INHOMOGENEOUS ISOPARAMETRIC HYPERSURFACES IN COMPLEX HYPERBOLIC SPACES

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ABSTRACT. We construct examples of inhomogeneous isoparametric real hypersurfaces in complex hyperbolic spaces.

1. INTRODUCTION

An isoparametric hypersurface of a Riemannian manifold is a hypersurface that is a level set of an isoparametric function. Cartan proved that a hypersurface is isoparametric if and only if the hypersurface itself and its sufficiently close parallel hypersurfaces have constant mean curvature. If the ambient manifold has constant sectional curvature, then Cartan also proved that a hypersurface is isoparametric if and only if it has constant principal curvatures. The study of isoparametric hypersurfaces is an active topic of research in Differential Geometry. See [6] for a survey.

In a more general ambient space, an isoparametric hypersurface might have nonconstant principal curvatures, and therefore it might be inhomogeneous. See [7] for isoparametric hypersurfaces with nonconstant principal curvatures in complex projective spaces. These examples are constructed from the inhomogeneous examples in spheres given by Ferus, Karcher and Münzner in [5] via the Hopf map. To our knowledge, the only examples of isoparametric hypersurfaces in complex hyperbolic spaces known so far were homogeneous. Homogeneous hypersurfaces in complex hyperbolic spaces were classified in [4] and their geometry was studied in [3].

The aim of this paper is to construct examples of isoparametric hypersurfaces in complex hyperbolic spaces that are in general not homogeneous. These are not related to the Ferus, Karcher and Münzner hypersurfaces in [5]. Our examples arise as tubes around certain homogeneous submanifolds that are in a way a modification of the homogeneous submanifolds introduced by Berndt and Brück in [2]. Indeed, the tubes around Berndt-Brück submanifolds are a particular case of our examples, and are the only ones being homogeneous hypersurfaces. Xiao claims in [8] that isoparametric real hypersurfaces in $\mathbb{C}H^n$ are homogeneous, although there is no proof of this fact to our knowledge. Furthermore, our examples show that inhomogeneous isoparametric examples do exist.

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The main result of our paper, the construction of these isoparametric hypersurfaces, is summarized in Theorem 3.1. In general, our examples are inhomogeneous. Pointwise, the principal curvatures of these hypersurfaces are the same as those of the Berndt and Brück examples. Then, we also show that the inhomogeneous hypersurfaces in our examples correspond precisely to those that have nonconstant principal curvatures.

2. Preliminaries

In this paper we follow the notation of [3].

Let $\mathbb{C}H^n$ be the complex hyperbolic space of constant holomorphic sectional curvature -1. We write $\mathbb{C}H^n$ as G/K where $G = SU(1,n)$ and $K = S(U(1)U(n))$. As usual, denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to a point $o \in \mathbb{C}H^n$ and choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, which is therefore 1-dimensional. Let $\mathfrak{g} = \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the root space decomposition of $\mathfrak{g}$ with respect to o and $\mathfrak{a}$. We introduce an ordering in the set of roots so that α is a positive root. Then o, $\mathfrak{a}$, and this ordering determine a point at infinity x in the ideal boundary $\mathbb{C}H^n(\infty)$ of $\mathbb{C}H^n$. Let $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ with respect to the point $o \in \mathbb{C}H^n$ and the point at infinity $x \in \mathbb{C}H^n(\infty)$. Let A, N and AN be the connected and simply connected subgroups of G whose Lie algebras are $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{a}\oplus\mathfrak{n}$, respectively. Thus G is diffeomorphic to $K \times A \times N$, and AN is diffeomorphic to $\mathbb{C}H^n$. The metric (resp., the complex structure J) on $\mathbb{C}H^n$ induces a left invariant metric (resp. a complex structure J) on the solvable Lie group AN, so that $\mathbb{C}H^n$ and AN become isometric as Kähler manifolds. We also have the isomorphism $T_o\mathbb{C}H^n \cong \mathfrak{a} \oplus \mathfrak{n}$.

From now on, let B be the unit left-invariant vector field of $\mathfrak{a}$ determined by x; that is, the geodesic through o whose initial speed is B converges to the point at infinity x. Set $Z = JB \in \mathfrak{g}_{2\alpha}$. We obviously have $\mathfrak{a} = \mathbb{R}B$ and $\mathfrak{g}_{2\alpha} = \mathbb{R}Z$. Moreover, $\mathfrak{g}_\alpha$ is J-invariant, so it is isomorphic to $\mathbb{C}^{n-1}$. The Lie algebra structure on $\mathfrak{a} \oplus \mathfrak{n}$ is given by the relations

$$[B,Z] = Z, \quad 2\,[B,U] = U, \quad [U,V] = \langle JU, V\rangle Z, \quad [Z,U] = 0,$$

where U, $V \in \mathfrak{g}_\alpha$. The Levi-Civita connection of AN is given by

$$\nabla_{aB+U+xZ}(bB+V+yZ) = \left(\frac{1}{2}\langle U,V\rangle + xy\right)B - \frac{1}{2}\,(bU + yJU + xJV) + \left(\frac{1}{2}\langle JU,V\rangle - bx\right)Z,$$

where a, b, x, $y \in \mathbb{R}$, U, $V \in \mathfrak{g}_\alpha$, and all vector fields are considered to be left-invariant.

3. The examples

In this section we present examples of isoparametric hypersurfaces in $\mathbb{C}H^n$, $n \geq 2$, that are in general not homogeneous. These hypersurfaces will be tubes around certain homogeneous submanifolds, so we proceed first with the construction of the latter.

Let $\mathfrak{w}$ be a subspace of $\mathfrak{g}_\alpha$ and define $\mathfrak{w}^\perp = \mathfrak{g}_\alpha \ominus \mathfrak{w}$, the orthogonal complement of $\mathfrak{w}$ in $\mathfrak{g}_\alpha$. We also define $k = \dim \mathfrak{w}^\perp$. Then, $\mathfrak{s}_\mathfrak{w} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{g}_{2\alpha}$ is a solvable Lie subalgebra of $\mathfrak{a}\oplus\mathfrak{n}$, as one can easily check from the bracket relations above. Let $S_\mathfrak{w}$ be the corresponding connected subgroup of AN whose Lie algebra is $\mathfrak{s}_\mathfrak{w}$. We define the submanifold $W_\mathfrak{w}$ as the orbit $S_{\mathfrak{w}} \cdot o$ of the group $S_{\mathfrak{w}}$ through the point o. Hence, $W_{\mathfrak{w}}$ is a homogeneous submanifold of $\mathbb{C}H^n$ of codimension k.

For each $\xi \in \mathfrak{w}^\perp$, we write $J\xi = P\xi + F\xi$, where $P\xi$ is the orthogonal projection of $J\xi$ onto $\mathfrak{w}$, and $F\xi$ is the orthogonal projection of $J\xi$ onto $\mathfrak{w}^\perp$. We define the Kähler angle of $\xi \in \mathfrak{w}^\perp$ with respect to $\mathfrak{w}^\perp$ as the angle $\varphi_\xi \in [0, \pi/2]$ between $J\xi$ and $\mathfrak{w}^\perp$; hence φ_ξ satisfies $\langle F\xi, F\xi\rangle = (\cos^2\varphi_\xi)\langle \xi, \xi\rangle$. It readily follows from $J^2 = -I$ that $\langle P\xi, P\xi\rangle = (\sin^2\varphi_\xi)\langle\xi,\xi\rangle$. Hence, if ξ has unit length, φ_ξ is determined by the fact that $\cos\varphi_\xi$ is the length of the orthogonal projection of $J\xi$ onto $\mathfrak{w}^\perp$. For $\xi \in \mathfrak{w}^\perp$ it is convenient to define

$$\bar{P}\xi = \frac{1}{\sin\varphi_\xi}P\xi, \text{ (if } P\xi \neq 0) , \quad \text{and} \quad \bar{F}\xi = \frac{1}{\cos\varphi_\xi}F\xi, \text{ (if } F\xi \neq 0).$$

Thus, if ξ is of unit length, so are $\bar{P}\xi$ and $\bar{F}\xi$ if they exist.

Let $\mathfrak{c}$ be the maximal complex subspace of $\mathfrak{s}_{\mathfrak{w}}$. Clearly, $\mathfrak{c} = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus \mathbb{C}\mathfrak{w}^\perp) \oplus \mathfrak{g}_{2\alpha}$, and since $\mathbb{C}\mathfrak{w}^\perp = \mathfrak{w}^\perp + J\mathfrak{w}^\perp = \mathfrak{w}^\perp + P\mathfrak{w}^\perp + F\mathfrak{w}^\perp = \mathfrak{w}^\perp \oplus P\mathfrak{w}^\perp$, we have the vector space direct sum decompositions $\mathfrak{s}_{\mathfrak{w}} = \mathfrak{c} \oplus P\mathfrak{w}^\perp$ and $\mathfrak{a} \oplus \mathfrak{n} = \mathfrak{c} \oplus P\mathfrak{w}^\perp \oplus \mathfrak{w}^\perp$. Denoting by $\mathfrak{C}$, $P\mathfrak{W}^\perp$, and $\mathfrak{W}^\perp$ the corresponding left-invariant distributions on AN, we get the tangent bundle $TW_{\mathfrak{w}} = \mathfrak{C} \oplus P\mathfrak{W}^\perp$ and the normal bundle $\nu W_{\mathfrak{w}} = \mathfrak{W}^\perp$ along $W_{\mathfrak{w}}$.

Recall that the shape operator $\mathcal{S}_\xi$ of $W_{\mathfrak{w}}$ with respect to a unit normal $\xi \in \nu W_{\mathfrak{w}}$ is defined by $\mathcal{S}_\xi X = -(\nabla_X\xi)^\top$, for any $X \in TW_{\mathfrak{w}}$, and where $(\cdot)^\top$ denotes orthogonal projection onto the tangent space. The expression for the Levi-Civita connection of AN allows us to calculate the shape operator of $W_{\mathfrak{w}}$ for left-invariant vector fields:

$$\begin{aligned}
&\mathcal{S}_\xi U = 0, \quad \text{if } U \in \mathfrak{c} \ominus \mathfrak{g}_{2\alpha}, \qquad \mathcal{S}_\xi Z = \frac{1}{2}P\xi = \frac{1}{2}(\sin\varphi_\xi)\bar{P}\xi,\\
&\mathcal{S}_\xi \bar{P}\eta = -\frac{1}{2}\langle J\bar{P}\eta, \xi\rangle Z = \frac{1}{2}\langle \bar{P}\eta, P\xi\rangle Z = 0, \quad \text{if } \bar{P}\eta \in P\mathfrak{w}^\perp \ominus \mathbb{R}P\xi,\\
&\mathcal{S}_\xi \bar{P}\xi = -\frac{1}{2}\langle J\bar{P}\xi, \xi\rangle Z = \frac{1}{2}\langle \bar{P}\xi, P\xi\rangle Z = \frac{1}{2}(\sin\varphi_\xi)Z.
\end{aligned}$$

Note that, if $\varphi_\xi = 0$ (that is, if $P\xi = 0$), then $\mathcal{S}_\xi = 0$. If $\varphi_\xi > 0$ then $W_{\mathfrak{w}}$ has three principal curvatures with respect to the unit normal vector ξ

$$\frac{1}{2}\sin\varphi_\xi, \quad -\frac{1}{2}\sin\varphi_\xi, \quad \text{and} \quad 0,$$

whose principal spaces are $\mathbb{R}(Z + \bar{P}\xi)$, $\mathbb{R}(Z - \bar{P}\xi)$, and $\mathfrak{a} \oplus (\mathfrak{w} \ominus \mathbb{R}\bar{P}\xi)$, respectively. In any case, the submanifold $W_{\mathfrak{w}}$ is minimal.

Now we construct isoparametric hypersurfaces in $\mathbb{C}H^n$. Denote by M^r the tube of radius r around the submanifold $W_{\mathfrak{w}}$. We claim that, for every $r > 0$, M^r is an isoparametric real hypersurface which has, in general, nonconstant principal curvatures. We may assume $k > 1$, as otherwise we just obtain parallel hypersurfaces to the hypersurface W^{2n-1} studied by Berndt in [1].

We will use Jacobi field theory to calculate the mean curvature of M^r. Given a geodesic γ in $\mathbb{C}H^n$, a field ζ along γ satisfies the Jacobi equation in $\mathbb{C}H^n$ if $4\zeta'' - \zeta - 3\langle\zeta, J\dot{\gamma}\rangle J\dot{\gamma} = 0$. Let $p \in W_{\mathfrak{w}}$, and $\xi \in \nu_p W_{\mathfrak{w}}$ be a unit normal vector. Denote by γ_ξ the geodesic such that $\gamma_\xi(0) = p$ and $\dot{\gamma}_\xi(0) = \xi$. We denote by $\mathcal{B}_v(t)$ the parallel translation of $v \in T_p\mathbb{C}H^n$ along

γ_ξ from $p=\gamma_\xi(0)$ to $\gamma_\xi(t)$. If $X \in T_pW_{\mathfrak{w}}$ is such that $\mathcal{S}_\xi X = \lambda X$, the solution ζ_X to the Jacobi equation with initial conditions $\zeta_X(0)=X$ and $\zeta_X'(0) = -\mathcal{S}_\xi X$ is given by

$$\zeta_X(t) = f_\lambda(t)\mathcal{B}_X(t) + \langle X, J\xi\rangle g_\lambda(t) J\dot{\gamma}_\xi(t),$$

where

$$f_\lambda(t) = \cosh\frac{t}{2} - 2\lambda\sinh\frac{t}{2}, \qquad g_\lambda(t) = \left(\cosh\frac{t}{2}-1\right)\left(1+2\cosh\frac{t}{2} - 2\lambda\sinh\frac{t}{2}\right).$$

On the other hand, for every $\eta \in \nu_pW_{\mathfrak{w}} \ominus \mathbb{R}\xi$, the solution ζ_η to the Jacobi equation with initial conditions $\zeta_\eta(0)=0$ and $\zeta_\eta'(0)=\eta$ is given by

$$\zeta_\eta(t) = p(t)\mathcal{B}_\eta(t) + \langle \eta, J\xi\rangle q(t) J\dot{\gamma}_\xi(t),$$

where

$$p(t) = 2\sinh\frac{t}{2}, \qquad q(t) = 2\sinh\frac{t}{2}\left(\cosh\frac{t}{2}-1\right).$$

Hence, the above expression for the shape operator $\mathcal{S}_\xi$ of $W_{\mathfrak{w}}$ with respect to a unit normal ξ, allows us to compute (we give the explicit calculations for $\varphi_\xi \in (0,\pi/2)$; notice that some adaptations are needed in case $P\xi = 0$ or $F\xi=0$):

$$\begin{aligned}
\zeta_X(t) =\ & \cosh\frac{t}{2}\, B_X(t), \qquad \text{if } X \in TW_{\mathfrak{w}} \ominus (\mathbb{R}Z \oplus \mathbb{R}\bar{P}\xi),\\
\zeta_Z(t) =\ & \cosh\frac{t}{2}\, B_Z(t) - \sin\varphi_\xi\left(\cos^2\varphi_\xi + \sin^2\varphi_\xi\cosh\frac{t}{2}\right)\sinh\frac{t}{2}\,\mathcal{B}_{\bar{P}\xi}(t)\\
& - \cos\varphi_\xi\sin^2\varphi_\xi\left(\cosh\frac{t}{2}-1\right)\sinh\frac{t}{2}\,\mathcal{B}_{\bar{F}\xi}(t),\\
\zeta_{\bar{P}\xi}(t) =\ & -\sin\varphi_\xi\sinh\frac{t}{2}\,\mathcal{B}_Z(t) + \left(\cos^2\varphi_\xi\cosh\frac{t}{2} + \sin^2\varphi_\xi\cosh t\right)\mathcal{B}_{\bar{P}\xi}(t)\\
& - \sin\varphi_\xi\cos\varphi_\xi\left(\cosh\frac{t}{2} - \cosh t\right)\mathcal{B}_{\bar{F}\xi}(t),\\
\zeta_{\bar{F}\xi}(t) =\ & 2\sin\varphi_\xi\cos\varphi_\xi\left(\cosh\frac{t}{2}-1\right)\sinh\frac{t}{2}\,\mathcal{B}_{\bar{P}\xi}(t)\\
& + 2\left(1+\cos^2\varphi_\xi\left(\cosh\frac{t}{2}-1\right)\right)\sinh\frac{t}{2}\,\mathcal{B}_{\bar{F}\xi}(t),\\
\zeta_X(t) =\ & 2\sinh\frac{t}{2}\,\mathcal{B}_X(t), \qquad \text{if } X \in \nu W_{\mathfrak{w}} \ominus (\mathbb{R}\xi \oplus \mathbb{R}\bar{F}\xi).
\end{aligned}$$

We define the endomorphism $D(r)$ of $T_{\gamma_\xi(r)}M^r$ by $D(r)\mathcal{B}_X(r) = \zeta_X(r)$ for each $X \in T_p\mathbb{C}H^n \ominus \mathbb{R}\xi$. If $\varphi_\xi = 0$ (that is, $P\xi=0$) then $D(r)$ can be represented by a diagonal matrix with blocks $\cosh(t/2)I_{2n-k}$, $2\sinh(t/2)I_{k-2}$, and $2\sinh(t/2)\cosh(t/2)I_1$, the last one corresponding to the vector $\bar{F}\xi = J\xi$. If $\varphi_\xi = \pi/2$ (that is, $F\xi = 0$) then $D(r)$ has two diagonal blocks $\cosh(t/2)I_{2n-k-2}$ and $2\sinh(t/2)I_{k-1}$, and a 2×2 block corresponding to the vectors Z, and $\bar{P}\xi = J\xi$. If $\varphi_\xi \in (0,\pi/2)$ then $D(r)$ has two diagonal blocks

$\cosh(t/2)I_{2n-k-2}$ and $2\sinh(t/2)I_{k-2}$, and another 3×3 block corresponding to the vectors Z, $\bar{P}\xi$, and $\bar{F}\xi$.

It is well known that if $D(r)$ is nonsingular for each $\xi \in \nu W_{\mathfrak{w}}$, then M^r is a hypersurface of $\mathbb{C}H^n$. In our case, regardless of the value of φ_ξ, we have

$$\det(D(r)) = 2^{k-1}\left(\cosh\frac{r}{2}\right)^{2n-k+1}\left(\sinh\frac{r}{2}\right)^{k-1},$$

and hence, M^r is a hypersurface for every $r > 0$. In this situation, Jacobi field theory shows that the shape operator of M^r at $\gamma_\xi(r)$ with respect to $-\dot{\gamma}_\xi(r)$ is given by $\mathcal{S}^r = D'(r)D(r)^{-1}$. Therefore, the mean curvature $\mathcal{H}^r$ of M^r is

$$\mathcal{H}^r(\gamma_\xi(r)) = \operatorname{tr}\mathcal{S}^r(\gamma_\xi(r)) = \frac{\frac{d}{dr}\det(D(r))}{\det(D(r))} = \frac{1}{2\sinh\frac{r}{2}\cosh\frac{r}{2}}\left(k-1+2n\sinh^2\frac{r}{2}\right).$$

Notice again that this value does not depend on the unit vector $\xi \in \nu W_{\mathfrak{w}}$. Therefore, for every $r > 0$, the tube M^r around $W_{\mathfrak{w}}$ is a hypersurface with constant mean curvature, and hence, tubes around the submanifold $W_{\mathfrak{w}}$ constitute an isoparametric family of hypersurfaces in $\mathbb{C}H^n$, that is, every tube M^r is an isoparametric hypersurface.

Moreover, it was proved in [2] that the tubes around $W_{\mathfrak{w}}$ are homogeneous precisely when $\mathfrak{w}^\perp$ has constant Kähler angle, that is, when φ_ξ is independent of the vector $\xi \in \mathfrak{w}^\perp$. Indeed, the Berndt-Brück submanifolds W^{2n-k}_φ [3] are precisely those $W_{\mathfrak{w}}$ for which $\mathfrak{w}^\perp$ has *constant* Kähler angle φ and $k = \dim\mathfrak{w}^\perp$. We summarize all this in the following

Theorem 3.1. *Let $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the Lie algebra of the isometry group $G = SU(1,n)$ of $\mathbb{C}H^n$ with respect to a point $o \in \mathbb{C}H^n$. Assume $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace and let $\mathfrak{g} = \mathfrak{g}_{-2\alpha}\oplus\mathfrak{g}_{-\alpha}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}$ be the root space decomposition with respect to $\mathfrak{a}$. Let $W_{\mathfrak{w}}$ be the orbit through o of the connected subgroup $S_{\mathfrak{w}}$ of G whose Lie algebra is $\mathfrak{s}_{\mathfrak{w}} = \mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{g}_{2\alpha}$, where $\mathfrak{w}$ is any proper subspace of $\mathfrak{g}_\alpha$.*

Then, the tubes around the submanifold $W_{\mathfrak{w}}$ are isoparametric hypersurfaces of $\mathbb{C}H^n$, and are homogeneous if and only if $\mathfrak{w}^\perp = \mathfrak{g}_\alpha\ominus\mathfrak{w}$ has constant Kähler angle.

It is feasible to calculate the shape operator $\mathcal{S}^r$ of the tube M^r at $\gamma_\xi(r)$ from the formula $\mathcal{S}^r = D'(r)D(r)^{-1}$, although calculations are very long. We do not give $\mathcal{S}^r$ but mention that its characteristic polynomial is

$$p_{r,\xi}(x) = (\lambda - x)^{2n-k-2}\left(\frac{1}{4\lambda} - x\right)^{k-2} q_{r,\xi}(x),$$

where $\lambda = \frac{1}{2}\tanh\frac{r}{2}$ and

$$q_{r,\xi}(x) = -x^3 + \left(3\lambda + \frac{1}{4\lambda}\right)x^2 - \frac{1}{2}\left(6\lambda^2+1\right)x + \frac{16\lambda^4 + 16\lambda^2 - 1 + (4\lambda^2-1)^2\cos 2\varphi_\xi}{32\lambda}.$$

(This is the same as [3, p. 146].)

It is important to remark that, *pointwise*, M^r has the same principal curvatures as the tubes around the Berndt-Brück submanifolds W^{2n-k}_φ, $\varphi \in [0,\pi/2]$, $k \in \{1,\dots,n-1\}$ (see [3]); notice that for $\varphi = 0$ these are tubes around a totally geodesic $\mathbb{C}H^k$, $k \in$

$\{1, \ldots, n-1\}$ in $\mathbb{C}H^n$. In other words, at each point, the tubes around $W_{\mathfrak{w}}$ have the same principal curvatures (with the same multiplicities) as the homogeneous hypersurfaces that arise as tubes around the W_{φ}^{2n-k}. However, in general, the principal curvatures, and even the number of principal curvatures, vary from point to point in M^r. Again, *the principal curvatures of M^r are constant if and only if $\mathfrak{w}^\perp$ has constant Kähler angle*, that is, if φ_ξ does not depend on ξ; this corresponds precisely to the homogeneous examples constructed by Berndt and Brück [2].

If $n = 2$, then either $\mathfrak{w}^\perp$ is 1-dimensional, in which case $W_{\mathfrak{w}}$ is the Lohnherr hypersurface W^{2n-1}, whose equidistant hypersurfaces are homogeneous [1], or $\mathfrak{w}^\perp = \mathfrak{g}_\alpha$, which gives a totally geodesic $\mathbb{C}H^1$ and thus the tubes around it are also homogeneous. In any case, for $n = 2$ we do not get inhomogeneous examples. However, for $n \geq 3$ this construction yields inhomogeneous hypersurfaces in $\mathbb{C}H^n$, for appropriate choices of $\mathfrak{w}$.

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