# Total curvatures of geodesic spheres associated to quadratic curvature invariants[1]

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*Dedicated to Professor L. Vanhecke*

**Abstract**

Total scalar curvatures of geodesic spheres obtained by integrating the second order linear invariants of the curvature tensor are investigated. The first terms in their power series expansions are derived and these results are used to characterize the two-point homogeneous spaces among Riemannian manifolds with adapted holonomy.

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# 1 Introduction

The main purpose of this paper is to contribute to the study of the following problem:

> *To what extent do the properties of sufficiently small geodesic spheres determine the Riemannian geometry of the ambient space?*

This program, originated mainly from the study of harmonic manifolds has been intensively studied over the last years, with special attention to the relation between the curvature of a Riemannian manifold and the volume of sufficiently small geodesic spheres. Indeed, it was conjectured by A. Gray and L. Vanhecke that the two-point homogeneous spaces should be locally detected by the volumes of their geodesic spheres [10]. Here, it is worth mentioning that although the answer is known to be true for many important cases, it is still an open problem. Other kinds of geometric objects, related to the extrinsic geometry of geodesic spheres (the shape operator, the mean curvature, the quadratic norm of the second fundamental form, etc.) have also been studied [8]. Moreover, many intrinsic analytic and algebraic objects of geodesic spheres have been used to understand the geometry of the ambient space. It was shown in [8] that two-point homogeneous spaces can be characterized by comparing the spectrum of the geodesic spheres with that of the geodesic spheres in these model spaces. Also, Ricci and Weyl tensors, as well as scalar curvature and quadratic curvature invariants were investigated by B. Y. Chen and L. Vanhecke in [8].

Motivated by the volume conjecture of A. Gray and L. Vanhecke above, many attempts have been done in searching for geometric quantities which would allow to characterize the two-point homogeneous spaces (which will be called model spaces) among Riemannian manifolds with holonomy group adapted to the model space. However, in all such characterizations two quantities are required. In fact, any combination of the volume, the total scalar curvature and the $L^2$-norm of the second fundamental form lead to the desired result.

Our purpose in this paper is to complete the previous studies by considering quadratic curvature invariants and investigating the corresponding $L^2$-norms on geodesic spheres. In doing that, we compute the first terms in their power series expansions. Several conclusions are obtained from those coefficients, specially we remark that *the $L^2$-norm of the curvature tensor of geodesic spheres suffices to characterize the model spaces.*

The paper is organized as follows. In Section 2 we recall some notation and use the Gauss equation of the geodesic spheres to write down the first terms in the power series expansions of their curvature tensor (Lemma 3). This allows us to obtain the power series expansions of $\widetilde{\tau}^2$, $\|\widetilde{\rho}\|^2$ and $\|\widetilde{R}\|^2$. The first terms in the power series expansions of the corresponding total invariants are derived in §3 and Section 4 is devoted to point out some applications of those expressions. Here we would like to emphasize that we tried to keep calculations at minimum, deleting almost all of them in order to make the paper more readable.

## 2 Preliminaries

Let $(M^n, g)$ be an $n$-dimensional smooth Riemannian manifold of class $C^\infty$. We will denote by $\nabla$ the Levi–Civita connection and put $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for the curvature tensor, where $X$, $Y$ are vector fields on $M$. Also, $R_{XYZW} = g(R_{XY}Z, W)$ and the Ricci tensor and the scalar curvature are given by $\rho_{XY} = \sum_{i=1}^n R_{Xe_iYe_i}$ and $\tau = \sum_{i=1}^n \rho_{e_ie_i}$, with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$. For simplicity, here and in what follows, we use the notation $\rho_{ij} = \rho_{e_ie_j}$, $R_{ijkl} = R_{e_ie_je_ke_l}$, $\nabla_{ijk\ldots} = \nabla_{e_ie_je_k\ldots}$ and so on. Also, by $\sum_{c,d\ldots} \cdots$ we mean $\sum_{c,d,\ldots=1}^n \cdots$.

A *scalar curvature invariant* is a polinomial in the components of the curvature tensor that does not depend on the choice of the orthonormal basis used to build it. The *order* of a scalar curvature invariant is, by definition, the number of derivatives of the metric tensor involved in it. Let $I(k,n)$ the vector space of curvature invariants of order $2k$, $m \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of the tangent space at $m$, $T_mM$.

For $n \geq 2$, $I(1,n)$ has dimension 1 and is generated by $\tau$. For $n \geq 4$, $I(2,n)$ has dimension 4 and a basis is given by

$$(1)\qquad \tau^2 \qquad \|\rho\|^2 = \sum \rho_{ij}^2 \qquad \|R\|^2 = \sum R_{ijkl}^2 \qquad \Delta\tau = \sum \nabla_{ii}^2\tau,$$

For $n \geq 6$, $I(3,n)$ is generated by the following basis [10]:

$$(2)\qquad \begin{array}{ll}
\tau^3 \qquad \tau\|\rho\|^2 & \tau\|R\|^2 \\
\check{\rho} = \sum \rho_{ij}\rho_{jk}\rho_{ik} & \langle \rho\otimes\rho, \overline{R}\rangle = \sum \rho_{ij}\rho_{kl}R_{ikjl} \\
\langle \rho, \dot{R}\rangle = \sum \rho_{ij}R_{iklr}R_{jklr} & \check{R} = \sum R_{ijkl}R_{ijrs}R_{klrs} \\
\breve{\check{R}} = \sum R_{ijkl}R_{irks}R_{jrls} & \|\nabla\tau\|^2 = \sum(\nabla_i\tau)^2 \\
\|\nabla\rho\|^2 = \sum(\nabla_i\rho_{jk})^2 & \alpha(\rho) = \sum \nabla_i\rho_{jk}\nabla_j\rho_{ik} \\
\|\nabla R\|^2 = \sum(\nabla_i R_{jklr})^2 & \langle \Delta\rho, \rho\rangle = \sum \rho_{ij}\nabla_{kk}^2\rho_{ij} \\
\langle \nabla^2\tau, \rho\rangle = \sum(\nabla_{ij}^2\tau)\,\rho_{ij} & \langle \Delta R, R\rangle = \sum R_{ijkl}\nabla_{rr}^2 R_{ijkl} \\
\tau\Delta\tau & \Delta^2\tau
\end{array}$$

### 2.1 Intrinsic geometry of geodesic spheres

In what follows, we denote by $G_m(r)$ the *geodesic sphere* with center $m$ and radius $r$, that is $G_m(r) = \{p \in M/\, d(m,p) = r\}$. It is always supposed that $r < i(m)$, the injectivity radius at the point $m \in M$. Due to this, $G_m(r)$ is a hypersurface of $M$ and moreover $G_m(r) = \exp_m(S^{n-1}(r))$, where $S^{n-1}(r) = \{x \in T_mM/\, \|x\| = r\}$ is the sphere of radius $r$ in the tangent space to $M$ at $m$.

¿From now on we are interested in studying the intrinsic geometry of geodesic spheres and its relation to the geometry of the ambient manifold. The Gauss equation provides us an explicit relation between the curvature tensor $\widetilde{R}$ of the submanifold and the curvature tensor $R$ of the ambient manifold by means of the second fundamental form $\sigma$:

$$\widetilde{R}_{XYZW} = R_{XYZW} + \sigma(X,Z)\sigma(Y,W) - \sigma(X,W)\sigma(Y,Z) \tag{3}$$

where $X, Y, Z, W$ are vector fields on $G_m(r)$. Further, let $(x_1, \ldots, x_n)$ be a system of normal coordinates on $M$ in a neighborhood around $m$. For our purposes, the following expansion of the components of the second fundamental form will be needed

**Lemma 1** [8] *We have the following series expansion for the second fundamental form of a geodesic sphere at $p = exp_m(ru)$:*

$$\begin{aligned}\sigma_{ij}(p) = \Big\{ & \tfrac{1}{r}\delta_{ij} - \tfrac{r}{3}R_{iuju} - \tfrac{r^2}{4}\nabla_u R_{iuju} - r^3\left(\tfrac{1}{10}\nabla^2_{uu}R_{iuju} + \tfrac{1}{45}\textstyle\sum_c R_{cuiu}R_{cuju}\right)\\ & - \tfrac{r^4}{24}\left(\tfrac{2}{3}\nabla^3_{uuu}R_{iuju} + \tfrac{1}{3}\textstyle\sum_c R_{cuju}\nabla_u R_{cuiu} + \tfrac{1}{3}\textstyle\sum_c R_{cuiu}\nabla_u R_{cuju}\right)\\ & - \tfrac{r^5}{720}\Big(\tfrac{30}{7}\nabla^4_{uuuu}R_{iuju} + \tfrac{45}{7}\textstyle\sum_c \nabla_u R_{cuiu}\nabla_u R_{cuju}\\ & \qquad + \tfrac{24}{7}\textstyle\sum_c R_{cuju}\nabla^2_{uu}R_{cuiu} + \tfrac{24}{7}\textstyle\sum_c R_{cuiu}\nabla^2_{uu}R_{cuju}\\ & \qquad + \tfrac{32}{21}\textstyle\sum_{c,d} R_{cudu}R_{cuiu}R_{duju}\Big)\Big\}(m) + O(r^6)\end{aligned}$$

Next, recall that the relation, at every point $p = \exp_m(ru)$, between the mean curvature $h_m$ of the geodesic sphere centered at $m$ and the the volume density function $\theta_m$ (with respect to normal coordinates at $m$) is ([7, p. 193])

$$h_m(p) = \frac{n-1}{r} + \frac{\partial}{\partial r}\log\theta_m(exp_m(ru)). \tag{4}$$

Now, one has

**Lemma 2** [8] *We have the following series expansion for the volume density function at $p = exp_m(ru)$:*

$$\begin{aligned}\theta_m(p) = \Big\{ & 1 - \tfrac{r^2}{6}\rho_{uu}(m) - \tfrac{r^3}{12}\nabla_u\rho_{uu}\\ & + \tfrac{r^4}{24}\left(-\tfrac{3}{5}\nabla^2_{uu}\rho_{uu} + \tfrac{1}{3}\rho^2_{uu} - \tfrac{2}{15}\textstyle\sum_{c,d} R^2_{cudu}\right)\\ & + \tfrac{r^5}{120}\left(-\tfrac{2}{3}\nabla^3_{uuu}\rho_{uu} + \tfrac{5}{3}\rho_{uu}\nabla_u\rho_{uu} - \tfrac{2}{3}\textstyle\sum_{c,d} R_{cudu}\nabla_u R_{cudu}\right)\\ & + \tfrac{r^6}{720}\Big(-\tfrac{5}{7}\nabla^4_{uuuu}\rho_{uu} + 3\rho_{uu}\nabla^2_{uu}\rho_{uu} + \tfrac{5}{2}\left(\nabla_u\rho_{uu}\right)^2 + \tfrac{2}{3}\rho_{uu}\textstyle\sum_{c,d} R^2_{cudu}\\ & \qquad - \tfrac{5}{9}\rho^3_{uu} - \tfrac{8}{7}\textstyle\sum_{c,d} R_{cudu}\nabla^2_{uu}R_{cudu} - \tfrac{15}{14}\textstyle\sum_{c,d}\left(\nabla_u R_{cudu}\right)^2\\ & \qquad - \tfrac{16}{63}\textstyle\sum_{c,d,e} R_{cudu}R_{cueu}R_{dueu}\Big)\Big\}(m) + O(r^7)\end{aligned}$$

Thus, proceeding as in [8, Theorem 4.2] one obtains the first terms in the power series expansion of the curvature tensor of a small geodesic sphere as follows

**Lemma 3** *Let $\widetilde{R}$ be the curvature tensor of a small geodesic sphere $G_m(r)$ and $p = exp_m(ru)$. Then:*

$$\widetilde{R}_{ijkl}(p) = \widetilde{R}_{-2}r^{-2} + \widetilde{R}_0(m) + \widetilde{R}_1(m)r + \widetilde{R}_2(m)r^2 + \widetilde{R}_3(m)r^3 + \widetilde{R}_4(m)r^4 + O(r^5)$$

*where*

$$\widetilde{R}_{-2} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

$$\widetilde{R}_0 = R_{ijkl} + \tfrac{1}{3}\left(\delta_{il}R_{juku} - \delta_{ik}R_{julu} + \delta_{jk}R_{iulu} - \delta_{jl}R_{iuku}\right)$$

$$\widetilde{R}_1 = \nabla_u R_{ijkl} + \tfrac{1}{4}\left(\delta_{il}\nabla_u R_{juku} - \delta_{ik}\nabla_u R_{julu} + \delta_{jk}\nabla_u R_{iulu} - \delta_{jl}\nabla_u R_{iuku}\right)$$

$$\begin{aligned}\widetilde{R}_2 = {}& \tfrac{1}{2}\nabla^2_{uu}R_{ijkl} + \tfrac{1}{10}\big(\delta_{il}\nabla^2_{uu}R_{juku} - \delta_{ik}\nabla^2_{uu}R_{julu} + \delta_{jk}\nabla^2_{uu}R_{iulu}\\ & -\delta_{jl}\nabla^2_{uu}R_{iuku}\big) + \tfrac{1}{45}\textstyle\sum_c\left(\delta_{il}R_{cuju}R_{cuku} - \delta_{ik}R_{cuju}R_{culu}\right.\\ & \left.+\delta_{jk}R_{cuiu}R_{culu} - \delta_{jl}R_{cuiu}R_{cuku}\right) + \tfrac{1}{9}\left(R_{iuku}R_{julu} - R_{iulu}R_{juku}\right)\end{aligned}$$

$$\begin{aligned}\widetilde{R}_3 = {}& \tfrac{1}{6}\nabla^3_{uuu}R_{ijkl} + \tfrac{1}{36}\big(\delta_{il}\nabla^3_{uuu}R_{juku} - \delta_{ik}\nabla^3_{uuu}R_{julu} + \delta_{jk}\nabla^3_{uuu}R_{iulu}\\ & -\delta_{jl}\nabla^3_{uuu}R_{iuku}\big) + \tfrac{1}{72}\textstyle\sum_c\left(\delta_{il}R_{cuku}\nabla_u R_{cuju} - \delta_{ik}R_{culu}\nabla_u R_{cuju}\right.\\ & + \delta_{il}R_{cuju}\nabla_u R_{cuku} - \delta_{ik}R_{cuju}\nabla_u R_{culu} + \delta_{jk}R_{cuiu}\nabla_u R_{culu}\\ & \left.+\delta_{jk}R_{culu}\nabla_u R_{cuiu} - \delta_{jl}R_{cuiu}\nabla_u R_{cuku} - \delta_{jl}R_{cuku}\nabla_u R_{cuiu}\right)\\ & + \tfrac{1}{12}(R_{iuku}\nabla_u R_{julu} - R_{iulu}\nabla_u R_{juku} - R_{juku}\nabla_u R_{iulu} + R_{julu}\nabla_u R_{iuku})\end{aligned}$$

$$\begin{aligned}\widetilde{R}_4 = {}& \tfrac{1}{24}\nabla^4_{uuuu}R_{ijkl} - \tfrac{1}{168}\big(\delta_{ik}\nabla^4_{uuuu}R_{julu} - \delta_{il}\nabla^4_{uuuu}R_{juku}\\ & -\delta_{jk}\nabla^4_{uuuu}R_{iulu} + \tfrac{1}{168}\delta_{jl}\nabla^4_{uuuu}R_{iuku}\big) + \tfrac{1}{30}\big(R_{iuku}\nabla^2_{uu}R_{julu}\\ & -R_{iulu}\nabla^2_{uu}R_{juku} - R_{juku}\nabla^2_{uu}R_{iulu} + R_{julu}\nabla^2_{uu}R_{iuku}\big)\\ & - \tfrac{1}{210}\textstyle\sum_c\big(\delta_{ik}R_{cuju}\nabla^2_{uu}R_{culu} + \delta_{ik}R_{culu}\nabla^2_{uu}R_{cuju} - \delta_{il}R_{cuju}\nabla^2_{uu}R_{cuku}\\ & -\delta_{il}R_{cuku}\nabla^2_{uu}R_{cuju} - \delta_{jk}R_{cuiu}\nabla^2_{uu}R_{culu} - \delta_{jk}R_{culu}\nabla^2_{uu}R_{cuiu}\\ & +\delta_{jl}R_{cuiu}\nabla^2_{uu}R_{cuku} + \delta_{jl}R_{cuku}\nabla^2_{uu}R_{cuiu}\big) - \tfrac{1}{16}\left(\nabla_u R_{iulu}\nabla_u R_{juku}\right.\\ & \left.-\nabla_u R_{iuku}\nabla_u R_{julu}\right) - \tfrac{1}{112}\textstyle\sum_c\left(\delta_{ik}\nabla_u R_{cuju}\nabla_u R_{culu}\right.\\ & \left.-\delta_{il}\nabla_u R_{cuju}\nabla_u R_{cuku} - \delta_{jk}\nabla_u R_{cuiu}\nabla_u R_{culu} + \delta_{jl}\nabla_u R_{cuiu}\nabla_u R_{cuku}\right)\\ & + \tfrac{1}{135}\textstyle\sum_c\left(R_{julu}R_{cuiu}R_{cuku} - R_{iulu}R_{cuju}R_{cuku} - R_{juku}R_{cuiu}R_{culu}\right.\\ & \left.+R_{iuku}R_{cuju}R_{culu}\right) - \tfrac{2}{945}\textstyle\sum_{c,d}\left(\delta_{ik}R_{cudu}R_{culu}R_{duju}\right.\\ & \left.-\delta_{il}R_{cudu}R_{cuku}R_{duju} - \delta_{jk}R_{cudu}R_{culu}R_{duiu} + \delta_{jl}R_{cudu}R_{cuku}R_{duiu}\right)\end{aligned}$$

Next lemmas show the first six terms in the power series expansions of the quadratic norms of the curvature and Ricci tensors and the quadratic scalar curvature of geodesic spheres. Although the first four terms were previously obtained in the work of B.Y. Chen and L. Vanhecke [8], we need those corresponding to degree one and two in order to write down the power series expansions of the corresponding total curvatures in the next section (Theorems 8, 10 and 11). All of them are obtained from Lemma 3 after some tedious but straightforward calculations.

**Lemma 4** *For any sufficiently small geodesic sphere $G_m(r)$ of a Riemannian manifold $M^n$, the quadratic norm of the curvature tensor of $G_m(r)$ at $p = exp_m(ru)$ satisfies*

$$\|\widetilde{R}\|^2(p) = \widetilde{\beta}_{-4}r^{-4}+\widetilde{\beta}_{-2}(m)r^{-2}+\widetilde{\beta}_{-1}(m)r^{-1}+\widetilde{\beta}_0(m)+\widetilde{\beta}_1(m)r+\widetilde{\beta}_2(m)r^2+O(r^3)$$

*where*

$$\widetilde{\beta}_{-4} = 2(n-1)(n-2)$$

$$\widetilde{\beta}_{-2} = 4\left(\tau - \tfrac{2(n+1)}{3}\rho_{uu}\right)$$

$$\widetilde{\beta}_{-1} = 4\left(\nabla_u\tau - \tfrac{n+2}{2}\nabla_u\rho_{uu}\right)$$

$$\begin{aligned}\widetilde{\beta}_0 =\; & 2\nabla^2_{uu}\tau - \tfrac{4(n+3)}{5}\nabla^2_{uu}\rho_{uu} + \tfrac{8}{9}\rho^2_{uu} - \tfrac{8}{3}\textstyle\sum_{c,d}\rho_{cd}R_{cudu}\\ & +\|R\|^2 - 4\textstyle\sum_{c,d,e}R^2_{cdeu} + \tfrac{4(3n+59)}{45}\textstyle\sum_{c,d}R^2_{cudu}\end{aligned}$$

$$\begin{aligned}\widetilde{\beta}_1 =\; & \tfrac{2}{3}\nabla^3_{uuu}\tau - \tfrac{2(n+4)}{9}\nabla^3_{uuu}\rho_{uu} + 2\textstyle\sum_{c,d,e,f}R_{cdef}\nabla_uR_{cdef}\\ & -2\textstyle\sum_{c,d}\rho_{cd}\nabla_uR_{cudu} - \tfrac{8}{3}\textstyle\sum_{c,d}R_{cudu}\nabla_u\rho_{cd} + \tfrac{4}{3}\rho_{uu}\nabla_u\rho_{uu}\\ & -8\textstyle\sum_{c,d,e}R_{cdeu}\nabla_uR_{cdeu} + \tfrac{2(2n+47)}{9}\textstyle\sum_{c,d}R_{cudu}\nabla_uR_{cudu}\end{aligned}$$

$$\begin{aligned}\widetilde{\beta}_2 =\; & \tfrac{1}{6}\nabla^4_{uuuu}\tau - \tfrac{n+5}{21}\nabla^4_{uuuu}\rho_{uu} + \tfrac{8}{15}\rho_{uu}\nabla^2_{uu}\rho_{uu}\\ & -\tfrac{4}{5}\textstyle\sum_{c,d}\rho_{cd}\nabla^2_{uu}R_{cudu} - \tfrac{4}{3}\textstyle\sum_{c,d}R_{cudu}\nabla^2_{uu}\rho_{cd} - \tfrac{8}{45}\rho_{uu}\textstyle\sum_{c,d}R^2_{cudu}\\ & -4\textstyle\sum_{c,d,e}R_{cdeu}\nabla^2_{uu}R_{cdeu} + \tfrac{4(5n+137)}{105}\textstyle\sum_{c,d}R_{cudu}\nabla^2_{uu}R_{cudu}\\ & -2\textstyle\sum_{c,d}\nabla_u\rho_{cd}\nabla_uR_{cudu} + \textstyle\sum_{c,d,e,f}(\nabla_uR_{cdef})^2 - 4\textstyle\sum_{c,d,e}(\nabla_uR_{cdeu})^2\\ & +\tfrac{5n+144}{28}\textstyle\sum^n_{c,d}(\nabla_uR_{cudu})^2 + \textstyle\sum_{c,d,e,f}R_{cdef}\nabla^2_{uu}R_{cdef}\\ & -\tfrac{8}{45}\textstyle\sum_{c,d,e}\rho_{cd}R_{cueu}R_{dueu} + \tfrac{4}{9}\textstyle\sum_{c,d,e,f}R_{cdef}R_{cueu}R_{dufu}\\ & +\tfrac{8(5n+32)}{945}\textstyle\sum_{c,d,e}R_{cudu}R_{cueu}R_{dueu} + \tfrac{1}{2}(\nabla_u\rho_{uu})^2\end{aligned}$$

**Lemma 5** *For any sufficiently small geodesic sphere $G_m(r)$ of a Riemannian manifold $M^n$, the quadratic norm of the Ricci tensor of $G_m(r)$ at $p = exp_m(ru)$ satisfies*

$$\|\widetilde{\rho}\|^2(p) = \widetilde{\varrho}_{-4}r^{-4}+\widetilde{\varrho}_{-2}(m)r^{-2}+\widetilde{\varrho}_{-1}(m)r^{-1}+\widetilde{\varrho}_0(m)+\widetilde{\varrho}_1(m)r+\widetilde{\varrho}_2(m)r^2+O(r^3)$$

*where*

$$\begin{aligned}
\widetilde{\varrho}_{-4} &= (n-1)(n-2)^2 \\
\widetilde{\varrho}_{-2} &= 2(n-2)\left(\tau - \tfrac{2(n+1)}{3}\rho_{uu}\right) \\
\widetilde{\varrho}_{-1} &= 2(n-2)\left(\nabla_u\tau - \tfrac{n+2}{2}\nabla_u\rho_{uu}\right) \\
\widetilde{\varrho}_{0} &= (n-2)\nabla^2_{uu}\tau - \tfrac{2(n-2)(n+3)}{5}\nabla^2_{uu}\rho_{uu} - \tfrac{2}{3}\tau\rho_{uu} + \|\rho\|^2 - 2\textstyle\sum_c \rho^2_{cu} \\
&\quad + \tfrac{5(n+2)}{9}\rho^2_{uu} - \tfrac{2n}{3}\textstyle\sum_{c,d}\rho_{cd}R_{cudu} + \tfrac{n^2+6n+4}{45}\textstyle\sum_{c,d}R^2_{cudu} \\
\widetilde{\varrho}_{1} &= \tfrac{n-2}{3}\nabla^3_{uuu}\tau - \tfrac{(n-2)(n+4)}{9}\nabla^3_{uuu}\rho_{uu} - \tfrac{2}{3}\rho_{uu}\nabla_u\tau \\
&\quad - \tfrac{1}{2}\tau\nabla_u\rho_{uu} + 2\textstyle\sum_{c,d}\rho_{cd}\nabla_u\rho_{cd} - 4\textstyle\sum_{c=1}^n \rho_{cu}\nabla_u\rho_{cu} \\
&\quad - \tfrac{n+1}{2}\textstyle\sum_{c,d}\rho_{cd}\nabla_u R_{cudu} - \tfrac{2n}{3}\textstyle\sum_{c,d}R_{cudu}\nabla_u\rho_{cd} \\
&\quad + \tfrac{(n+1)(n+4)}{18}\textstyle\sum_{c,d}R_{cudu}\nabla_u R_{cudu} + \tfrac{5(n+3)}{6}\rho_{uu}\nabla_u\rho_{uu} \\
\widetilde{\varrho}_{2} &= \tfrac{n-2}{12}\nabla^4_{uuuu}\tau - \tfrac{(n-2)(n+5)}{42}\nabla^4_{uuuu}\rho_{uu} - \tfrac{1}{5}\tau\nabla^2_{uu}\rho_{uu} \\
&\quad + \textstyle\sum_{c,d}\rho_{cd}\nabla^2_{uu}\rho_{cd} - 2\textstyle\sum_c \rho_{cu}\nabla^2_{uu}\rho_{cu} + \tfrac{n+4}{3}\rho_{uu}\nabla^2_{uu}\rho_{uu} \\
&\quad - \tfrac{n+2}{5}\textstyle\sum_{c,d}\rho_{cd}\nabla^2_{uu}R_{cudu} - \tfrac{n}{3}\textstyle\sum_{c,d}R_{cudu}\nabla^2_{uu}\rho_{cd} - \tfrac{1}{3}\rho_{uu}\nabla^2_{uu}\tau \\
&\quad + \tfrac{3n^2+16n+12}{105}\textstyle\sum_{c,d}R_{cudu}\nabla^2_{uu}R_{cudu} - \tfrac{1}{2}\nabla_u\tau\nabla_u\rho_{uu} \\
&\quad + \tfrac{5n+21}{16}\left(\nabla_u\rho_{uu}\right)^2 + \tfrac{3n^2+16n+19}{112}\textstyle\sum_{c,d}\left(\nabla_u R_{cudu}\right)^2 \\
&\quad - 2\textstyle\sum_c\left(\nabla_u\rho_{cu}\right)^2 + \textstyle\sum_{c,d}\left(\nabla_u\rho_{cd}\right)^2 - \tfrac{n+1}{2}\textstyle\sum_{c,d}\nabla_u\rho_{cd}\nabla_u R_{cudu} \\
&\quad - \tfrac{2}{45}\tau\textstyle\sum_{c,d}R^2_{cudu} + \tfrac{2(3n^2+16n+12)}{945}\textstyle\sum_{c,d,e}R_{cudu}R_{cueu}R_{dueu} \\
&\quad - \tfrac{2(n+2)}{45}\textstyle\sum_{c,d,e}\rho_{cd}R_{cueu}R_{dueu} - \tfrac{2}{27}\rho^3_{uu} + \tfrac{2}{9}\textstyle\sum_{c,d}\rho_{uu}\rho_{cd}R_{cudu}
\end{aligned}$$

**Lemma 6** *For any sufficiently small geodesic sphere $G_m(r)$ of a Riemannian manifold $M^n$, the quadratic scalar curvature $\widetilde{\tau}^2$ of $G_m(r)$ at $p = exp_m(ru)$ satisfies*

$$\widetilde{\tau}^2(p) = \widetilde{t}_{-4}r^{-4} + \widetilde{t}_{-2}(m)r^{-2} + \widetilde{t}_{-1}(m)r^{-1} + \widetilde{t}_0(m) + \widetilde{t}_1(m)r + \widetilde{t}_2(m)r^2 + O(r^3)$$

*where*

$$\begin{aligned}
\widetilde{t}_{-4} &= (n-1)^2(n-2)^2 \\
\widetilde{t}_{-2} &= 2(n-2)(n-1)\left(\tau - \tfrac{2(n+1)}{3}\rho_{uu}\right) \\
\widetilde{t}_{-1} &= 2(n-2)(n-1)\left(\nabla_u\tau - \tfrac{n+2}{2}\nabla_u\rho_{uu}\right) \\
\widetilde{t}_{0} &= (n-1)(n-2)\nabla^2_{uu}\tau - \tfrac{2(n-1)(n-2)(n+3)}{5}\nabla^2_{uu}\rho_{uu} + \tau^2 \\
&\quad - \tfrac{4(n+1)}{3}\tau\rho_{uu} + \tfrac{2(3n^2+n+4)}{9}\rho^2_{uu} - \tfrac{2(n-2)(n-1)(2n+1)}{45}\textstyle\sum_{c,d}R^2_{cudu}
\end{aligned}$$

$$
\begin{aligned}
\widetilde{t}_1 &= \tfrac{(n-1)(n-2)}{3}\nabla^3_{uuu}\tau - \tfrac{(n-2)(n-1)(n+4)}{9}\nabla^3_{uuu}\rho_{uu} \\
&\quad +2\tau\nabla_u\tau + (n^2+n+2)\rho_{uu}\nabla_u\rho_{uu} - \tfrac{4(n+1)}{3}\rho_{uu}\nabla_u\tau \\
&\quad -(n+2)\tau\nabla_u\rho_{uu} - \tfrac{(n-1)(n-2)(n+1)}{9}\textstyle\sum_{c,d} R_{cudu}\nabla_u R_{cudu}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{t}_2 &= \tfrac{(n-1)(n-2)}{12}\nabla^4_{uuuu}\tau - \tfrac{(n-1)(n-2)(n+5)}{42}\nabla^4_{uuuu}\rho_{uu} + \tau\nabla^2_{uu}\tau \\
&\quad -\tfrac{2(n+3)}{5}\tau\,\nabla^2_{uu}\rho_{uu} - \tfrac{2(n+1)}{3}\rho_{uu}\nabla^2_{uu}\tau - \tfrac{4(n+1)}{27}\rho^3_{uu} \\
&\quad +\tfrac{2(3n^2+5n+8)}{15}\rho_{uu}\nabla^2_{uu}\rho_{uu} - \tfrac{2(n-1)(n-2)(3n+2)}{105}\textstyle\sum_{c,d} R_{cudu}\nabla^2_{uu}R_{cudu} \\
&\quad +(\nabla_u\tau)^2 - (n+2)\nabla_u\tau\nabla_u\rho_{uu} + \tfrac{3n^2+5n+10}{8}(\nabla_u\rho_{uu})^2 \\
&\quad -\tfrac{(n-1)(n-2)(2n+3)}{56}\textstyle\sum_{c,d}(\nabla_u R_{cudu})^2 + \tfrac{2}{9}\tau\rho^2_{uu} \\
&\quad +\tfrac{4(n^2+1)}{45}\textstyle\sum_{c,d}\rho_{uu}R^2_{cudu} - \tfrac{2(2n+1)}{45}\tau\textstyle\sum^n_{c,d}R^2_{cudu} \\
&\quad -\tfrac{4(n-1)(n-2)(2n+3)}{945}\textstyle\sum_{c,d,e} R_{cudu}R_{cueu}R_{dueu}
\end{aligned}
$$

# 3 Power series expansions of the total quadratic curvature invariants

Our purpose here is to obtain the first terms in the power series expansions of the integrals of the curvature invariants of order two, which will be called total quadratic curvature invariants.

First, note that we will not consider the Laplacian of the scalar curvature since, by the Divergence Theorem, $\int_{G_m(r)}\widetilde{\Delta}\widetilde{\tau}du = 0$. Therefore, we proceed with the other curvature invariants $\widetilde{\tau}^2$, $\|\widetilde{\rho}\|^2$ and $\|\widetilde{R}\|^2$. In what follows $c_{n-1} = \frac{n\pi^{n/2}}{(\frac{n}{2})!}$, $(\frac{n}{2})! = \Gamma(\frac{n}{2}+1)$ stands for the volume of the unit sphere in the Euclidean $n$-space, $c_{n-1} = Vol(S^{n-1})$ (cf. [9]).

**Definition 7** Define the $L^2$-*norm of the curvature tensor* of a geodesic sphere $G_m(r)$ by

$$\int_{G_m(r)}\|\widetilde{R}\|^2 \tag{5}$$

**Theorem 8** *Let $(M,g)$ be a Riemannian manifold and $m\in M$. Then*

$$\int_{G_m(r)}\|\widetilde{R}\|^2 = c_{n-1}\,r^{n-1}\left\{A_{-4}r^{-4} + A_{-2}r^{-2} + A_0 + A_2r^2 + O\left(r^3\right)\right\}$$

*where*

$$
\begin{aligned}
A_{-4} &= 2\,(n-2)\,(n-1) \\
A_{-2} &= -\tfrac{(n-5)(n-2)}{3n}\,\tau(m) \\
A_0 &= \tfrac{1}{n\,(n+2)}\left\{\tfrac{59n^2-93n-10}{60}\,\|R\|^2 + \tfrac{2\left(n^2-37n+60\right)}{45}\,\|\rho\|^2\right. \\
&\qquad\left. + \tfrac{n^2-11n+2}{36}\,\tau^2 - \tfrac{(n-5)(n-2)}{10}\,\Delta\tau\right\}(m)
\end{aligned}
$$

$$
\begin{aligned}
A_2 \;=\; & \frac{1}{n\,(n+2)\,(n+4)}\Big\{-\tfrac{n^2-15n-22}{648}\,\tau^3-\tfrac{n^2-41n+4}{135}\,\tau\|\rho\|^2\\
&-\tfrac{59n^2+151n+126}{360}\,\tau\|R\|^2+\tfrac{8\left(n^2+357n-1705\right)}{2835}\,\breve{\rho}\\
&-\tfrac{4\left(2n^2+49n-960\right)}{945}\,\langle\rho\otimes\rho,\overline{R}\rangle+\tfrac{4\left(n^2+427n-95\right)}{315}\,\langle\rho,\dot{R}\rangle\\
&-\tfrac{11n^2+4851n-1682}{2268}\,\breve{R}-\tfrac{5n^2+4473n-2414}{567}\,\breve{\breve{R}}\\
&+\tfrac{(n-2)(n-5)}{56}\,\|\nabla\tau\|^2+\tfrac{n^2-1239n+626}{112}\,\|\nabla\rho\|^2\\
&+\tfrac{n^2+553n-270}{56}\,\alpha(\rho)+\tfrac{3\left(37n^2+21n-22\right)}{112}\,\|\nabla R\|^2\\
&+\tfrac{n^2-11n-6}{60}\,\tau\Delta\tau+\tfrac{2\left(n^2-133n+220\right)}{105}\,\langle\Delta\rho,\rho\rangle-\tfrac{(n-5)\,(n-2)}{56}\,\Delta^2\tau\\
&+\tfrac{9n^2+273n-470}{420}\,\langle\nabla^2\tau,\rho\rangle+\tfrac{83n^2-133n-10}{84}\,\langle\Delta R,R\rangle\Big\}\,(m)
\end{aligned}
$$

**Proof.** The $L^2$-norm norm of the curvature tensor of $G_m(r)$ is given by

$$
\int_{G_m(r)}\|\widetilde{R}\|^2=r^{n-1}\int_{S^{n-1}}\left(\|\widetilde{R}\|^2\,\theta_m\right)(exp_m(ru))du.
$$

Using the formal expansion of $\|\widetilde{R}\|^2$ obtained in Lemma 4, and the power series expansion for the volume density function in Lemma 2, one obtains the formal expansion of $\|\widetilde{R}\|^2\,\theta_m$. Then the result follows by integration. We skip the calculations, which are similar to those in [9], [10]. □

The total curvatures associated to the other quadratic curvature invariants are defined, as in the previous case, as follows.

**Definition 9** Define the $L^2$*-norm of the Ricci tensor* of a geodesic sphere $G_m(r)$ by

$$
\int_{G_m(r)}\|\widetilde{\rho}\|^2=r^{n-1}\int_{S^{n-1}}\left(\|\widetilde{\rho}\|^2\,\theta_m\right)(exp_m(ru))du. \tag{6}
$$

Also, the $L^2$*-norm of the scalar curvature* is defined by

$$
\int_{G_m(r)}\widetilde{\tau}^2=r^{n-1}\int_{S^{n-1}}\left(\widetilde{\tau}^2\,\theta_m\right)(exp_m(ru))du. \tag{7}
$$

Proceeding in an analogous way as before, one gets the first terms in the power series expansion for the $L^2$-norm of the Ricci tensor. Once again, we delete all the calculations.

**Theorem 10** *Let $(M,g)$ be a Riemannian manifold and $m\in M$. Then*

$$
\int_{G_m(r)}\|\widetilde{\rho}\|^2=c_{n-1}\,r^{n-1}\left\{B_{-4}r^{-4}+B_{-2}r^{-2}+B_0+B_2r^2+O\left(r^3\right)\right\}
$$

*where*

$$B_{-4} = (n-2)^2\,(n-1)$$

$$B_{-2} = -\tfrac{(n-5)(n-2)^2}{6n}\,\tau(m)$$

$$\begin{aligned} B_0 = \tfrac{1}{n\,(n+2)} \Big\{ & -\tfrac{n^3-9n^2-16n-20}{120}\,\|R\|^2 + \tfrac{n^3+31n^2-16n-120}{45}\,\|\rho\|^2 \\ & + \tfrac{n^3-13n^2-16n+44}{72}\,\tau^2 - \tfrac{(n-5)(n-2)^2}{20}\,\Delta\tau \Big\}(m) \end{aligned}$$

$$\begin{aligned} B_2 = \tfrac{1}{n\,(n+2)\,(n+4)} \Big\{ & -\tfrac{n^3-17n^2-112n-4}{1296}\,\tau^3 \\ & - \tfrac{(n+2)\left(n^2+25n+14\right)}{270}\,\tau\|\rho\|^2 + \tfrac{(n+2)(n^2-15n-66)}{720}\,\tau\|R\|^2 \\ & + \tfrac{4(n^3+187n^2+395n+1310)}{2835}\,\check{\rho} + \tfrac{2(n+2)(n^2+10n-10)}{315}\,\langle\rho,\dot{R}\rangle \\ & - \tfrac{2(2n^3+94n^2-225n-180)}{945}\,\langle\rho\otimes\rho,\overline{R}\rangle + \tfrac{9n^3+143n^2-680n+940}{840}\,\langle\nabla^2\tau,\rho\rangle \\ & - \tfrac{(n+2)\left(11n^2+103n+338\right)}{4536}\,\check{R} - \tfrac{(n+2)(5n^2+253n+566)}{1134}\,\check{\check{R}} \\ & + \tfrac{n^3-9n^2-116n+120}{112}\,\|\nabla\tau\|^2 + \tfrac{n^3+61n^2-60n-1238}{224}\,\|\nabla\rho\|^2 \\ & + \tfrac{n^3+61n^2+164n+330}{112}\,\alpha(\rho) - \tfrac{n^3-23n^2-88n-118}{224}\,\|\nabla R\|^2 \\ & + \tfrac{n^3-13n^2-56n+108}{120}\,\tau\Delta\tau + \tfrac{n^3+47n^2+80n-440}{105}\,\langle\Delta\rho,\rho\rangle \\ & - \tfrac{(n+2)(n^2-11n-10)}{168}\,\langle\Delta R,R\rangle - \tfrac{(n-5)(n-2)^2}{112}\,\Delta^2\tau \Big\}(m) \end{aligned}$$

The scalar curvature of sufficiently small geodesic spheres is also settled here for the sake of completeness, and it will also be used in the characterization of the model spaces among the class of Einstein manifolds.

**Theorem 11** *Let $(M,g)$ be a Riemannian manifold and $m\in M$. Then*

$$\int_{G_m(r)} \widetilde{\tau}^2 = c_{n-1}\,r^{n-1}\left\{D_{-4}r^{-4} + D_{-2}r^{-2} + D_0 + D_2 r^2 + O\left(r^3\right)\right\}$$

*where*

$$D_{-4} = (n-2)^2\,(n-1)^2$$

$$D_{-2} = -\tfrac{(n-5)\,(n-2)^2\,(n-1)}{6n}\,\tau(m)$$

$$\begin{aligned} D_0 = \tfrac{1}{n\,(n+2)} \Big\{ & -\tfrac{(n-2)\,(n-1)\,(n^2+13\,n+10)}{120}\,\|R\|^2 + \tfrac{n^4+10\,n^3+43\,n^2-14\,n+120}{45}\,\|\rho\|^2 \\ & + \tfrac{n^4-14\,n^3+29\,n^2-60\,n-188}{72}\,\tau^2 - \tfrac{(n-5)\,(n-2)^2\,(n-1)}{20}\,\Delta\tau \Big\}(m) \end{aligned}$$

$$
\begin{aligned}
D_2 = {} & \tfrac{1}{n\,(n+2)\,(n+4)} \Big\{ -\tfrac{n^4-18\,n^3+n^2+132\,n-428}{1296}\,\tau^3 \\
& - \tfrac{(n+2)\,(n^3+4\,n^2+7\,n-248)}{270}\,\tau\|\rho\|^2 + \tfrac{n^4+6\,n^3-55\,n^2-324\,n-156}{720}\,\tau\|R\|^2 \\
& + \tfrac{4\,(n^4+18\,n^3+61\,n^2+2028\,n+580)}{2835}\,\check{\rho} - \tfrac{4\,(n^4+18\,n^3+131\,n^2+978\,n+720)}{945}\,\langle\rho\otimes\rho,\overline{R}\rangle \\
& + \tfrac{2\left(n^4+18\,n^3-9\,n^2+138\,n+20\right)}{315}\,\langle\rho,\dot{R}\rangle + \tfrac{n^4-10\,n^3-23\,n^2-100\,n-540}{112}\,\|\nabla\tau\|^2 \\
& - \tfrac{(n-2)\,(n-1)\left(11\,n^2+231\,n-338\right)}{4536}\,\check{R} - \tfrac{(n-2)\,(n-1)\left(5\,n^2+105\,n-566\right)}{1134}\,\check{\check{R}} \\
& + \tfrac{n^4+18\,n^3+425\,n^2-632\,n+1196}{224}\,\|\nabla\rho\|^2 + \tfrac{n^4+18\,n^3-23\,n^2+712\,n+300}{112}\,\alpha(\rho) \\
& - \tfrac{(n-2)\,(n-1)\,(n+2)\,(n+19)}{224}\,\|\nabla R\|^2 + \tfrac{n^4-14\,n^3+5\,n^2-4\,n-396}{120}\,\tau\Delta\tau \\
& + \tfrac{n^4+18\,n^3+131\,n^2-142\,n+440}{105}\,\langle\Delta\rho,\rho\rangle - \tfrac{(n-2)\,(n-1)\left(n^2+21\,n+10\right)}{168}\,\langle\Delta R,R\rangle \\
& + \tfrac{(n-2)\left(9\,n^3+40\,n^2-71\,n+470\right)}{840}\,\langle\nabla^2\tau,\rho\rangle - \tfrac{(n-5)(n-2)^2\,(n-1)}{112}\,\Delta^2\tau \Big\}\,(m)
\end{aligned}
$$

# 4 Characterizations of model spaces

In this section we will present some applications of the expansions derived in the previous section. We focus on the characterization of the model spaces by the total quadratic curvature invariants. Recall that by *model space* we mean one of the two-point homogeneous spaces, that is: the Euclidean $n$-space, the $n$-dimensional sphere and the hyperbolic space, the projective and hyperbolic $n$-spaces over complex numbers or over quaternions, and the Cayley projective or hyperbolic plane. Furthermore, we will say that the holonomy of a Riemannian manifold $(M,g)$ is *adapted* to one of these models if the holonomy group of $(M,g)$ is a subgroup of the holonomy group of the given model space, that is, the holonomy of $(M,g)$ is contained in $O(n)$, $U(n)$, $Sp(1)\cdot Sp(n)$ or $Spin(9)$, respectively.

Recall that the volume conjecture of A. Gray and L. Vanhecke [10] states that two-point homogeneous spaces should be characterized by the volumes of their small geodesic spheres among Riemannian manifolds with holonomy group adapted to the model space. Analogous conjectures were formulated by B.Y. Chen and L. Vanhecke for the total scalar curvature and $L^2$-norm of the second fundamental form of geodesic spheres in [8], where they solved the characterization problem completely by combining any two of the three functions above.

Note that if $(M,g)$ is a model space, then the total quadratic curvatures of geodesic spheres can be computed explicitly. The results in the table below follow after some calculations by using the Gauss equation (3) and the fact that the shape operators of sufficiently small geodesic spheres satisfy

$$
(8) \qquad T_m(exp_m(ru)) = \begin{pmatrix} \sqrt{\lambda}\cot r\sqrt{\lambda}\,I_\nu & 0 \\ 0 & \frac{\sqrt{\lambda}}{2}\cot r\frac{\sqrt{\lambda}}{2}\,I_\mu \end{pmatrix}
$$

and the volume-density function is

$$\theta_m(exp_m(ru)) = \left(\frac{\sin r\sqrt{\lambda}}{r\sqrt{\lambda}}\right)^{\nu}\left(\frac{2}{r\sqrt{\lambda}}\sin r\frac{\sqrt{\lambda}}{2}\right)^{\mu} \tag{9}$$

where $1+\nu+\mu = dimM$ and $\nu = dimM-1, 1, 3, 7$ whenever $(M, g)$ is of constant curvature $\lambda > 0$, a Kähler manifold of constant holomorphic sectional curvature $\lambda > 0$, a quaternionic Kähler manifold of constant $Q$–sectional curvature $\lambda > 0$ and the Cayley projective plane, respectively (cf. [13]).

Note that we have only considered the cases of positive curvature and omit the Cayley plane since its holonomy group completely characterizes its local geometry. Nonpositive curvature cases are obtained by replacing the trigonometric functions by the corresponding hyperbolic ones. Also, the $L^2$-norm of the scalar curvatures are omitted in the table below, since those can be obtained from [8, Theorem 6.13].

| | $\int_{G_m(r)} \|\widetilde{R}\|^2$ | $\int_{G_m(r)} \|\widetilde{\rho}\|^2$ |
|---|---|---|
| $\mathbb{R}^n$ | $2c_{n-1}(n-1)(n-2)r^{n-5}$ | $c_{n-1}(n-1)(n-2)^2 r^{n-5}$ |
| $S^n(\lambda)$ | $2\,c_{n-1}\,(n-1)\,(n-2)\,\left(\frac{\sin r\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-5}$ | $c_{n-1}\,(n-1)\,(n-2)^2\,\left(\frac{\sin r\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-5}$ |
| $\mathbb{C}P^n(\lambda)$ | $4(n-1)c_{2n-1}\Big[2n-1+(6n-1)\sin^4\frac{r\sqrt{\lambda}}{2}$ $+2\sin^2\frac{r\sqrt{\lambda}}{2}\Big]\cos\frac{r\sqrt{\lambda}}{2}\left(\frac{2\sin\frac{r\sqrt{\lambda}}{2}}{\sqrt{\lambda}}\right)^{2n-5}$ | $4(n-1)c_{2n-1}\Big[(n-1)(2n-1)+(n+1)\sin^4\frac{r\sqrt{\lambda}}{2}$ $+2(n-1)\sin^2\frac{r\sqrt{\lambda}}{2}\Big]\cos\frac{r\sqrt{\lambda}}{2}\left(\frac{2\sin\frac{r\sqrt{\lambda}}{2}}{\sqrt{\lambda}}\right)^{2n-5}$ |
| $\mathbb{H}P^n(\lambda)$ | $4c_{4n-1}\Big[(2n-1)(4n-1)$ $+6(2n-1)\sin^2 r\frac{\sqrt{\lambda}}{2}$ $+3\sin^4 r\frac{\sqrt{\lambda}}{2}\Big((6n-5)(4n-1)$ $+4\tan^2 r\frac{\sqrt{\lambda}}{2}+\tan^4 r\frac{\sqrt{\lambda}}{2}\Big)\Big]\cdot$ $\cdot\cos^3 r\frac{\sqrt{\lambda}}{2}\left(\frac{2\sin r\frac{\sqrt{\lambda}}{2}}{\sqrt{\lambda}}\right)^{4n-5}$ | $4c_{4n-1}\Big[(2n-1)^2(4n-1)$ $+6(2n-1)^2\sin^2\frac{r\sqrt{\lambda}}{2}$ $+3\sin^4 r\frac{\sqrt{\lambda}}{2}\Big((4n^2+4n-5)$ $+4\tan^2\frac{r\sqrt{\lambda}}{2}+\tan^4\frac{r\sqrt{\lambda}}{2}\Big)\Big]\cdot$ $\cdot\cos^3\frac{r\sqrt{\lambda}}{2}\left(\frac{2\sin\frac{r\sqrt{\lambda}}{2}}{\sqrt{\lambda}}\right)^{4n-5}$ |

Now, as a consequence of Theorem 8, we get the following characterization of the model spaces.

**Theorem 12** *Let $M$ be a Riemannian manifold with dim $M = n \geq 3$, $n \neq 5$ and holonomy group adapted to a model space and suppose that for all $m \in M$ and all sufficiently small $r$, the $L^2$-norm of the curvature tensor of geodesic spheres is the same as that in the model space. Then $M$ is locally isometric to that model space.*

**Proof.** First of all, note that if a manifold has holonomy group contained in Spin(9), then it is flat or locally isometric to the Cayley plane or its noncompact dual [1], [5]. Therefore, we only need to deal with real, complex and quaternionic space forms. Next, suppose the $L^2$–norm of the curvature tensor of sufficiently small geodesic spheres is the same as in a space of constant sectional curvature $\lambda$, that is

$$\begin{aligned}\int_{G_m(r)} \|\widetilde{R}\|^2 \;&=\; c_{n-1} r^{n-1}\Big\{\tfrac{2\,(n-1)(n-2)}{r^4} - \tfrac{(n-1)(n-2)(n-5)\lambda}{3\,r^2} \\ &\quad + \tfrac{(n-1)(n-2)(n-5)(5\,n-27)\lambda^2}{180} + O\left(r^2\right)\Big\}(m).\end{aligned} \tag{10}$$

Then, if $\dim M \neq 5$, it follows from $A_{-2}$ in Theorem 8 and the expansion above that the scalar curvature is constant $\tau = n(n-1)\lambda$. Hence, the coefficient $A_0$ in the power series expansion of $\int_{G_m(r)} \|\widetilde{R}\|^2$ given by Theorem 8 becomes

$$\begin{aligned}A_0 \;&=\; \tfrac{1}{n(n+2)}\Big\{\tfrac{59\,n^2-93\,n-10}{60}\,\|R\|^2 + \tfrac{2\left(n^2-37\,n+60\right)}{45}\,\|\rho\|^2 \\ &\quad + \tfrac{n^2(n-1)^2(n^2-11\,n+2)}{36}\,\lambda^2\Big\}(m) \\ &=\; \tfrac{59\,n^2-93\,n-10}{60\,n(n+2)}\,\Big(\|R\|^2 - \tfrac{2}{n-1}\|\rho\|^2\Big)(m) \\ &\quad + \tfrac{4\,n^3+25\,n^2+109\,n-270}{90\,n(n-1)(n+2)}\,\left(\|\rho\|^2 - \tfrac{1}{n}\,\tau^2\right)(m) \\ &\quad + \tfrac{(n-1)(n-2)(n-5)(5\,n-27)}{180}\,\lambda^2.\end{aligned}$$

Now, comparing again the coefficient of $A_0$ above with the corresponding in (10), one obtains

$$\begin{aligned}0 &= \tfrac{59\,n^2-93\,n-10}{60\,n(n+2)}\,\Big(\|R\|^2 - \tfrac{2}{n-1}\|\rho\|^2\Big) \\ &\quad + \tfrac{4\,n^3+25\,n^2+109\,n-270}{90\,n(n-1)(n+2)}\,\left(\|\rho\|^2 - \tfrac{1}{n}\,\tau^2\right).\end{aligned} \tag{11}$$

Now, since $\|\rho\|^2 \geq \frac{1}{n}\tau^2$ for any Riemannian manifold (with equality if and only if the manifold is Einstein) and $\|R\|^2 \geq \frac{2}{n-1}\|\rho\|^2$ (with equality if and only if the manifold has constant sectional curvature) and $\frac{59\,n^2-93\,n-10}{60\,n(n+2)}$ and $\frac{4\,n^3+25\,n^2+109\,n-270}{90\,n(n-1)(n+2)}$ are both positive ($n > 2$), it follows that $M$ has constant sectional curvature, which must be exactly $\lambda$ due to the value of the scalar curvature.

The other cases are derived in a similar way, using that $\|R\|^2 \geq \frac{4}{n+1}\|\rho\|^2$ for manifolds with holonomy adapted to the complex space forms (with the equality holding if and only if $M$ has constant holomorphic sectional curvature), and that $\|R\|^2 \geq \frac{5\,n+1}{(n+2)^2}\|\rho\|^2$ for manifolds with holonomy adapted to quaternionic space forms (with the equality holding precisely for $P^n(\mathbb{H})$ and $H^n(\mathbb{H})$) [4], [6], [12].$\square$

Next, we will explain the reason to exclude $\dim M = 5$ in the previous theorem. It is well-known ([2, pag. 82]) that the Euler characteristic $\chi(M)$ of a compact four-dimensional manifold satisfies

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{\|R\|^2 - 4\|\rho\|^2 + \tau^2\}dV. \tag{12}$$

Now, if $(M,g)$ is a 5-dimensional space of constant sectional curvature, its geodesic spheres are Einstein [8, Theorem 7.3], and thus the $L^2$-norm of the curvature tensor of geodesic spheres becomes a topological invariant of geodesic spheres, therefore meaningless for our purpose of characterizing the curvature of the ambient manifold. However, a characterization of real space forms can be given, although the sign and precise value of the sectional curvature cannot be detected.

**Theorem 13** *Let $M$ be a Riemannian manifold of $dim M = 5$. If the $L^2$-norm of the curvature tensor of each small geodesic sphere is the same as for a 5-dimensional real space form, then $M$ has constant sectional curvature.*

**Proof.** First of all, note that if $M^5$ is a space of constant sectional curvature, then the $L^2$-norm of the curvature tensor of sufficiently small geodesic spheres satisfies $\int_{G_m(r)} \|\widetilde{R}\|^2 = 24\, c_4$, and thus it follows from Theorem 8 that

$$\begin{aligned} 24c_4 &= 24c_4 + \tfrac{c_4 r^4}{35}\left(\tfrac{50}{3}\|R\|^2 - \tfrac{40}{9}\|\rho\|^2 - \tfrac{7}{9}\tau^2\right) + O(r^6) \\ &= 24c_4 + \tfrac{c_4 r^4}{35}\left\{\tfrac{50}{3}\left(\|R\|^2 - \tfrac{1}{2}\|\rho\|^2\right) + \tfrac{35}{9}\left(\|\rho\|^2 - \tfrac{1}{5}\tau^2\right)\right\} + O(r^6). \end{aligned}$$

Hence

$$\frac{50}{3}\left(\|R\|^2 - \frac{1}{2}\|\rho\|^2\right) + \frac{35}{9}\left(\|\rho\|^2 - \frac{1}{5}\tau^2\right) = 0 \tag{13}$$

from where it follows that the sectional curvature is constant. $\square$

For the $L^2$-norm of the Ricci tensor we have similar results, but only for low dimensions.

**Theorem 14** *Let $M$ be a Riemannian manifold with dim $M = n$, $3 \leq n \leq 10$, $n \neq 5$ and holonomy group adapted to a model space and suppose that for all $m \in M$ and all sufficiently small $r$ the $L^2$-norm of the Ricci tensor of geodesic spheres is the same as those in the model space. Then $M$ is locally isometric to that model space.*

**Proof.** The proof goes as in Theorem 12. Suppose the $L^2$–norm of the Ricci tensor of a sufficiently small geodesic sphere is the same as that in a space of constant curvature $\lambda$, that is,

$$\begin{aligned} \int_{G_m(r)} \|\widetilde{\rho}\|^2 &= c_{n-1} r^{n-1} \left\{ \tfrac{(n-1)(n-2)^2}{r^4} - \tfrac{(n-1)(n-2)^2(n-5)\lambda}{6\, r^2} \right. \\ &\quad \left. + \tfrac{(n-1)(n-2)^2(n-5)(5\,n-27)\lambda^2}{360} + O\left(r^2\right) \right\} (m). \end{aligned} \tag{14}$$

Then, it follows from $B_{-2}$ in Theorem 10 that $\tau = n(n-1)\lambda$, provided that $n \neq 5$, and thus the coefficient $B_0$ in Theorem 10 becomes

$$\begin{aligned} B_0 &= -\tfrac{n^3 - 9\,n^2 - 16\,n - 20}{120\, n(n+2)} \left(\|R\|^2 - \tfrac{2}{n-1}\|\rho\|^2\right)(m) \\ &\quad + \tfrac{4\,n^4 + 117\,n^3 - 161\,n^2 - 368\,n + 540}{180\, n(n-1)(n+2)} \left(\|\rho\|^2 - \tfrac{1}{n}\tau^2\right)(m) \\ &\quad + \tfrac{(n-1)(n-2)^2(n-5)(5\,n-27)}{360}\lambda^2. \end{aligned}$$

Comparing again with (14) we have

$$0 = -\tfrac{n^3-9\,n^2-16\,n-20}{120\,n(n+2)}\left(\|R\|^2 - \tfrac{2}{n-1}\|\rho\|^2\right) + \tfrac{4\,n^4+117\,n^3-161\,n^2-368\,n+540}{180\,n(n-1)(n+2)}\left(\|\rho\|^2 - \tfrac{1}{n}\,\tau^2\right) \tag{15}$$

Now the result follows from (15) proceeding in the same way as in Theorem 12 and using that $-\frac{n^3-9\,n^2-16\,n-20}{120\,n(n+2)}$ and $\frac{4\,n^4+117\,n^3-161\,n^2-368\,n+540}{180\,n(n-1)(n+2)}$ are both positive for $3 \leq n \leq 10$. The complex and quaternionic cases are derived in a similar way. □

**Remark 15** By Theorem 10 one also deduces that, if dim $M = 5$ and the $L^2$-norm of the Ricci tensor of each small geodesic sphere is the same as for those in a 5-dimensional real space form, then $M$ has constant sectional curvature.

We recall here that the volume conjecture has a positive answer in the class of Einstein manifolds [10]. Analogously, if $M$ is assumed to be an Einstein manifold, the $L^2$-norm of the Ricci tensor or the $L^2$-norm of the scalar curvature of sufficiently small geodesic spheres suffice to characterize the model spaces.

**Theorem 16** *Let $M$ be an Einstein manifold with dim $M = n$, $n \neq 5$ and holonomy group adapted to a model space. If for all $m \in M$ and all sufficiently small $r$ the $L^2$-norm of the Ricci tensor or the $L^2$-norm of the scalar curvature of geodesic spheres is the same as those in the model space, then $M$ is locally isometric to that model space.*

The proof can be sketched as follows. If $(M, g)$ is assumed to be Einstein, the coefficient $B_0$ in the power series expansion of the $L^2$-norm of the Ricci tensor of geodesic spheres (Theorem 10) becomes

$$B_0 = \tfrac{1}{n(n+2)}\left(-\tfrac{n^3-9n^2-16n-20}{120}\|R\|^2 + \tfrac{5n^4-57n^3+168n^2+92n-960}{360n}\tau^2\right) \tag{16}$$

and the result follows proceeding as in previous theorems. Also, if $(M, g)$ is Einstein, then the coefficient $D_0$ of the $L^2$-norm of the scalar curvature of sufficiently small geodesic spheres in Theorem 11 satisfies

$$D_0 = \tfrac{1}{n(n+2)}\left(-\tfrac{(n-2)(n-1)(n^2+13n+10)}{120}\|R\|^2 + \tfrac{(n-2)(5n^4-52n^3+121n^2+286n-480)}{360n}\tau^2\right) \tag{17}$$

and the desired characterization is obtained as in previous theorems.

**Remark 17** It follows from Theorem 16 that the assumption on the dimension in Theorem 14 can be dropped if the holonomy group is contained in $Sp(1) \cdot Sp(n)$, since any $4n$-dimensional quaternionic Kähler manifold is Einstein for $n > 1$.

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