $-1/2$. According to the classification in [1] of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$ such a hypersurface does not exist. We conclude that the orthogonal projection of the Hopf vector field $J\xi$ onto T_0 is zero everywhere.

Now define M^+ as the set of all points $p \in M$ at which the orthogonal projections of $J\xi_p$ onto $T_{1/2}(p)$ and $T_{-1/2}(p)$ are both nonzero. Clearly, M^+ is an open subset of M. Using again the classification in [1] of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$, we see that M^+ is nonempty.

Let X and Y be local unit vector fields on M with $X \in \Gamma(T_{1/2})$ and $Y \in \Gamma(T_{-1/2})$. Then we can write $J\xi = aX + bY$ with $a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$. We may assume that X and Y are chosen such that $a, b \geq 0$. As we have seen above, $T_0(p)$ cannot be a real subspace at any point $p \in M$. Thus there exist vector fields $U, V \in \Gamma(T_0)$ with $\langle JU, V\rangle \neq 0$. Since

$\bar\nabla J = 0$ we have $\bar\nabla_U J\xi = J\bar\nabla_U\xi = -JSU = 0$, and thus Lemma 2.5 implies

$$0 = U\langle V, J\xi\rangle = \langle\nabla_U V, J\xi\rangle = a\langle\nabla_U V, X\rangle + b\langle\nabla_U V, Y\rangle = \frac{1}{2}(a^2 - b^2)\langle JU, V\rangle\ .$$

As $\langle JU, V\rangle \neq 0$ this gives $a^2 = b^2$ and hence $a = b = 1/\sqrt{2}$. This shows that M^+ is a closed subset of M. As M^+ is open and nonempty, we see that $M^+ = M$. In particular, the length of the orthogonal projections of the Hopf vector field $J\xi$ onto $T_{1/2}$ and $T_{-1/2}$ is constant and equal to $1/\sqrt{2}$. We now define $Z = a(X - Y)$. Then the second fundamental form of M is of the form as in Theorem 3.1, and the result now follows from that theorem. □

4. Principal curvatures

Let M be an orientable connected real hypersurface of $\mathbb{C}H^n$ and ξ a global unit normal vector field on M. We assume that M has three distinct constant principal curvatures λ_1, λ_2, λ_3 and denote by m_i the multiplicity of λ_i. If M is a Hopf hypersurface, it was shown in [1] that M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for some $k \in \{1, \ldots, n-2\}$, or of a tube with radius $r \neq \ln(2+\sqrt{3})$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. We can therefore assume that M is not a Hopf hypersurface. Then there exists an open subset of M on which at least two of the three orthogonal projections of the Hopf vector field $J\xi$ onto the principal curvature distributions T_{λ_i} are nontrivial.

In the first part of this section we will prove that there cannot be three nontrivial projections. We then derive some equations relating the principal curvatures and obtain some geometrical information about the principal curvature distributions.

Lemma 4.1. *Assume that there exists a point $p \in M$ such that $J\xi_p = \sum b_i u_i$ with some unit vectors $u_i \in T_{\lambda_i}(p)$ and $0 \neq b_i \in \mathbb{R}$, $i = 1, 2, 3$. Then $\mathbb{R}\xi_p \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_3$ is a complex subspace of $T_p\mathbb{C}H^n$ and, by a suitable orientation of u_1, u_2, u_3, we have $b_i = \langle Ju_j, u_k\rangle$ for all cyclic permutations (i, j, k) of $(1, 2, 3)$.*

Proof. According to Lemma 2.2, each of the three principal curvature spaces $T_{\lambda_i}(p)$ is a real subspace of $T_p\mathbb{C}H^n$. Thus we can write

$$Ju_i = \sum_{j=1}^{3}\langle Ju_i, u_j\rangle u_j + \sum_{j=1}^{3} w_{ij} - b_i\xi_p, \tag{1}$$

with some vectors $w_{ij} \in T_{\lambda_j}(p) \ominus \mathbb{R}u_j$, $w_{ii} = 0$. Then we have

$$0 = \langle u_i, \xi_p\rangle = \langle Ju_i, J\xi_p\rangle = \sum_{j=1}^{3} b_j\langle Ju_i, u_j\rangle, \tag{2}$$

and hence

$$-\xi_p = J^2\xi_p = J(J\xi_p) = \sum_{i=1}^{3} b_i Ju_i = \sum_{j=1}^{3}\left(\sum_{i=1}^{3} b_i w_{ij}\right) - \xi_p.$$

This implies $\sum_{i=1}^{3} b_i w_{ij} = 0$ for all $j \in \{1, 2, 3\}$. Thus for each $j \in \{1, 2, 3\}$ the two vectors w_{ij} with $i \neq j$ are either both zero, or both nonzero and collinear. From (1) and

$J\xi_p = \sum b_i u_i$, we therefore see that $\mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_3 \oplus \mathbb{R}w_{12} \oplus \mathbb{R}w_{23} \oplus \mathbb{R}w_{31} \oplus \mathbb{R}\xi_p$ is a complex subspace of $T_p\mathbb{C}H^n$. As the real dimension of a complex vector space is even, at least one of the three vectors w_{12}, w_{23}, w_{31} must be zero, say $w_{23} = 0$, which implies also $w_{13} = 0$. Moreover, for dimension reasons, the vectors w_{12}, w_{31} are either both zero or both nonzero. Then, using (1), we get

$$\begin{aligned} 0 &= \langle u_1, w_{12}\rangle = \langle Ju_1, Jw_{12}\rangle = -\langle Ju_1, u_3\rangle\langle w_{32}, w_{12}\rangle, \\ 0 &= \langle u_2, w_{21}\rangle = \langle Ju_2, Jw_{21}\rangle = -\langle Ju_2, u_3\rangle\langle w_{31}, w_{21}\rangle. \end{aligned}$$

If w_{12}, w_{31} are both nonzero, then w_{32}, w_{21} are nonzero as well, and we get $\langle Ju_1, u_3\rangle = 0 = \langle Ju_2, u_3\rangle$ using the collinearity of w_{12}, w_{32} and w_{31}, w_{21}. From (2) we then get $\langle Ju_1, u_2\rangle = 0$ as well. This implies $Ju_1 = w_{12} - b_1\xi_p$ and hence $b_1 J\xi_p = u_1 + Jw_{12}$. As $T_{\lambda_2}(p)$ is a real subspace of $T_p\mathbb{C}H^n$, the previous equation shows that $J\xi_p \in T_{\lambda_1}(p) \oplus T_{\lambda_3}(p)$, which contradicts the assumption on $J\xi_p$. Hence w_{12}, w_{31} are both zero. Altogether this shows that $\mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_3 \oplus \mathbb{R}\xi_p$ is a complex subspace of $T_p\mathbb{C}H^n$.

Finally, solving the system of equations (2), we see that the vector (b_1, b_2, b_3) is in the real span of $(\langle Ju_2, u_3\rangle, \langle Ju_3, u_1\rangle, \langle Ju_1, u_2\rangle)$. From $b_1^2 + b_2^2 + b_3^2 = 1$ and (1) we get

$$3 = \sum_{i=1}^{3} \langle u_i, u_i\rangle^2 = \sum_{i=1}^{3} \langle Ju_i, Ju_i\rangle^2 = 2\Big(\langle Ju_2, u_3\rangle^2 + \langle Ju_3, u_1\rangle^2 + \langle Ju_1, u_2\rangle^2\Big) + 1.$$

Thus $(\langle Ju_2, u_3\rangle, \langle Ju_3, u_1\rangle, \langle Ju_1, u_2\rangle)$ is a unit vector in $\mathbb{R}^3$, and the lemma now follows. $\square$

Lemma 4.2. *Assume that there exists a point $p \in M$ such that the orthogonal projections of $J\xi_p$ onto $T_{\lambda_i}(p)$, $i = 1, 2, 3$, are nontrivial. Then we have*

$$(2\lambda_i(\lambda_i - \lambda_j) - 1)\langle Jw_i, w_j\rangle = 0$$

for all distinct $i, j \in \{1, 2, 3\}$, $w_i \in T_{\lambda_i}(p) \ominus \mathbb{R}u_i$ and $w_j \in T_{\lambda_j}(p) \ominus \mathbb{R}u_j$.

Proof. By continuity, the orthogonal projections of $J\xi$ onto T_{λ_i}, $i = 1, 2, 3$, must be nontrivial on an open neighborhood of p in M. The following calculations hold on this open neighborhood. It follows from Lemma 4.1 that there exist unit vector fields $U_\nu \in \Gamma(T_{\lambda_\nu})$ such that $J\xi = \sum b_\nu U_\nu$ with $b_\nu = \langle JU_\mu, U_\rho\rangle$, where (ν, μ, ρ) is a cyclic permutation of $(1, 2, 3)$. We note that $\mathfrak{D} = TM \ominus (\mathbb{R}U_1 \oplus \mathbb{R}U_2 \oplus \mathbb{R}U_3)$ is a J-invariant subbundle of TM by Lemma 4.1.

In the following we denote by W_ν and $\widetilde{W}_\nu$ vector fields with values in $T_{\lambda_\nu} \cap \mathfrak{D} = T_{\lambda_\nu} \ominus \mathbb{R}U_\nu$, $\nu = 1, 2, 3$. Using $\bar{\nabla} J = 0$ and the Weingarten formula we get $\bar{\nabla}_{W_i} J\xi = J\bar{\nabla}_{W_i}\xi = -JSW_i = -\lambda_i JW_i$. Since T_{λ_i} is real, and using Lemma 2.5, we get

$$0 = W_i\langle \widetilde{W}_i, J\xi\rangle = \sum_{\nu=1}^{3} b_\nu \langle \bar{\nabla}_{W_i}\widetilde{W}_i, U_\nu\rangle + \langle \widetilde{W}_i, \bar{\nabla}_{W_i} J\xi\rangle = b_i\langle \nabla_{W_i}\widetilde{W}_i, U_i\rangle.$$

Hence, $\langle \nabla_{W_i}\widetilde{W}_i, U_i\rangle = 0$. As T_{λ_j} is real and $\mathfrak{D}$ is complex, we can write $JW_j = \widetilde{W}_i + \widetilde{W}_k$ with $k \neq i, j$. Then, using $\langle \nabla_{W_i}\widetilde{W}_i, U_i\rangle = 0$, Lemma 2.5, and the fact that T_{λ_i} is real, we get

$$\langle \bar{\nabla}_{W_i} JU_i, W_j\rangle = -\langle \nabla_{W_i} U_i, JW_j\rangle = \langle \nabla_{W_i}\widetilde{W}_i, U_i\rangle - \langle \nabla_{W_i} U_i, \widetilde{W}_k\rangle = 0. \tag{3}$$

Next, Lemma 2.5 implies

$$\begin{aligned} 0 &= W_i\langle W_j, J\xi\rangle = \sum_\nu b_\nu\langle\bar\nabla_{W_i}W_j, U_\nu\rangle + \langle W_j, \bar\nabla_{W_i}J\xi\rangle \\ &= \frac{b_i^2}{2(\lambda_i-\lambda_j)}\langle JW_i, W_j\rangle + \sum_{\nu\neq i} b_\nu\langle\nabla_{W_i}W_j, U_\nu\rangle - \lambda_i\langle JW_i, W_j\rangle \\ &= \frac{b_i^2-2\lambda_i(\lambda_i-\lambda_j)}{2(\lambda_i-\lambda_j)}\langle JW_i, W_j\rangle + b_j\langle\nabla_{W_i}W_j, U_j\rangle + b_k\langle\nabla_{W_i}W_j, U_k\rangle. \end{aligned}$$

On the other hand, replacing JU_i by $\sum_\nu\langle JU_i, U_\nu\rangle U_\nu - b_i\xi$ and using (3) we get $0 = W_i\langle W_j, JU_i\rangle = \sum_\nu\langle JU_i, U_\nu\rangle\langle\nabla_{W_i}W_j, U_\nu\rangle$ and hence

$$0 = b_k\langle\nabla_{W_i}W_j, U_j\rangle - b_j\langle\nabla_{W_i}W_j, U_k\rangle.$$

The last two equations provide a system of linear equations with unknowns $\langle\nabla_{W_i}W_j, U_j\rangle$ and $\langle\nabla_{W_i}W_j, U_k\rangle$. This linear system has a unique solution which is given by

$$\langle\nabla_{W_i}W_j, U_\nu\rangle = \frac{b_\nu(2\lambda_i(\lambda_i-\lambda_j)-b_i^2)}{2(\lambda_i-\lambda_j)(1-b_i^2)}\langle JW_i, W_j\rangle \quad (\nu\neq i). \tag{4}$$

As T_{λ_i} is real, we have $\langle JW_i, \widetilde{W}_k\rangle = \langle JW_i, JW_j\rangle = 0$, and using Lemma 2.5 and equation (4) (with j and k interchanged) we get

$$\langle\bar\nabla_{W_i}W_j, JU_k\rangle = \langle\nabla_{W_i}U_k, JW_j\rangle = -\langle\nabla_{W_i}\widetilde{W}_i, U_k\rangle - \langle\nabla_{W_i}\widetilde{W}_k, U_k\rangle = 0.$$

Replacing JU_k by $\sum_\nu\langle JU_k, U_\nu\rangle U_\nu - b_k\xi$, this implies

$$0 = \langle\bar\nabla_{W_i}W_j, JU_k\rangle = \sum_\nu\langle\nabla_{W_i}W_j, U_\nu\rangle\langle JU_k, U_\nu\rangle,$$

from which we easily get

$$\langle\nabla_{W_i}W_j, U_j\rangle = \frac{b_j}{2(\lambda_i-\lambda_j)}\langle JW_i, W_j\rangle$$

by using Lemma 2.5 once again. By comparison of this equation with equation (4) for $\nu = j$ we eventually get the result. □

Proposition 4.3. *If $n \geq 3$, then there exists no point $p \in M$ such that the orthogonal projections of $J\xi_p$ onto $T_{\lambda_i}(p)$, $i = 1, 2, 3$, are nontrivial.*

Proof. As $n \geq 3$, the complex vector space $\mathfrak{D}_p = \bigoplus_i(T_{\lambda_i}(p)\ominus\mathbb{R}u_i)$ has dimension ≥ 1. Since each $T_{\lambda_i}(p)\ominus\mathbb{R}u_i$ is real, there exist $i \neq j$ such that $\langle Jw_i, w_j\rangle \neq 0$ for some $w_i \in T_{\lambda_i}(p)\ominus\mathbb{R}u_i$, $w_j \in T_{\lambda_j}(p)\ominus\mathbb{R}u_j$. From Lemma 4.2 we therefore get $2\lambda_i(\lambda_i-\lambda_j)-1 = 2\lambda_j(\lambda_j-\lambda_i)-1 = 0$, and thus $\lambda_i^2 = \lambda_j^2 = 1/4$. This argument shows that $T_{\lambda_k}(p)\ominus\mathbb{R}u_k$ must be trivial, that is, the third eigenvalue λ_k has multiplicity one. Since the eigenspaces are real it also implies that $J(T_{\lambda_i}(p)\ominus\mathbb{R}u_i) = T_{\lambda_j}(p)\ominus\mathbb{R}u_j$.

Let $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$ be a unit vector field which is defined in an open neighborhood of p in M, and define $W_j = JW_i \in \Gamma(T_{\lambda_j} \ominus \mathbb{R}U_j)$, where U_ν is as in the previous proof. Applying Lemma 2.7 to W_i and W_j, and using Corollary 2.6, we obtain

$$3 - 4\lambda_i\lambda_j + 8\langle \nabla_{W_i}W_j, \nabla_{W_j}W_i\rangle = 0.$$

We have $\nabla_{W_i}W_j \in \Gamma(T_{\lambda_j} \oplus \mathbb{R}U_i \oplus \mathbb{R}U_k)$ and $\nabla_{W_j}W_i \in \Gamma(T_{\lambda_i} \oplus \mathbb{R}U_j \oplus \mathbb{R}U_k)$ by Lemma 2.5, and therefore $\langle \nabla_{W_i}W_j, \nabla_{W_j}W_i\rangle = \sum_\nu \langle \nabla_{W_i}W_j, U_\nu\rangle\langle \nabla_{W_j}W_i, U_\nu\rangle$. The latter sum can be calculated easily by using Lemma 2.5 and equation (4). Using the fact that $4\lambda_i^2 = 4\lambda_j^2 = 1$ this gives $4\langle \nabla_{W_i}W_j, \nabla_{W_j}W_i\rangle = 1$. Inserting this into the above equation yields $5 - 4\lambda_i\lambda_j = 0$. From $4\lambda_i^2 = 4\lambda_j^2 = 1$ and $\lambda_i \neq \lambda_j$ we know that $4\lambda_i\lambda_j = -1$, which gives a contradiction. Therefore there exists no point $p \in M$ such that the orthogonal projections of $J\xi_p$ onto $T_{\lambda_i}(p)$ are nontrivial. $\square$

Lemma 4.4. *Assume that there exists a point $p \in M$ such that $J\xi_p = b_1u_1 + b_2u_2$ with some unit vectors $u_i \in T_{\lambda_i}(p)$ and $0 \neq b_i \in \mathbb{R}$, $i = 1, 2$. Then there exists a unit vector $a \in T_{\lambda_3}(p)$ such that $\mathbb{R}\xi_p \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}a$ is a complex subspace of $T_p\mathbb{C}H^n$ and, by a suitable orientation of a, we have $b_1 = \langle Ju_2, a\rangle$, $b_2 = -\langle Ju_1, a\rangle$, $\langle Ju_1, u_2\rangle = 0$ and $Ju_i = (-1)^i b_j a - b_i\xi$ with distinct $i, j \in \{1, 2\}$.*

Proof. The eigenspaces $T_{\lambda_1}(p)$ and $T_{\lambda_2}(p)$ are real subspaces of $T_p\mathbb{C}H^n$ by Lemma 2.2. Therefore we can write

$$\begin{aligned} Ju_1 &= \langle Ju_1, u_2\rangle u_2 + w_{12} + w_{13} - b_1\xi, \\ Ju_2 &= \langle Ju_2, u_1\rangle u_1 + w_{21} + w_{23} - b_2\xi, \end{aligned}$$

with $w_{21} \in T_{\lambda_1}(p) \ominus \mathbb{R}u_1$, $w_{12} \in T_{\lambda_2}(p) \ominus \mathbb{R}u_2$, and $w_{13}, w_{23} \in T_{\lambda_3}(p)$. Hence,

$$\begin{aligned} -\xi_p &= J^2\xi_p = J(J\xi_p) = b_1Ju_1 + b_2Ju_2 \\ &= b_2\langle Ju_2, u_1\rangle u_1 + b_1\langle Ju_1, u_2\rangle u_2 + b_2w_{21} + b_1w_{12} + (b_1w_{13} + b_2w_{23}) - \xi_p. \end{aligned}$$

This shows that $\langle Ju_1, u_2\rangle = 0$, $w_{12} = w_{21} = 0$ and $b_1w_{13} + b_2w_{23} = 0$. As $b_1, b_2 \neq 0$, the vectors w_{13}, w_{23} are either both zero or both nonzero. If $w_{13} = w_{23} = 0$, then $Ju_1 = -b_1\xi_p$ and $Ju_2 = -b_2\xi_p$, which is impossible. Hence w_{13}, w_{23} are both nonzero and collinear. Let a be a unit vector in $\mathbb{R}w_{13} = \mathbb{R}w_{23} \subset T_{\lambda_3}(p)$. Since $Jw_{13} = b_1J\xi_p - u_1 \in \mathbb{R}u_1 \oplus \mathbb{R}u_2$ we get $Ja \in \mathbb{R}u_1 \oplus \mathbb{R}u_2$, which shows that $\mathbb{R}\xi_p \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}a$ is a complex subspace of $T_p\mathbb{C}H^n$. The two vectors $Ja, J\xi_p \in \mathbb{R}u_1 \oplus \mathbb{R}u_2$ are orthonormal and $J\xi_p = b_1u_1 + b_2u_2$. Therefore, by a suitable orientation of a, we can write $Ja = b_2u_1 - b_1u_2$. As $Ja = \langle Ja, u_1\rangle u_1 + \langle Ja, u_2\rangle u_2$, the result now follows. $\square$

In view of Proposition 4.3 we can assume from now on that there exists an open subset of M on which the orthogonal projection of $J\xi$ onto T_{λ_3} is trivial. The following calculations are done on this open subset. It follows from Lemma 4.4 that there exist unit vector fields $U_1 \in \Gamma(T_{\lambda_1})$, $U_2 \in \Gamma(T_{\lambda_2})$ and $A \in \Gamma(T_{\lambda_3})$ such that

$$\begin{gathered} J\xi = b_1U_1 + b_2U_2\ ,\ JU_i = (-1)^i b_jA - b_i\xi\ (i \neq j)\ , \\ b_1 = \langle JU_2, A\rangle\ ,\ b_2 = \langle JA, U_1\rangle\ ,\ \langle JU_1, U_2\rangle = 0. \end{gathered}$$

Below we will use these relations frequently without referring to them explicitly. Moreover,

$$\mathfrak{D} = TM \ominus (\mathbb{R}U_1 \oplus \mathbb{R}U_2 \oplus \mathbb{R}A) = (T_{\lambda_1} \ominus \mathbb{R}U_1) \oplus (T_{\lambda_2} \ominus \mathbb{R}U_2) \oplus (T_{\lambda_3} \ominus \mathbb{R}A)$$

is a J-invariant subbundle.

Lemma 4.5. *For $i, j \in \{1, 2\}$ with $i \neq j$ we have*

$$\begin{aligned}
\nabla_{U_i} U_i &= (-1)^i \frac{3b_1 b_2}{4(\lambda_3 - \lambda_i)} A, && (5)\\
\nabla_{U_i} U_j &= (-1)^j \left(\lambda_i + \frac{3b_i^2}{4(\lambda_3 - \lambda_i)}\right) A, && (6)\\
\nabla_{U_i} A &= (-1)^j \frac{3b_1 b_2}{4(\lambda_3 - \lambda_i)} U_i + (-1)^i \left(\lambda_i + \frac{3b_i^2}{4(\lambda_3 - \lambda_i)}\right) U_j, && (7)\\
\nabla_A U_i &= \frac{(-1)^j}{\lambda_i - \lambda_j} \left(\frac{b_i^2 - 2b_j^2}{4} + (\lambda_j - \lambda_3)\Big(\lambda_i + \frac{3b_i^2}{4(\lambda_3 - \lambda_i)}\Big)\right) U_j, && (8)\\
\nabla_A A &= 0. && (9)
\end{aligned}$$

Proof. Let $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$, $W_j \in \Gamma(T_{\lambda_j} \ominus \mathbb{R}U_j)$ and $W_3 \in \Gamma(T_{\lambda_3} \ominus \mathbb{R}A)$. Since U_i has constant length, we have $\langle \nabla_{U_i} U_i, U_i \rangle = 0$. From Lemma 2.5 we easily get

$$\langle \nabla_{U_i} U_i, U_j \rangle = \langle \nabla_{U_i} U_i, W_j \rangle = \langle \nabla_{U_i} U_i, W_3 \rangle = 0\ ,\ \ \langle \nabla_{U_i} U_i, A \rangle = (-1)^i \frac{3b_i b_j}{4(\lambda_3 - \lambda_i)} A.$$

As T_{λ_i} is real, we have $\langle W_i, \bar\nabla_{U_i} J\xi \rangle = \langle W_i, J\bar\nabla_{U_i} \xi \rangle = -\lambda_i \langle W_i, JU_i \rangle = 0$, and using Lemma 2.5 once again we then get

$$\begin{aligned}
0 &= U_i \langle W_i, J\xi \rangle \ = \ \langle \bar\nabla_{U_i} W_i, J\xi \rangle + \langle W_i, \bar\nabla_{U_i} J\xi \rangle \\
&= b_i \langle \nabla_{U_i} W_i, U_i \rangle + b_j \langle \nabla_{U_i} W_i, U_j \rangle \ = \ -b_i \langle \nabla_{U_i} U_i, W_i \rangle.
\end{aligned}$$

Since $b_i \neq 0$, this implies $\langle \nabla_{U_i} W_i, U_i \rangle = 0$, and equation (5) now follows.

Since U_j has constant length, we have $\langle \nabla_{U_i} U_j, U_j \rangle = 0$, from (5) we get $\langle \nabla_{U_i} U_j, U_i \rangle = 0$, and Lemma 2.5 implies $\langle \nabla_{U_i} U_j, W_i \rangle = -\langle \nabla_{U_i} W_i, U_j \rangle = 0$. Let $\nu \in \{j, 3\}$. Using (5) and $\langle W_\nu, \bar\nabla_{U_i} J\xi \rangle = \langle W_\nu, J\bar\nabla_{U_i} \xi \rangle = -\lambda_i \langle W_\nu, JU_i \rangle = 0$ we get

$$0 = U_i \langle W_\nu, J\xi \rangle = \langle \bar\nabla_{U_i} W_\nu, J\xi \rangle + \langle W_\nu, \bar\nabla_{U_i} J\xi \rangle = b_j \langle \nabla_{U_i} W_\nu, U_j \rangle,$$

which gives $\langle \nabla_{U_i} U_j, W_\nu \rangle = 0$. Finally, $0 = U_i \langle JU_i, U_j \rangle = \langle \bar\nabla_{U_i} JU_i, U_j \rangle + \langle JU_i, \bar\nabla_{U_i} U_j \rangle = -\langle \bar\nabla_{U_i} U_i, JU_j \rangle + \langle JU_i, \bar\nabla_{U_i} U_j \rangle$. Replacing now JU_i and JU_j by the corresponding expressions in terms of A and ξ we obtain

$$0 = b_j \left(\frac{3b_i^2}{4(\lambda_3 - \lambda_i)} + \lambda_i + (-1)^i \langle \nabla_{U_i} U_j, A \rangle\right).$$

Altogether this now implies equation (6).

Since A has constant length, we have $\langle \nabla_{U_i} A, A \rangle = 0$. For $\nu \in \{1, 2, 3\}$ we get $0 = U_i \langle JU_i, W_\nu \rangle = \langle \bar\nabla_{U_i} JU_i, W_\nu \rangle + \langle JU_i, \bar\nabla_{U_i} W_\nu \rangle = -\langle \bar\nabla_{U_i} U_i, JW_\nu \rangle + \langle JU_i, \bar\nabla_{U_i} W_\nu \rangle$. The first term vanishes because of equation (5). For the second term we replace JU_i by $(-1)^i b_j A -$

$b_i\xi$, which leads to $0 = \langle \nabla_{U_i} A, W_\nu \rangle$. It follows that $\nabla_{U_i} A = \langle \nabla_{U_i} A, U_i \rangle U_i + \langle \nabla_{U_i} A, U_j \rangle U_j$, which allows to determine equation (7) from equations (5) and (6).

Since U_i has constant length, we have $\langle \nabla_A U_i, U_i \rangle = 0$, and from Lemma 2.5 we get $\langle \nabla_A U_i, A \rangle = -\langle \nabla_A A, U_i \rangle = 0$ and $\langle \nabla_A U_i, W_3 \rangle = -\langle \nabla_A W_3, U_i \rangle = 0$. Using Lemma 2.1 and (7) we obtain $0 = \bar{R}_{AU_iW_j\xi} = (\lambda_i - \lambda_j)\langle \nabla_A U_i, W_j \rangle$ and hence $\langle \nabla_A U_i, W_j \rangle = 0$. Using this equality (with i and j interchanged) we get $0 = A\langle W_i, J\xi \rangle = \langle \bar{\nabla}_A W_i, J\xi \rangle + \langle W_i, \bar{\nabla}_A J\xi \rangle = b_i \langle \nabla_A W_i, U_i \rangle$, which yields $\langle \nabla_A U_i, W_i \rangle = 0$. Thus we have $\nabla_A U_i = \langle \nabla_A U_i, U_j \rangle U_j$. The latter inner product can be calculated by using the explicit expression for $\bar{R}$, Lemma 2.1 and (7) from

$$\frac{(-1)^j}{4}(b_i^2 - 2b_j^2) = \bar{R}_{AU_iU_j\xi} = (\lambda_i - \lambda_j)\langle \nabla_A U_i, U_j \rangle - (\lambda_3 - \lambda_j)(-1)^i \left(\lambda_i + \frac{3b_i^2}{4(\lambda_3 - \lambda_i)} \right).$$

Altogether this now gives equation (8).

Since A has constant length, we have $\langle \nabla_A A, A \rangle = 0$. Let $\nu \in \{1,2\}$. From (8) we get $\langle \nabla_A A, U_\nu \rangle = 0$, and from Lemma 2.5 we get $\langle \nabla_A A, W_\nu \rangle = 0$. Next, we consider $0 = A\langle JU_i, W_3 \rangle = \langle \bar{\nabla}_A JU_i, W_3 \rangle + \langle JU_i, \bar{\nabla}_A W_3 \rangle = -\langle \nabla_A U_i, JW_3 \rangle + \langle JU_i, \bar{\nabla}_A W_3 \rangle$. The first term vanishes because of (8), and in the second term we replace JU_i by its expression in terms of A and ξ to obtain $0 = \langle \nabla_A A, W_3 \rangle$. This eventually implies equation (9). □

Corollary 4.6. *The integral curves of A are geodesics in M and the three vector fields A, U_1, U_2 span an autoparallel distribution $\mathfrak{D}^\perp$, that is, $\mathfrak{D}^\perp$ is integrable and its leaves are totally geodesic submanifolds of M.*

Corollary 4.7. *The principal curvatures $\lambda_1, \lambda_2, \lambda_3$ and the functions b_1, b_2 satisfy the equation*

$$0 = 3\big((\lambda_3 - \lambda_2)^2 b_1^2 + (\lambda_3 - \lambda_1)^2 b_2^2\big) + (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\big(1 + 4\lambda_2(\lambda_3 - \lambda_1) + 4\lambda_1(\lambda_3 - \lambda_2)\big).$$

Proof. From Lemma 2.1 we get $1 = 4\bar{R}_{U_1U_2A\xi} = 4(\lambda_2 - \lambda_3)\langle \nabla_{U_1} U_2, A \rangle - 4(\lambda_1 - \lambda_3)\langle \nabla_{U_2} U_1, A \rangle$. The assertion then follows by using equation (6). □

Lemma 4.8. *If $i \in \{1,2\}$ and $m_i > 1$, then $4\lambda_3\lambda_i = 1$.*

Proof. Let $W_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$ be a local unit vector field. Applying Lemma 2.7 with $X = W_i$ and $Y = A$, and taking into account (9), yields

$$0 = 1 - 4\lambda_3\lambda_i + 8\langle \nabla_{W_i} A, \nabla_A W_i \rangle.$$

We thus need to prove $\langle \nabla_{W_i} A, \nabla_A W_i \rangle = 0$. From (8) and (9) we see that $\nabla_A W_i \in \Gamma(\mathfrak{D})$, and Lemma 2.5 shows that $\nabla_A W_i$ is perpendicular to $T_{\lambda_3} \cap \mathfrak{D}$. From Lemma 2.5 we also see that $\nabla_{W_i} A$ is perpendicular to $T_{\lambda_i} \cap \mathfrak{D}$. It thus suffices to prove that $\langle \nabla_{W_i} A, W_j \rangle = 0$ for all $W_j \in \Gamma(T_{\lambda_j} \cap \mathfrak{D})$, where $j \in \{1,2\}$ with $j \neq i$.

Let $\nu, \mu \in \{1,2\}$ with $\nu \neq \mu$. Then $0 = W_i\langle U_\nu, JW_j \rangle = \langle \bar{\nabla}_{W_i} U_\nu, JW_j \rangle + \langle U_\nu, \bar{\nabla}_{W_i} JW_j \rangle = \langle \nabla_{W_i} U_\nu, JW_j \rangle - \langle JU_\nu, \bar{\nabla}_{W_i} W_j \rangle$. As $JU_\nu = (-1)^\nu b_\mu A - b_\nu \xi$, this implies

$$\langle \nabla_{W_i} W_j, A \rangle = \frac{(-1)^\nu}{b_\mu} \langle \nabla_{W_i} U_\nu, JW_j \rangle. \tag{10}$$

As T_{λ_j} is real, we can write $JW_j = \widetilde{W}_i + \widetilde{W}_3$ with $\widetilde{W}_i \in \Gamma(T_{\lambda_i} \ominus \mathbb{R}U_i)$ and $\widetilde{W}_3 \in \Gamma(T_{\lambda_3} \ominus \mathbb{R}A)$. We have $\langle \nabla_{W_i} U_j, \widetilde{W}_i \rangle = 0$ and $0 = W_i \langle \widetilde{W}_i, J\xi \rangle = \langle \bar{\nabla}_{W_i} \widetilde{W}_i, J\xi \rangle + \langle \widetilde{W}_i, \bar{\nabla}_{W_i} J\xi \rangle = b_i \langle \nabla_{W_i} \widetilde{W}_i, U_i \rangle$ by Lemma 2.5, which implies $\langle \nabla_{W_i} U_\nu, \widetilde{W}_i \rangle = 0$ and hence $\langle \nabla_{W_i} U_\nu, JW_j \rangle = \langle \nabla_{W_i} U_\nu, \widetilde{W}_3 \rangle$. From $0 = W_i \langle \widetilde{W}_3, J\xi \rangle = \langle \bar{\nabla}_{W_i} \widetilde{W}_3, J\xi \rangle + \langle \widetilde{W}_3, \bar{\nabla}_{W_i} J\xi \rangle = b_i \langle \bar{\nabla}_{W_i} \widetilde{W}_3, U_i \rangle + b_j \langle \bar{\nabla}_{W_i} \widetilde{W}_3, U_j \rangle - \lambda_i \langle JW_i, \widetilde{W}_j \rangle$ we obtain

$$b_j \langle \nabla_{W_i} U_j, \widetilde{W}_3 \rangle = -\left(\frac{b_i^2}{2(\lambda_3 - \lambda_i)} + \lambda_i \right) \langle JW_i, \widetilde{W}_3 \rangle$$

by using Lemma 2.5. From the same lemma it follows that

$$\langle \nabla_{W_i} U_i, \widetilde{W}_3 \rangle = \frac{b_i}{2(\lambda_3 - \lambda_i)} \langle JW_i, \widetilde{W}_3 \rangle.$$

Taking into account the last two equations, (10) becomes

$$\frac{(-1)^i b_i}{2b_j(\lambda_3 - \lambda_i)} \langle JW_i, \widetilde{W}_3 \rangle = \langle \nabla_{W_i} W_j, A \rangle = \frac{(-1)^i}{b_i\, b_j} \Big(\frac{b_i^2}{2(\lambda_3 - \lambda_i)} + \lambda_i \Big) \langle JW_i, \widetilde{W}_3 \rangle. \tag{11}$$

This readily implies $\lambda_i \langle JW_i, \widetilde{W}_3 \rangle = 0$. Since at least one of the two eigenvalues λ_1, λ_2 must be nonzero, it follows that $\langle JW_1, \widetilde{W}_3 \rangle = 0$ or $\langle JW_1, \widetilde{W}_3 \rangle = 0$. From (11) we thus see that $\langle \nabla_{W_i} W_j, A \rangle = 0$ or $\langle \nabla_{W_j} W_i, A \rangle = 0$. But from Lemma 2.1 we know that

$$0 = \bar{R}_{W_i W_j A \xi} = (\lambda_j - \lambda_3) \langle \nabla_{W_i} W_j, A \rangle - (\lambda_i - \lambda_3) \langle \nabla_{W_j} W_i, A \rangle,$$

which implies that $\langle \nabla_{W_i} W_j, A \rangle = 0$ in both cases. This finishes the proof. $\square$

From Lemma 4.8 we immediately get

Corollary 4.9. *$m_1 = 1$ or $m_2 = 1$.*

According to Corollary 4.9 we may assume that $m_2 = 1$, that is, $T_{\lambda_2} = \mathbb{R}U_2$. We will now distinguish the two cases $m_1 > 1$ and $m_1 = 1$.

Case 1: $m_1 > 1$. Then we have $4\lambda_1\lambda_3 = 1$ by Lemma 4.8 and $J(T_{\lambda_1} \ominus \mathbb{R}U_1) \subset T_{\lambda_3} \ominus \mathbb{R}A$ because T_{λ_1} is real. Let $W_1 \in \Gamma(T_{\lambda_1} \ominus \mathbb{R}U_1)$ and $W_3 \in \Gamma(T_{\lambda_3} \ominus \mathbb{R}A)$. Then $0 = W_1 \langle W_3, J\xi \rangle = \langle \bar{\nabla}_{W_1} W_3, J\xi \rangle + \langle W_3, \bar{\nabla}_{W_1} J\xi \rangle = b_1 \langle \nabla_{W_1} W_3, U_1 \rangle + b_2 \langle \nabla_{W_1} W_3, U_2 \rangle - \lambda_1 \langle JW_1, W_3 \rangle$. Using Lemma 2.5 this implies

$$\langle \nabla_{W_1} U_2, W_3 \rangle = -\frac{1}{b_2} \left(\frac{b_1^2}{2(\lambda_3 - \lambda_1)} + \lambda_1 \right) \langle JW_1, W_3 \rangle. \tag{12}$$

Next, using $JU_2 = b_1 A - b_2 \xi$, we get $0 = W_1 \langle W_1, JU_2 \rangle = \langle \bar{\nabla}_{W_1} W_1, JU_2 \rangle + \langle W_1, \bar{\nabla}_{W_1} JU_2 \rangle = \langle \nabla_{W_1} U_2, JW_1 \rangle + b_1 \langle W_1, \nabla_{W_1} A \rangle - b_2 \langle W_1, \bar{\nabla}_{W_1} \xi \rangle$. We now assume that W_1 has length one. Using Corollary 2.6 this implies

$$\langle \nabla_{W_1} U_2, JW_1 \rangle = -b_2 \lambda_1. \tag{13}$$

Comparing (12) with $W_3 = JW_1$ and (13), and using $b_1^2 + b_2^2 = 1$, implies $2\lambda_1(\lambda_1 - \lambda_3) = 1$. Together with $4\lambda_1\lambda_3 = 1$ this implies $\lambda_1 = \sqrt{3}/2$ and $\lambda_3 = \sqrt{3}/6$, where we assume that the orientation of ξ is such that $\lambda_3 > 0$.

We now apply Lemma 2.7 with $X = W_1$ and $Y = U_2$, and use Corollary 2.6 and (5), to obtain

$$0 = (\lambda_2 - \lambda_1)(1 - 4\lambda_1\lambda_2 + 8\langle\nabla_{W_1}U_2, \nabla_{U_2}W_1\rangle) - b_2\langle\nabla_{W_1}U_2, JW_1\rangle + 2b_2\langle\nabla_{U_2}W_1, JW_1\rangle. \quad (14)$$

Using Lemma 2.5 we easily get $\nabla_{W_1}U_2 \in \Gamma(T_{\lambda_3})$, and (7) shows that $\langle\nabla_{U_2}W_1, A\rangle = 0$. From (12) we thus get

$$\langle\nabla_{W_1}U_2, \nabla_{U_2}W_1\rangle = \langle\nabla_{W_1}U_2, JW_1\rangle\langle\nabla_{U_2}W_1, JW_1\rangle. \quad (15)$$

From Lemma 2.1 and (13) we obtain $b_2 = 4\bar{R}_{U_2W_1JW_1\xi} = 4(\lambda_1 - \lambda_3)\langle\nabla_{U_2}W_1, JW_1\rangle + 4(\lambda_2 - \lambda_3)b_2\lambda_1$ and hence

$$\langle\nabla_{U_2}W_1, JW_1\rangle = \frac{b_2}{4(\lambda_1 - \lambda_3)}(1 - 4\lambda_1(\lambda_2 - \lambda_3)) = b_2\lambda_1(1 - 2\lambda_1\lambda_2). \quad (16)$$

Inserting (12), (15) and (16) into (14) yields

$$0 = 12(3b_2^2 - 1)\lambda_2^2 + 4\sqrt{3}(2 - 9b_2^2)\lambda_2 + 3(9b_2^2 - 1).$$

On the other hand, inserting the above particular values for λ_1 and λ_3 into the equation in Corollary 4.7, and replacing b_1^2 by $1 - b_2^2$, yields

$$0 = 12(9b_2^2 + 1)\lambda_2^2 - 4\sqrt{3}(2 + 9b_2^2)\lambda_2 - 3(9b_2^2 - 1).$$

Adding up the previous two equations gives $\lambda_2(2\lambda_2 - \sqrt{3}) = 0$. As $\lambda_2 \neq \lambda_1 = \sqrt{3}/2$ we therefore get $\lambda_2 = 0$, which implies $b_2^2 = 1/9$ and $b_1^2 = 8/9$.

Case 2: $m_1 = 1$. In this case we have $T_{\lambda_1} = \mathbb{R}U_1$, and $T_{\lambda_3} \ominus \mathbb{R}A = \mathfrak{D}$ is a J-invariant distribution. Let $W_3 \in \Gamma(\mathfrak{D})$ be of unit length. We have $0 = W_3\langle JW_3, J\xi\rangle = \langle\bar{\nabla}_{W_3}JW_3, J\xi\rangle + \langle JW_3, \bar{\nabla}_{W_3}J\xi\rangle$, and applying Lemma 2.5 this yields

$$(\lambda_3 - \lambda_2)b_1^2 + (\lambda_3 - \lambda_1)b_2^2 + 4\lambda_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) = 0. \quad (17)$$

Together with $b_1^2 + b_2^2 = 1$ this implies

$$b_i^2 = \frac{\lambda_3 - \lambda_i}{\lambda_j - \lambda_i}(1 + 4\lambda_3(\lambda_3 - \lambda_j)) \qquad (i, j \in \{1, 2\},\ i \neq j). \quad (18)$$

Inserting these expressions for b_1^2 and b_2^2 into the equation in Corollary 4.7 yields

$$(\lambda_1 - \lambda_2)^2 - (\lambda_1 + \lambda_2 - 4\lambda_3)^2 = 1 - 4\lambda_3^2. \quad (19)$$

We now apply Lemma 2.7 with $X = W_3$ and $Y = U_i$, which gives

$$\begin{aligned} 0 \;=\; & (\lambda_i - \lambda_3)\big(1 - 4\lambda_i\,\lambda_3 + 8\langle\nabla_{W_3}U_i, \nabla_{U_i}W_3\rangle - 4\langle\nabla_{W_3}W_3, \nabla_{U_i}U_i\rangle\big) \\ & -b_i\langle\nabla_{W_3}U_i, JW_3\rangle + 2b_i\langle\nabla_{U_i}W_3, JW_3\rangle. \end{aligned} \quad (20)$$

Let $i, j \in \{1, 2\}$ and $i \neq j$. Then we have $0 = W_3\langle U_i, J\xi\rangle = \langle\bar{\nabla}_{W_3}U_i, J\xi\rangle + \langle U_i, \bar{\nabla}_{W_3}J\xi\rangle = b_j\langle\nabla_{W_3}U_i, U_j\rangle$ and hence $\langle\nabla_{W_3}U_i, U_j\rangle = 0$. For $\widetilde{W_3} \in \Gamma(\mathfrak{D})$ we have $4(\lambda_3 - \lambda_i)\langle\nabla_{W_3}U_i, \widetilde{W_3}\rangle = 4(\lambda_i - \lambda_3)\langle\nabla_{W_3}\widetilde{W_3}, U_i\rangle = b_i\langle JW_3, \widetilde{W_3}\rangle$ and $\langle\nabla_{W_3}U_i, A\rangle = -\langle\nabla_{W_3}A, U_i\rangle = 0$ by Lemma 2.5. Altogether this gives $4(\lambda_3 - \lambda_i)\nabla_{W_3}U_i = b_iJW_3$, and together with (5) equation (20) now becomes

$$0 = 4(\lambda_3 - \lambda_i)^2(1 - 4\lambda_i\lambda_3) - 12(-1)^i b_1b_2(\lambda_3 - \lambda_i)\langle\nabla_{W_3}W_3, A\rangle + b_i^2.$$

Multiplying this equation with $\lambda_3 - \lambda_j$, then adding the two equations for $i = 1$ and $i = 2$, and then using (17) yields

$$4\lambda_3(1 + \lambda_1^2 + \lambda_2^2) - (\lambda_1 + \lambda_2)(1 + 4\lambda_3^2) = 0. \tag{21}$$

If $\lambda_3 = 0$, we immediately get $\lambda_1, \lambda_2 \in \{\pm 1/2\}$ from (19) and (21). From now on we assume $\lambda_3 \neq 0$. If we put $x = \lambda_1 - \lambda_2$ and $y = \lambda_1 + \lambda_2 - 4\lambda_3$, equations (19) and (21) are equivalent to

$$x^2 - y^2 = 1 - 4\lambda_3^2 \ , \ x^2 + \left(y - \frac{1 - 12\lambda_3^2}{4\lambda_3}\right)^2 = \frac{1 + 16\lambda_3^4}{16\lambda_3^2}.$$

Obviously, these are the equations of a hyperbola and a circle. It is straightforward to calculate their common points, namely

$$(x, y) = \left(\pm\sqrt{1 - 3\lambda_3^2} \ , \ -\lambda_3\right) \quad \text{and} \quad (x, y) = \left(\pm\frac{1}{4\lambda_3} \ , \ \frac{1 - 8\lambda_3^2}{4\lambda_3}\right) \ ,$$

where the first possibility only arises if $3\lambda_3^2 \leq 1$. Taking into account that λ_1 and λ_2 are different from λ_3, this eventually implies

$$\lambda_1 = \frac{1}{2}\left(3\lambda_3 - \sqrt{1 - 3\lambda_3^2}\right) \ , \ \lambda_2 = \frac{1}{2}\left(3\lambda_3 + \sqrt{1 - 3\lambda_3^2}\right) \tag{22}$$

where we assume without loss of generality that $\lambda_1 < \lambda_2$. Obviously, we get a solution only if $3\lambda_3^2 \leq 1$. If $|\lambda_3| = 1/2$ or $|\lambda_3| = 1/\sqrt{3}$, then the three principal curvatures cannot be different. Suppose that $1/2 < |\lambda_3| < 1/\sqrt{3}$. From (17) and (22) we get

$$\frac{b_1^2}{2\lambda_3(\lambda_3 - \sqrt{1 - 3\lambda_3^2})} + \frac{b_2^2}{2\lambda_3(\lambda_3 + \sqrt{1 - 3\lambda_3^2})} = 1.$$

If $1/2 < |\lambda_3| < 1/\sqrt{3}$, elementary calculations show that $0 < 2\lambda_3(\lambda_3 - \sqrt{1 - 3\lambda_3^2}) < 1$ and $0 < 2\lambda_3(\lambda_3 + \sqrt{1 - 3\lambda_3^2}) < 1$. Therefore the last equation is the equation of an ellipse centered at the origin and with axes of length less than 1. Obviously such an ellipse has no points of intersection with the circle $b_1^2 + b_2^2 = 1$. This shows that $|\lambda_3| < 1/2$.

We summarize the discussion in this section in

Theorem 4.10. *Let M be a connected real hypersurface in $\mathbb{C}H^n$, $n \geq 3$, with three distinct constant principal curvatures $\lambda_1, \lambda_2, \lambda_3$, and suppose that M is not a Hopf hypersurface. Then, with a suitable labelling of the principal curvatures, we have $J\xi = b_1U_1 + b_2U_2$ with some real numbers $b_1, b_2 > 0$, where U_i denotes the orthogonal projection of $J\xi$ onto T_{λ_i} normalized to length one. There exists a unit vector field $A \in \Gamma(T_{\lambda_3})$ such that $JA = b_2U_1 - b_1U_2$. The subbundle $\mathbb{R}U_1 \oplus \mathbb{R}U_2$ is real, and the subbundle $\mathbb{R}A \oplus \mathbb{R}U_1 \oplus \mathbb{R}U_2 \oplus \mathbb{R}\xi$ is complex. Moreover, $m_2 = 1$ and one the following two cases holds:*

(i) *$m_1 > 1$, $\lambda_1 = \sqrt{3}/2$, $\lambda_2 = 0$, $\lambda_3 = \sqrt{3}/6$, $b_1 = 2\sqrt{2}/3$, $b_2 = 1/3$, the subbundle $T_{\lambda_1} \ominus \mathbb{R}U_1$ is real, and $J(T_{\lambda_1} \ominus \mathbb{R}U_1) \subset T_{\lambda_3}$.*

(ii) $m_1 = 1$, $-1/2 < \lambda_3 < 1/2$, $\lambda_1 = \frac{1}{2}(3\lambda_3 - \sqrt{1 - 3\lambda_3^2})$, $\lambda_2 = \frac{1}{2}(3\lambda_3 + \sqrt{1 - 3\lambda_3^2})$, *and*

$$b_i^2 = \frac{\lambda_3 - \lambda_i}{\lambda_j - \lambda_i}(1 + 4\lambda_3(\lambda_3 - \lambda_j)) \qquad (i, j \in \{1, 2\},\ i \neq j).$$

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let M be a connected real hypersurface in $\mathbb{C}H^n$ with three distinct constant principal curvatures. If M is a Hopf hypersurface, it was shown in [1] that M is an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for some $k \in \{1, \ldots, n-2\}$, or of a tube with radius $r \neq \ln(2 + \sqrt{3})$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$. We can therefore assume that M is not a Hopf hypersurface. Then M must satisfy one of the two possibilities described in Theorem 4.10. The result will follow from a thorough investigation of the possible focal sets and equidistant hypersurfaces of M by means of Jacobi field theory.

For $r \in \mathbb{R}$ we define the smooth map $\Phi^r : M \to \mathbb{C}H^n$, $p \mapsto \Phi^r(p) = \exp_p(r\xi_p)$, where $\exp_p$ is the exponential map of $\mathbb{C}H^n$ at p. Geometrically this means that we assign to p the point in $\mathbb{C}H^n$ which is obtained by travelling for the distance r along the geodesic $c_p(t) = \exp_p(t\xi_p)$ in direction of the normal vector ξ_p (for $r > 0$; for $r < 0$ one sets off in direction $-\xi_p$; and for $r = 0$ there is no movement at all). For $v \in T_pM$ we denote by B_v the parallel vector field along the geodesic c_p with $B_v(0) = v$, and by ζ_v the Jacobi field along c_p with $\zeta_v(0) = v$ and $\zeta_v'(0) = -S_p v$. Note that ζ_v is the unique solution of the linear differential equation

$$4\zeta_v'' - \zeta_v - 3\langle \zeta_v, J\dot{c}_p\rangle J\dot{c}_p = 0\ ,\ \zeta_v(0) = v\ ,\ \zeta_v'(0) = -S_p v,$$

where $\dot{c}_p$ denotes the tangent vector field of c_p and the prime $'$ indicates the covariant derivative of a vector field along c_p. For $v \in T_{\lambda_i}(p)$ we have the explicit expression

$$\zeta_v(t) = f_i(t)B_v(t) + \langle v, J\xi\rangle g_i(t) J\dot{c}_p(t)$$

with

$$\begin{aligned} f_i(t) &= \cosh(t/2) - 2\lambda_i \sinh(t/2), \\ g_i(t) &= (\cosh(t/2) - 1)\,(1 + 2\cosh(t/2) - 2\lambda_i \sinh(t/2))\,. \end{aligned}$$

Finally, we define a vector field η^r along the map Φ^r by $\eta_p^r = \dot{c}_p(r)$. The relation between the map Φ^r, the vector field η^r and the Jacobi field ζ_v is given by

$$\zeta_v(r) = \Phi_*^r v\ ,\ \zeta_v'(r) = \bar{\nabla}_v \eta^r,$$

where Φ_*^r denotes the differential of Φ^r. The singularities of Φ^r are focal points of M and can be calculated using Jacobi fields from the equation $\zeta_v(r) = \Phi_*^r v$. We will see that in case (i) of Theorem 4.10 there exists a particular distance r at which the map Φ^r has constant rank $2n - m_1$, which means that the image of Φ^r forms locally a submanifold of codimension m_1. In case (ii) of Theorem 4.10 there exists a particular distance r at which the map Φ^r has constant rank $2n-1$ and the image is locally a minimal real hypersurface. We then use the equation $\zeta_v'(r) = \bar{\nabla}_v \eta^r$ to obtain some information about the second fundamental form of these submanifolds. We continue using the notation introduced in Section 4.

Case 1: $m_1 > 1$. We define $u_i = (U_i)_p$ and $r = \ln(2+\sqrt{3})$. For $v \in T_pM$ we denote by v_i the orthogonal projection of v onto $T_{\lambda_i}(p)$. Using the equation $\Phi^r_* v = \zeta_v(r)$ and the explicit expression for the Jacobi fields, we obtain

$$\begin{aligned} 9\Phi^r_* v &= 3\sqrt{6}B_{v_3}(r) + \big(4\langle v_1, u_1\rangle + (4\sqrt{2}-2\sqrt{3})\langle v_2, u_2\rangle\big)B_{u_1}(r) \\ &\quad + \big(\sqrt{2}\langle v_1, u_1\rangle + (2+4\sqrt{6})\langle v_2, u_2\rangle\big)B_{u_2}(r). \end{aligned}$$

This shows that $\Phi^r_* v = 0$ if and only if $v \in T_{\lambda_1}(p) \ominus \mathbb{R}u_1$. Therefore the rank of Φ^r is constant and equal to $2n - m_1$. This means that for every point in M there exists an open neighborhood $\mathcal{V}$ such that $\mathcal{W} = \Phi^r(\mathcal{V})$ is an embedded submanifold of $\mathbb{C}H^n$ and $\Phi^r : \mathcal{V} \to \mathcal{W}$ is a submersion. Let $p \in \mathcal{V}$ and $q = \Phi^r(p) \in \mathcal{W}$. The above expression for the differential of Φ^r shows that the tangent space $T_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $T_{\lambda_3}(p) \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2$ along the geodesic c_p from $p = c_p(0)$ to $q = c_p(r)$. Hence, the normal space $\nu_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $\mathbb{R}\xi_p \oplus (T_{\lambda_1}(p) \ominus \mathbb{R}u_1)$ along c_p from p to q. This shows in particular that $\mathcal{W}$ has totally real normal bundle.

Clearly, $\eta^r_p = B_{\xi_p}(r)$ is a unit normal vector of $\mathcal{W}$ at q. For the shape operator S^r of $\mathcal{W}$ we have $S^r_{\eta^r_p}\Phi^r_* v = -(\bar\nabla_v \eta^r)^\top = -(\zeta'_v(r))^\top$, where $(\cdot)^\top$ denotes the component tangent to $\mathcal{W}$. Using the explicit expression for the Jacobi fields we easily get

$$S^r_{\eta^r_p} B_{v_3}(r) = 0 \text{ for all } v_3 \in T_{\lambda_3}(p). \tag{23}$$

Moreover, $S^r_{\eta^r_p}$ leaves $\mathbb{R}B_{u_1}(r) \oplus \mathbb{R}B_{u_2}(r)$ invariant and has the matrix representation

$$\frac{1}{18}\begin{pmatrix} 4\sqrt{2} & -7 \\ -7 & -4\sqrt{2} \end{pmatrix},$$

with respect to $B_{u_1}(r), B_{u_2}(r)$. Since $3JA_p = u_1 - 2\sqrt{2}u_2$ and $3J\xi_p = 2\sqrt{2}u_1 + u_2$, the above matrix representation yields

$$2S^r_{\eta^r_p} B_{JA_p}(r) = J\eta^r_p\ ,\ 2S^r_{\eta^r_p} J\eta^r_p = B_{JA_p}(r). \tag{24}$$

As $J(\nu_q\mathcal{W} \ominus \mathbb{R}\eta^r_p)$ is contained in the parallel translate of $T_{\lambda_3}(p)$ along c_p from p to q, (23) and the linearity of $S^r_{\eta^r_p}$ show that

$$2S^r_{\eta^r_p} J\tilde\eta = \langle \eta^r_p, \tilde\eta\rangle B_{JA_p}(r) \text{ for all } \tilde\eta \in \nu_q\mathcal{W}. \tag{25}$$

As a special case we get $S^r_{\eta^r_p} J\tilde\eta = 0$ for all $p \in \mathcal{V}$ and $\tilde\eta \in \nu_q\mathcal{W} \ominus \mathbb{R}\eta^r_p$. From the Gauß formula and $\bar\nabla J = 0$ one easily gets $S^r_{\tilde\eta} J\eta^r_p = S^r_{\eta^r_p} J\tilde\eta$ and hence

$$S^r_{\tilde\eta} J\eta^r_p = 0 \text{ for all } \tilde\eta \in \nu_q\mathcal{W} \ominus \mathbb{R}\eta^r_p. \tag{26}$$

Now let γ be any curve in $(\Phi^r)^{-1}(\{q\}) \cap \mathcal{V}$ with $\gamma(0) = p$. Since η^r_p and $\eta^r_{\gamma(t)} - \langle \eta^r_{\gamma(t)}, \eta^r_p\rangle \eta^r_p$ are perpendicular, (26), the linearity of $\eta \mapsto S^r_\eta$ and (25) imply

$$0 = 2S_{\eta^r_{\gamma(t)} - \langle \eta^r_{\gamma(t)}, \eta^r_p\rangle \eta^r_p} J\eta^r_p = 2S_{\eta^r_{\gamma(t)}} J\eta^r_p - \langle \eta^r_{\gamma(t)}, \eta^r_p\rangle B_{JA_p}(r).$$

On the other hand, (25) with $\gamma(t)$ instead of p gives

$$2S^r_{\eta^r_{\gamma(t)}} J\eta^r_p = \langle \eta^r_{\gamma(t)}, \eta^r_p \rangle B_{JA_{\gamma(t)}}(r).$$

The previous two equations show that the map $\tilde{p} \mapsto B_{JA_{\tilde{p}}}(r)$ is of constant value $z \in T_q\mathcal{W}$ on the connected component $\mathcal{V}_o$ of $(\Phi^r)^{-1}(\{q\}) \cap \mathcal{V}$ containing p. Note that z has length one because of $z = B_{JA_p}(r)$. For all $v_1 \in T_{\lambda_1}(p) \ominus \mathbb{R}u_1$ we have $\bar{\nabla}_{v_1}\eta^r = \zeta'_{v_1}(r) = (-1/\sqrt{2})B_{v_1}(r)$, which implies that η^r is a local diffeomorphism from $\mathcal{V}_o$ into the unit sphere in $\nu_q\mathcal{W}$. Thus $\eta^r(\mathcal{V}_o)$ is an open subset of the unit sphere in $\nu_q\mathcal{W}$. Since S^r_η depends analytically on $\eta \in \nu_q\mathcal{W}$, we conclude from (23) and (24) that

$$2S^r_\eta J\eta = z \ , \ 2S^r_\eta z = J\eta \ , \ S^r_\eta v = 0 \text{ for all } \eta \in \nu_q\mathcal{W}, v \in T_q\mathcal{W} \ominus J(\nu_q\mathcal{W} \ominus \mathbb{R}\eta) \ominus \mathbb{R}z.$$

Therefore the second fundamental form $I\!I^r_q$ of $\mathcal{W}$ at q is given by the trivial bilinear extension of $2I\!I^r_q(z, J\eta) = \eta$ for all $\eta \in \nu_q\mathcal{W}$. The construction of z shows that it depends smoothly on the point $q \in \mathcal{W}$. Hence there exists a unit vector field Z on $\mathcal{W}$ such that the second fundamental form $I\!I^r$ of $\mathcal{W}$ is given by the trivial bilinear extension of $2I\!I^r(Z, J\eta) = \eta$ for all $\eta \in \Gamma(\nu\mathcal{W})$. From Theorem 3.1 we see that $\mathcal{W}$ is holomorphically congruent to an open part of the ruled minimal submanifold W^{2n-m_1}. Thus we have proved that locally M lies on a tube with radius $r = \ln(2+\sqrt{3})$ around a ruled minimal submanifold holomorphically congruent to W^{2n-m_1}. This finally implies that M is holomorphically congruent to an open part of the tube with radius $r = \ln(2+\sqrt{3})$ around W^{2n-m_1}.

Case 2: $m_1 = 1$. If $\lambda_3 = 0$, then $\lambda_1 = -1/2$ and $\lambda_2 = 1/2$, and it follows from Theorem 3.2 that M is holomorphically congruent to an open part of the ruled minimal hypersurface W^{2n-1}. If $0 < |\lambda_3| < 1/2$, we can write $2\lambda_3 = \tanh(r/2)$ with some $0 \neq r \in \mathbb{R}$.

Let $p \in M$ and define $u_i = (U_i)_p$. Using the equation $\Phi^r_* v = \zeta_v(r)$ and the explicit expression for the Jacobi fields, we obtain

$$\Phi^r_* v_3 = \operatorname{sech}(r/2) B_{v_3}(r) \text{ for all } v_3 \in T_{\lambda_3}(p)$$

and

$$\begin{pmatrix} \Phi^r_* u_1 \\ \Phi^r_* u_2 \end{pmatrix} = D(r) \begin{pmatrix} B_{u_1}(r) \\ B_{u_2}(r) \end{pmatrix}$$

with

$$D(t) = \begin{pmatrix} f_1(t) + b_1^2 g_1(t) & b_1 b_2 g_1(t) \\ b_1 b_2 g_2(t) & f_2(t) + b_2^2 g_2(t) \end{pmatrix}.$$

As $\det(D(r)) = \operatorname{sech}^3(r/2)$, we can now conclude that Φ^r_* has maximal rank everywhere. This means that for every point in M there exists an open neighborhood $\mathcal{V}$ such that $\mathcal{W} = \Phi^r(\mathcal{V})$ is an embedded real hypersurface of $\mathbb{C}H^n$ and $\Phi^r : \mathcal{V} \to \mathcal{W}$ is a diffeomorphism. Let $p \in \mathcal{V}$ and $q = \Phi^r(p) \in \mathcal{W}$. The tangent space $T_q\mathcal{W}$ of $\mathcal{W}$ at q is obtained by parallel translation of $T_p\mathcal{V}$ along the geodesic c_p from $p = c_p(0)$ to $q = c_p(r)$, and η^r_p is a unit normal vector of $\mathcal{W}$ at q.

For the shape operator S^r of $\mathcal{W}$ we have $S^r_{\eta^r_p}\Phi^r_* v = -\bar{\nabla}_v \eta^r = -\zeta'_v(r)$. Since $f_3'(r) = 0$ we immediately get

$$S^r_{\eta^r_p} B_{v_3}(r) = 0 \text{ for all } v_3 \in T_{\lambda_3}(p),$$

and for $\Phi^r_* u_1$ and $\Phi^r_* u_2$ we get

$$\begin{pmatrix} S^r_{\eta^r_p}\Phi^r_* u_1 \\ S^r_{\eta^r_p}\Phi^r_* u_2 \end{pmatrix} = C(r)\begin{pmatrix} B_{u_1}(r) \\ B_{u_2}(r) \end{pmatrix}$$

with $C(r) = -D'(r)D(r)^{-1}$. A tedious calculation shows that $\det(D'(r)) = -\mathrm{sech}^3(r/2)/4$ and $(\det(D))'(r) = 0$, which implies

$$\det(C(r)) = \frac{\det(D'(r))}{\det(D(r))} = -\frac{1}{4} \quad \text{and} \quad \mathrm{tr}(C(r)) = -\frac{(\det(D))'(r)}{\det(D(r))} = 0.$$

From this we easily see that the eigenvalues of $C(r)$ are $\pm 1/2$. Altogether we now get that $\mathcal{W}$ has three distinct constant principal curvatures 0, $+1/2$ and $-1/2$ with corresponding multiplicities $2n-3$, 1 and 1, respectively. It follows from Theorem 3.2 that $\mathcal{W}$ is holomorphically congruent to an open part of the ruled real hypersurface W^{2n-1}. From this we eventually conclude that M is holomorphically congruent to an open part of an equidistant hypersurface to W^{2n-1}.

This finishes the proof of Theorem 1.1.

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