Intersection of quadrics in $\mathbb{C}^n$, moment-angle manifolds, complex manifolds and convex polytopes

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LECTURES

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Definition

Let W be a complex manifold of complex dimension n.
A holomorphic foliation $\mathcal{F}$ of complex dimension p (or complex codimension $n-p$) is given by a *foliated atlas* i.e. a covering $\{U_i\}_{i\in\mathcal{I}}$ of W and homeomorphisms $\Phi_i : U_i \to \mathbb{C}^n = \mathbb{C}^{n-p} \times \mathbb{C}^p$ such that for overlapping pairs U_i, U_j the transition functions $\varphi_{ij} = \Phi_j {\Phi_i}^{-1}$ are of the form:

$$\varphi_{ij}(x,y) = (\Phi^1_{ij}(x), \Phi^2_{ij}(x,y)) \quad x \in \mathbb{C}^{n-p}, \quad y \in \mathbb{C}^p$$

where Φ^1_{ij} and Φ^2_{ij} are holomorphic and Φ^1_{ij} is a local biholomorphism between open sets of $\mathbb{C}^{n-p}$ and Φ^2_{ij} is a local holomorphic submersion from an open set in $\mathbb{C}^n$ onto an open set of $\mathbb{C}^p$.

Let m and n be two positive natural numbers such that $n > 2m$ Let $(\Lambda_1, \cdots, \Lambda_n)$ be an n-tuple of vectors in $\mathbb{C}^m$ where $\Lambda_i = (\lambda_i^1, \cdots, \lambda_i^m)$ for $i = 1, \cdots, n$.

To the configuration $(\Lambda_1, \cdots, \Lambda_n)$ we can associate the linear (singular) foliation of $\mathbb{C}^n$ generated by the m holomorphic linear commuting vector fields $(1 \leq j \leq m)$

$$\xi_j : (z_1, \cdots, z_n) \in \mathbb{C}^n \longmapsto \sum_{i=1}^{n} \lambda_i^j z_i \frac{\partial}{\partial z_i} \qquad (\star)$$

$$\frac{d\mathbf{Z}}{dT} = \begin{bmatrix} \lambda_1^j & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^j & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & & \dots & \lambda_n^j \end{bmatrix} \mathbf{Z},$$

$$\mathbf{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \quad j = 1, \cdots, m, \quad T \in \mathbb{C}$$

Let us start with the construction of an infinite family of compact complex manifolds. Let m be a positive integer and n and integer such that $n > 2m$. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ be a configuration of n vectors in $\mathbb{C}^m$. Let $\mathcal{H}(\Lambda_1, \cdots, \Lambda_n)$ be the convex hull of $(\Lambda_1, \cdots, \Lambda_n)$

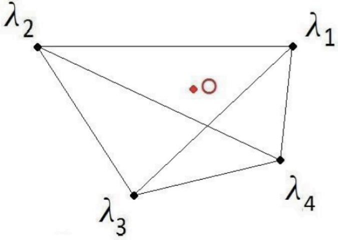
We say that Λ is *admissible* if

1. The Siegel condition: 0 belongs to the convex hull $\mathcal{H}(\Lambda_1, \ldots, \Lambda_n)$ of $(\Lambda_1, \ldots, \Lambda_n)$ in $\mathbb{C}^m \simeq \mathbb{R}^{2m}$.
2. The weak hyperbolicity condition: for every $2m$-tuple of integers $i_1, \cdots, i_{2m}$ such that $1 \leq i_1 < \cdots < i_{2m} \leq n$ we have $0 \notin \mathcal{H}(\Lambda_{i_1}, \cdots, \Lambda_{i_{2m}})$

This definition can be reformulated geometrically in the following way: the convex polytope $\mathcal{H}(\Lambda_1, \cdots, \Lambda_n)$ contains 0, but neither external nor internal facet of this polytope (that is to say hyperplane passing through $2m$ vertices) contains 0. An admissible configuration satisfies the following regularity property

Lemma

Let $\Lambda_i' = (\Lambda_i, 1) \in \mathbb{C}^{n+1}$, for $i \in \{1, \cdots, n\}$. For all set of integers $J \subset \{1, \cdots, n\}$ such that $0 \in \mathcal{H}((\Lambda_j)_{j \in J})$ the complex rank of the matrix whose columns are the vectors $(\Lambda_j)_{j \in J}$ is equal to $m+1$, therefore it is of maximal rank.



λ_2
λ_1
O
λ_4
λ_3

One considers the holomorphic (singular) foliation $\mathcal{F}$ in projective space $\mathbb{P}^{n-1}$ given by the orbits of the linear action of $\mathbb{C}^n$ induced by the linear vector fields $(\star)$.

$$(T,[z]) \in \mathbb{C}^m \times \mathbb{P}^{n-1} \longmapsto [z_1 \cdot \exp\langle \Lambda_1, T\rangle, \ldots, z_n \cdot \exp\langle \Lambda_n, T\rangle] \in \mathbb{P}^{n-1}$$

where $T = (t_1, \cdots, t_m) \in \mathbb{C}^m$, $[z_1, \cdots, z_n]$ are projective coordinates and $\langle -, - \rangle$ is inner product. One can lift this foliation to a foliation $\tilde{\mathcal{F}}$ in $\mathbb{C}^n$ given by the linear action

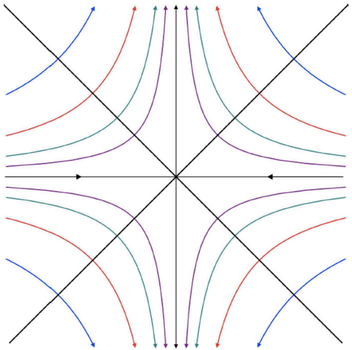
$$(T,z) \in \mathbb{C}^m \times \mathbb{C}^n \longmapsto (z_1 \cdot \exp\langle \Lambda_1, T\rangle, \ldots, z_n \cdot \exp\langle \Lambda_n, T\rangle) \in \mathbb{C}^n$$

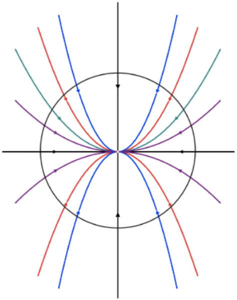
The so-defined foliation is degenerate, in particular 0 is a singular point. The behaviour in the neighbourhood of 0 determines two different sorts of leaves.

Definition. Let L be a leaf of the previous foliation. If 0 belongs to the closure of L, we say that L is a *Poincaré* leaf. In the opposite case, we talk of a *Siegel leaf.*

If $0 \notin \mathcal{H}(\Lambda_1, \ldots, \Lambda_n)$ then every leaf is of Poincaré type.

The set of Siegel leaves is nonempty if and only if $0 \in \mathcal{H}(\Lambda_1, \ldots, \Lambda_n)$





For $z = (z_1, \cdots, z_n)$ let $I_z \subset \{1, \cdots, n\}$ defined by $I_z = \{j : \ z_j \neq 0\}$. Let $\Lambda_{I_z} = \{\Lambda_j \ : \ j \in I_z\}$ One defines:

$$\mathcal{S} = \{z \in \mathbb{C}^n \quad | \quad 0 \in \mathcal{H}(\Lambda_{I_z})\}.$$

Let $\mathcal{V} \subset \mathbb{P}^{n-1}$ be the image of $\mathcal{S}$ in $\mathbb{P}^{n-1}$ under the canonical projection $\pi : \mathbb{C}^n - \{0\} \to \mathbb{P}^{n-1}$.

Finally, let

$$\mathcal{T} = \{z \in \mathbb{C}^n \quad | \quad z \neq 0, \ \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0 \,\}$$

and

$$\mathcal{N} = \{[z] \in \mathbb{P}^{n-1} \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0\}$$

One can verify that $\mathcal{S}$ is the union of the Siegel leaves and that $\mathcal{S}$ is an open set of the form $\mathcal{S} = \mathbb{C}^n - E$ where E is an analytic set, whose different components correspond to subspaces of C^n where some coordinates vanish.
The leaf space of the foliation restricted to $\mathcal{S}$, that we call M, is identified with $\mathcal{T}$.
Since $\mathcal{S}$ contains $(\mathbb{C}^*)^n$ we see that $\mathcal{S}$ is dense in $\mathbb{C}^n$.

If L is a Siegel leaf then the distance from that leaf to the origin is positive and one can show that there exists a unique point $\mathbf{z} = (z_1, \cdots, z_n) \in L$ which minimizes the distance to the origin and this point satisfies

$$\sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0$$

The weak hyperbolicity condition implies that the system of quadrics given by the preceding equations which define $\mathcal{T}$ and $\mathcal{N}$ have maximal rank in every point

The Siegel condition implies that $\mathcal{T}$ and $\mathcal{N}$ are nonempty. One can show also that $\tilde{\mathcal{F}}$ is regular in $\mathcal{S}$ and that $\mathcal{T}$ is a smooth manifold transverse to the restriction of $\tilde{\mathcal{F}}$ to $\mathcal{S}$. In other words the quotient space of $\tilde{\mathcal{F}}$ restricted to $\mathcal{S}$ can be identified with $\mathcal{T}$. Therefore $\mathcal{T}$ has the structure on a (non-compact) complex manifold which we call M.

Also $\mathcal{N}$ can be identified to the quotient space of $\mathcal{F}$ resrticted to $\mathcal{V}$ and therefore it inherits a complex structure. Let us denote this complex manifold by N. The complex dimension of M is $n - m$ and of N is $n - m - 1$.

The natural projection $M \to N$, induced by the projection $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}_{\mathbb{C}}^{n-1}$, is in fact a principal $\mathbb{C}^*$ fibration. Let M_1 denote the total space of the associated circle fibration It has the same homotopy type as M but it has the advantage of being compact.

Let us observe that M_1 can be identified with the transverse intersection of the cone $\mathcal{T}$ (with the vertex at the origin delated) and the unit sphere $\mathbb{S}^{2n-1}$ in $\mathbb{C}^n$. For this reason we take as definition

$$M_1 = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0,\ \sum_{i=1}^{n} |z_i|^2 = 1\}$$

Another characterization of $\mathcal{S}$ is the following:

$$\mathcal{S} = \{z \in \mathbb{C}^n \mid 0 \text{ is not in the closure of the leaf of } \tilde{\mathcal{F}} \text{ through } z\}$$

in other words $\mathcal{S}$ is the union of the Siegel Leaves and it open and invariant under the action of $\mathbb{C}^m$

Remark

The space of Siegel leaves $\mathcal{S}$ has the same homotopy type as M and therefore also as M_1.

Remark

The linear holomorphic action of $(\mathbb{C}^*)^m$ commutes with the diagonal action (by diagonal matrices) hence $(\mathbb{C}^*)^n$ acts on M.

The set $\mathcal{S}$ is a deleted complex cone in $\mathbb{C}^n$: i.e. if $Z \in \mathcal{S}$ then $\lambda Z \in \mathcal{S}$ for all $\lambda \in \mathbb{C}^*$. Therefore $\mathcal{S} = \pi(\mathcal{S})$ is an open set of $\mathbb{P}_{\mathbb{C}}^{n-1}$.

$\pi(\mathcal{T})$ is a smooth manifold of dimension equal to the codimension of $\mathcal{F}$ and transversal to the leaves.

By a lemma by André Haefliger it is a complex manifold:

Lemma

Let U be a complex manifold of complex dimension $n \geq 2$ and $\mathcal{F}$ a holomorphic foliation of codimension $m \geq 1$. Let $W \subset U$ be a smooth manifold of real dimension $2m$. Then W is in a natural way a complex manifold.

Bogomolov has conjectured that every compact complex manifold W can be obtained by this process for a singular holomorphic foliation of projective space and W transversal to the foliation outside of the singularities.

EXAMPLES

(i) If $n = 2m + 1$, the convex hull $\{\Lambda_i\}_{i \in \{1,\cdots,2n+1\}}$ is a simplex in $\mathbb{C}^m \simeq \mathbb{R}^{2m}$.

In fact if one removes one the Λ's then 0 is not in the complex hull of the remaining.

In other words $\mathcal{S}$ is equal to $(\mathbb{C}^*)^n$ and one can show that N is a complex torus. We will show later this result.

Remark

Every compact complex torus is obtained by this process. In particular, if $n = 3$ and $m = 1$ we obtain every elliptic curve.

(ii) If $m = 1$ let us define for $n \geq 4$

$$\Lambda_1 = 1 \qquad \Lambda_2 = i \qquad \Lambda_3 = \ldots = \Lambda_n = -1 - i\ .$$

It is easy to verify that under these conditions $\mathcal{S}$ is equal to $(\mathbb{C}^*)^2 \times \mathbb{C}^{n-2} \setminus \{0\}$.
Consider the two real equations that are used to define $\mathcal{T}$:

$$|z_1|^2 = |z_3|^2 + \ldots + |z_n|^2, \qquad |z_2|^2 = |z_3|^2 + \ldots + |z_n|^2$$

If we fix the modules of z_1 and z_2 (by the definition of $\mathcal{S}$ they cannot be 0) the above equations imply that these modules are equal and that $(z_3, \ldots, z_n)$ belong to a sphere $\mathbb{S}^{2n-5}$. Therefore these equations define a manifold which is diffeomorphic to $S^{2n-5} \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^+_*$. The manifold M_1 obtained as the intersection of $\mathcal{T}$ and the unit sphere of $\mathbb{C}^n$ is diffeomorphic to $\mathbb{S}^{2n-5} \times \mathbb{S}^1 \times \mathbb{S}^1$ and N is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{2n-5}$. In particular for $n = 4$, on has all the linear Hopf surfaces.

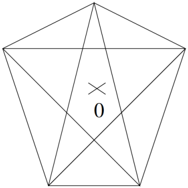
(iii) Let $m = 1, \; n = 5$ and

$$\Lambda_1 = 1 \qquad \Lambda_2 = \Lambda_3 = i \qquad \Lambda_4 = \Lambda_5 = -1 - i \; .$$

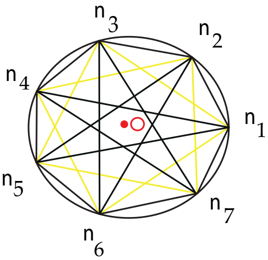
An argument similar to the previous one shows that N is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3$. One obtains an example of Calabi-Eckmann of non Kähler manifolds.

In general one obtains complex structures in products of odd dimensional spheres $\mathbb{S}^{2r+1} \times \mathbb{S}^{2l+1}$ like in the classical Calabi-Eckmann manifolds.

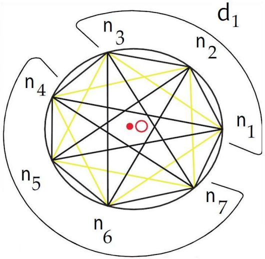
(iv) , S. López de Medrano has shown that for the pentagon in the picture below M_1 is diffeomorphic to the connected sum of five copies of $\mathbb{S}^3 \times \mathbb{S}^4$. The complex manifold N is the quotient of this connected sum under a non-trivial action of $\mathbb{S}^1$.



When $m = 1$ it can be assumed Λ is one of the following normal forms: Take $n = n_1 + \cdots + n_{2\ell+1}$ a partition of n into an odd number of positive integers. Consider the configuration consisting of the vertices of a regular polygon with $(2\ell + 1)$ vertices, where the i-th vertex in the cyclic order appears with multiplicity n_i.



The topology of M_1 and N can be completely described in terms of the numbers $d_i = n_i + \cdots + n_{i+\ell-1}$, i.e., the sums of ℓ consecutive n_i in the cyclic order of the partition:



For $\ell = 1$: $\quad M_1 = \mathbb{S}^{2n_1-1} \times \mathbb{S}^{2n_2-1} \times \mathbb{S}^{2n_3-1}$.

For $\ell > 1$: $\quad M_1 = \mathop{\#}_{j=1}^{2\ell+1} \left(\mathbb{S}^{2d_i-1} \times \mathbb{S}^{2n-2d_i-2}\right)$.

Definition. Let $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ be an admissible configuration. For $i \in \{1, \cdots, n\}$ we say that Λ_i is *indispensable* if 0 does not belong to the convex hull of $\Lambda - \Lambda_i$. Otherwise we say that Λ_i is *eliminable*.
Let k denote the number of indispensable points. By Carathéodory $k \leq 2m+1$ the maximum is attained only when $n = 2m+1$. One has:

Lemma

1. *$\mathcal{S} = (\mathbb{C}^*)^k \times (\mathbb{C}^{n-k} \setminus A)$ with A an analytic set of codimension at least two in every point.*
2. *This decomposition descends to a decomposition $M_1 = (\mathbb{S}^1)^k \times M_0$ where M_0 is a real compact manifold which is 2-connected.*

Sketch of proof. Let $\mathcal{S} = \mathbb{C}^n \setminus E$, where E is a union of subspaces

$$E = \{z \in \mathbb{C}^n \quad | \quad 0 \notin \mathcal{H}(\Lambda_{I_z})\}.$$

The components of codimension one are given by indices corresponding to indispensable points in the configuration. This proves the first part. Since A is of complex codimension at least 2 in every point ($\mathbb{C}^{n-k} \setminus A$) is 2-connected, donc M_0 est 2-connexe, since they have the same homotopy type. □

In examples (ii) and (iii) one obtains compact complex manifolds which are not symplectic because the second de Rham cohomology group is trivial. This is in fact a general property of the manifolds we obtain:

Theorem. The following are equivalent:

1. N is symplectic.
2. N is Kähler.
3. N is a complex torus.
4. $n = 2m + 1$.

Sketch of the proof. It is easy to prove the equivalence of (iii) and (iv). If N is a complex torus, one must have $S = (\mathbb{C}^*)^n$ hence all the Λ_i must be indispensable and in this case the comvex hull must be a simplex and $n = 2m + 1$. If the convex hull is a simplex then as in example (i) N is a compact complex torus.

The most difficult part is that (i) implies (iv). One proves that by contradiction. Suppose $n > 2m + 1$. As in the examples one must study the de Rham cohomology of N and to prove that it is incompactible with the existence of a symplectic form.

We consider two cases: <u>1st case</u> There exists indispensable

points. From here one can deduce that the fibration $M \to N$ is trivial. Hence the decomposition $M_1 = (\mathbb{S}^1)^k \times M_0$ of the previous lemma gives a decomposition $N = (\mathbb{S}^1)^{k-1} \times M_0$. IN other words if N has a symplectic structure it must be supported by $(\mathbb{S}^1)^{k-1}$. The maximal power of this symplectic form must be a volume form in N but that is only possible only if $k - 1$ is equal to the real dimension of N, i.e. to $2n - 2m - 2$. since $k \leq 2m + 1$ and $n > 2m + 1$

second case: If there are not indispensable points then M is 2-connected and the fibration $M \to N$ is not topologically trivial. Therefore the second de Rham cohomology group of N is generated by the Euler class of that fibration. Analizing carefully this fibration one shows that the Euler class is trivial. Therefore this class is not symplectic

$\square$

BASIC EXAMPLE

In $\mathbb{C}$ consider a non-degenerate triangle with vertices λ_1, λ_2 and λ_3. Suppose that the origin is in the interior of this triangle. Then the open set $\mathcal{U} \subset \mathbb{C}^3$ is the complement of the three coordinate hyperplanes $z_1 = 0$, $z_2 = 0$ and $z_3 = 0$.
The set in $\mathbb{C}^3 - \{0\}$ given by the equation $(\star)$

$$\lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_3|^2 = 0 \qquad (\star)$$

meets every leaf in $\mathcal{U}$ in exactly one point. So that the space of leaves in $\mathcal{U}$ can be identified with the set, also denoted by M, satisfying this equation.
The set M is a complex cone with the origin deleted so that if $Z \in M$ also $cZ \in M$ for al $c \in \mathbb{C}^*$.
Hence one has a free action of $\mathbb{C}^*$ and the quotient $N := M/\mathbb{C}^*$, then a complex, compact manifold of dimension one. In fact N is an elliptic curve.

Any elliptic curve is obtained this way.

We see that N is the projectivization of M and therefore N can be identified is the set of points satisfying the following two equations:

$$\lambda_1|z_1|^2 + \lambda_2|z_2|^2 + \lambda_3|z_3|^2 = 0$$

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$$

modulo the natural action of the circle given by

$$(z_1, z_2, z_3) \mapsto (\mu z_1, \mu z_2, \mu z_3),\ |\mu| = 1,\ (z_1, z_2, z_3) \in N.$$

Toric varieties and Generalized Calabi-Eckmann fibrations

Let $(\Lambda_1, \cdots, \Lambda_n)$ be a configuration admissible i.e. it satisfies both the Siegel and weak hyperbolicity conditions as before. Consider the system of equations:

$$\sum_{i=1}^{n} s_i \Lambda_i = 0$$

$$\sum_{i=1}^{n} s_i = 0$$

We say that the configuration satisfies condition (K) if the dimension over $\mathbb{Q}$ of the vector space of rational solutions of the system above is maximal, in other words is of dimension $n - 2m - 1$.

Theorem

Let N be one of our manifolds corresponding to a configuration which satisfies condition (K).
Then N is a Seifert fibration in complex torii of dimension m over a quasi-regular, projective, toric variety of dimension $n - 2m - 1$.

This theorem has the following:

Corollary

Let N satisfy the conditions of the above theorem. Then the algebraic reduction of N is a quasi-regular, projective, toric variety of dimension $n - 2m - 1$.

As a particular case of the previous theorem one recovers the elliptic fibrations used by E. Calabi et B. Eckmann to provide the product of spheres $\mathbb{S}^{2p-1} \times \mathbb{S}^{2q-1}$ (for $p > 1$ et $q > 1$) with a complex structure. This generalization is given by the following

Definition

A generalized Calabi-Eckmann fibration is the fibration obtained by the previous theorem.

Since we know, fixing m and n, that the set of configurations satisfying condition (K) is dense in the space of admissible configurations on obtains:

Corollary

Every manifold N corresponding to an admissible configuration is a small deformation of a generalized Calabi-Eckmann fibration

Recall that a n-dimensional toric variety W (possibly singular) is an algebraic variety with an open and dense subset biholomorphic to $(\mathbb{C}^*)^n$ such that the natural action of $(\mathbb{C}^*)^n$ extends to a holomorphic action on all af W. In other words: a toric variety of complex dimension n is an algebraic variety which is an equivariant compactification of the abelian algebraic torus $(\mathbb{C}^*)^n$.

Theorem

Let X be a projective, quasi-regular, toric variety. Then there exists $m > 0$ and a manifold N corresponding to an admissible configuration which admits a generalized Calabi-Eckmann over X and whose fibres are complex torii of complex dimension m. Furthermore, if X is nonsingular (smooth), one can choose m and N such that the fibration is a holomorphic principal fibration

.

The previous theorem motivated a possible definition of non-commutative toric varieties and its deformations (usual toric varieties are rigid).

The definition of a non-commutative toric variety.
Katzarkov, Ludmil; Lupercio, Ernesto; Meersseman, Laurent; Verjovsky, Alberto. Algebraic topology: applications and new directions, 223–250, *Contemp. Math.*, 620, Amer. Math. Soc., Providence, RI, 2014

Some contact structures on moment-angle manifolds

The complex manifolds which we have obtained are not symplectic. We have the circle fibration

$$\Pi : M_1(\Lambda) \to \mathcal{N}(\Lambda)$$

Using a recent result by Borman, M.S., Eliashberg, Y. and Murphy, E., implying that every elmost-contact manifold is a contact manifold we can prove that the manifolds manifolds $M_1(\Lambda)$ always have contact structures,

Theorem

$M_1(\Lambda)$ is a contact manifold.

First we show that $M_1(\Lambda)$ is an almost-contact manifold. Recall that a $(2n+1)$-dimensional manifold $\mathcal{M}$ is called *almost contact* if its tangent bundle admits a reduction to $\mathbf{U}(n) \times \mathbb{R}$. This is seen as follows: consider the fibration $\pi : M_1(\Lambda) \to \mathcal{N}(\boldsymbol{\Lambda})$ with fibre the circle, given by taking the quotient by the diagonal action. Since $\mathcal{N}(\Lambda)$ is a complex manifold, the foliation defined by the diagonal circle action is transversally holomorphic.

Therefore, $M_1(\Lambda)$ has an atlas modeled on $\mathbb{C}^{n-2} \times \mathbb{R}$ with changes of coordinates of the charts of the form

$$((z_1, \cdots, z_{n-2}), t) \mapsto (h(z_1, \cdots, z_{n-2}, t), g(z_1, \cdots, z_{n-2}, t)),$$

where $h : U \to \mathbb{C}^{n-2}$ and $g : U \to \mathbb{R}$ where U is an open set in $\mathbb{C}^{n-2} \times \mathbb{R}$ and, for each fixed t the function $(z_1, \cdots, z_{n-2}) \mapsto h(z_1, \cdots, z_{n-2}, t)$ is a biholomorphism onto an open set of $\mathbb{C}^{n-2} \times \{t\}$. This means that the differential, in the given coordinates, is represented by a matrix of the form

$$\left[\begin{array}{ccc|c} & A & & * \\ \hline 0 & \ldots & 0 & r \end{array}\right]$$

where $*$ denotes a column $(n-2)$-real vector and $A \in \mathbf{GL}(n-2, \mathbb{C})$. The set of matrices of the above type form a subgroup of $\mathbf{GL}(2n-3, \mathbb{R})$. By Gram-Schmidt this group retracts onto $\mathbf{U}(n-2) \times \mathbb{R}$.

Now it follows that $M_1(\Lambda)$ is a contact manifold and the Theorem is proved.

From polytopes to quadrics

Let $\mathbb{R}^n$ be given the standard inner product$\langle\cdot,\cdot\rangle$ and consider consider convex polyhedrons defined as intersections of m-halfspaces:

$$P = \{x \in \mathbb{R}^n \ : \ \langle a_i, x\rangle + b_i \geq 0, \quad for \quad i = 1, \ldots, m\}$$

$a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. Assume that the hyperplanes defined by the equations $\langle a_i, x\rangle + b_i = 0$ are in general position, i. e. at least n of them meet at a single point. Assume further that $dimP = n$ and P is bounded (which implies that $m > n$). Then P is an n-dimensional simple polytope. Set

$$F_i = \{x \in P : \langle a_i, x\rangle + b_i = 0\}$$

Since the hyperplanes are in general position F_i is either empty or a facet of P. If it is empty the linear equation is redundant and we can remove the corresponding inequality without changing P.

Let A_P be the $m \times n$ matrix of row vectors a_i, and b_P be the column vector of scalars $b_i \in \mathbb{R}^n$. Then we can write:

$$P = \{x \in \mathbb{R}^n \ : A_P x + b_P \geq 0\}$$

and consider the affine map

$$i_P : \mathbb{R}^n \to \mathbb{R}^m,$$

$$i_P(x) = A_P x + b_P.$$

It embeds P into the first orthant

$$\mathbb{R}^m_{\geq 0} = \{(y_1, \cdots, y_m) \in \mathbb{R}^m \ : \ y_i > 0 \quad , \textit{for} \quad i \in \{1, \ldots, m\}\}.$$

We identify C^m (as a real vector space) with R^{2m} as usual using the map $z = (z_1, \ldots, z_m) \mapsto (x_1, y_1, \ldots, x_m, y_m)$, where $z_k = x_k + iy_k$ for $k = 1, \ldots, m$.

Consider the following commutative diagram where $\mathcal{Z}_P$ its obtained by pull-back and $\mu : \mathbb{C}^m \to \mathbb{R}^m_{\geq 0}$ is given by $\mu(z_1, \ldots, z_m) = (|z_1|, \ldots, |z_m|)$

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_P^*} & \mathbb{C}^m \\ \pi \big\downarrow & & \big\downarrow \mu \\ P & \xrightarrow[i_P]{} & \mathbb{R}^m_{\geq 0} \end{array}$$

The map μ may be thought of as the quotient map for the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| = 1 \text{ for } 1 \leq i \leq m\}$$

on C^m.

Therefore, $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient P, and i_P^* is a $\mathbb{T}^m$-equivariant embedding. The image of R^n under i_P is an n-dimensional affine plane in R^m, which can be written as

$$i_P(\mathbb{R}^n) = \{y \in \mathbb{R}^m \,:\, y = A_P(x) + b_P \textit{ for some } x \in \mathbb{R}^n\} =$$

$$= \{y \in \mathbb{R}^m \,:\, \Gamma y = \Gamma b_P\},$$

where $\Gamma = ((\gamma_{jk}))$ is an $(m - n) \times m$ matrix whose rows form a basis of linear relations between the vectors a_i. That is, Γ is of full rank and satisfies the identity $\Gamma A_P = 0$.

Then we obtain that $\mathcal{Z}_P$ embeds into C^m as the set of common zeros of $m - n$ real quadratic equations:

$$i_P^*(\mathcal{Z}_P) = \left\{ z \in \mathbb{C}^m \ : \sum_{k=1}^{m} \gamma_{jk}|z_k|^2 = \sum_{k=1}^{m} \gamma_{jk} b_k, \ \mathit{for}\ 1 \leq j \leq m - n \right\}$$

The following properties of $\mathcal{Z}_P$ easily follow from its construction.

1. Given a point $z \in \mathcal{Z}_P$, the i^{th} coordinate of $i_P^*(z) \in \mathbb{C}^m$ vanishes if and only if z projects onto a point $x \in P$ such that $x \in F_i$ for some facet F_i.
2. Adding a redundant inequality to results in multiplying $\mathcal{Z}_P$ by a circle.
3. $\mathcal{Z}_P$ is a smooth manifold of dimension $m + n$. The embedding $i_P^* : \mathcal{Z}_P \to \mathbb{C}^m$ has $\mathbb{T}^m$-equivariantly trivial normal bundle.

Associated Polytope of LV-M manifolds (from quadrics to polytopes)

Let N be as before. Let, as before,

$$M_1 = \{z \in \mathbb{C}^n \quad | \quad \sum_{i=1}^{n} \Lambda_i |z_i|^2 = 0,\ \sum_{i=1}^{n} |z_i|^2 = 1\}$$

Let us remark that the standard action of the torus $(\mathbb{S}^1)^n$ on $\mathbb{C}^n$

$$((\exp i\theta_1, \cdots, \exp i\theta_n), z) \longmapsto (\exp i\theta_1 \cdot z_1, \ldots, \exp i\theta_n \cdot z_n) \quad (\star\star)$$

leaves M_1 invariant. The quotient of M_1 by this action can be identified, via the difféomorphism $r \in \mathbb{R}^+ \to r^2 \in \mathbb{R}^+$, to

$$K = \{r \in (\mathbb{R}^+)^n \quad | \quad \sum_{i=1}^{n} r_i \Lambda_i = 0,\ \sum_{i=1}^{n} r_i = 1\}$$

Lemma

The quotient K is a convex polytope of dimension $n - 2m - 1$ with $n - k$ facets.

Proof. By definition K is the intersection of the space A of solutions of an affine system with the closed sets $r_i \geq 0$. Each one of these closed sets defines an affine half-space $A \cap \{r_i \geq 0\}$ in the affine space A. In other words, K is the intersection of a finite number of affine half-spaces. Since this intersection is bounded (since M_1 is compact), one obtains indeed a convex polytope. The weak hyperbolicity condition implies that the affine system that defines K is of maximal rank. Hence, K is of dimension $n - 2m - 1$.

Let us consider in more detail the definition of K. The points $r \in K$ verifying $r_i > 0$ for all i are the points which belong to the interior of the convex polytope. They correspond to the points z de M_1 which also belong to $(\mathbb{C}^*)^n$, i.e. to the points of M_1 such that the orbit under the action $(\star\star)$ is isomorphic to $(\mathbb{S}^1)^n$. The points which belong to a hyperface are exactly the points r of K having all of its coordinates *except one* equal to zero. They correspond to the points z de M_1 which have a unique coordinate equal to zero, i.e. such that its orbit under the action $(\star\star)$ is isomorphic to $(\mathbb{S}^1)^{n-1}$. One obtains from the definition of K that there exist points of K having all coordinates different from zero except the ith coordinate if and only 0 belongs to the convex envelope of the configuration formed by the Λ_j with j different from i; hence if and only if Λ_i is a point which can be eliminated keeping the conditions of Siegel and weak hyperbolicity. therefore one has $n - k$ hyperfaces. $\square$

One calls the convex polytope K *the associated polytope* . One central idea is that the topology of the manifolds M_1, and therefore of the manifolds N, is codified by the combinatorial type of the polytope K. To make this idea more precise, it is interesting to push to the end the reasoning involved in the proof of the preceding lemma. One had seen that

$$K_i = K \cap \{r_i = 0,\ r_j > 0 \text{ for } j \neq i\}$$

is nonempty, and therefore is a hyperface de K, if and only if

$$0 \in \mathcal{H}((\Lambda_j)_{j \neq i}) \ .$$

Analogously, given I a subset of $\{1, \ldots, n\}$, the set

$$K_I = K \cap \{r_i = 0 \text{ for } i \in I,\ r_j > 0 \text{ for } j \notin I\}$$

is nonempty, and therefore it is a facet of K of codimension equal the cardinality of I, if and only if

$$0 \in \mathcal{H}((\Lambda_j)_{j \notin I})$$

One has therefore stablished a very important correspondence between two convex polytopes: the polytope K on one hand and the convex hull of the Λ_i's on the other hand.
This correspondence allows us to to prove the following result:

Theorem

(i) The polytope K is simple, in other words, it is the dual of a simplicial polytope.
(ii) Let P be a simple convex polytope. Then there exists manifolds N, as described before, whose associated polytope is combinatorially equivalent to P.

Sketch of the proof. The first part is a direct consequence of the existence of the correspondence. One translates the weak hyperbolicity condition in the combinatorics of K to deduce that each vertex of K is a vertex of exactly $n - 2m - 1$ edges, and this number is precisely the dimension of K. This property characterizes simple polytopes. To prove (ii), one needs to reconstruct the convex hull of the Λ_i's from the polytope P. The correspondence described before can be expressed in the following way: The convex hull of the Λ_i's must be a Gale diagram of the polytope which is dual to P . There are classical methods in combinatorics and convex geometry to construct such diagrams and this permits to finish the proof. $\square$

Recall

Given $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$ a n-uple of vectors of $\mathbb{C}^m$ fulfilling

1. the *Siegel condition*: 0 belongs to the convex hull $\mathcal{H}(\Lambda)$ of the vectors Λ_i in $\mathbb{C}^n$.
2. the *Weak Hyperbolicity condition*: take I a subset of $\{1, \ldots, n\}$ and let Λ_I be the corresponding set. Then, if $0 \in \mathcal{H}(\Lambda_I)$, we must have Card $I \geq 2m + 1$.

we associate to it a LVM manifold N_Λ constructed as follows. Define

$$\mathcal{S}_\Lambda := \{z \in \mathbb{C}^n \quad | \quad 0 \in \mathcal{H}(\Lambda_{I_z})\} \tag{1}$$

where

$$i \in I_z \iff z_i \neq 0. \tag{2}$$

Then N_λ is the quotient of the projectivization $\mathbb{P}(\mathcal{S}_\Lambda)$ by the holomorphic action

$$(T, [z]) \in \mathbb{C}^m \times \mathbb{P}(\mathcal{S}_\Lambda) \longmapsto \left[z_i \exp\langle \Lambda_i, T\rangle\right]_{i=1,\dots,n} \tag{3}$$

where $\langle -, -\rangle$ denotes the inner product of $\mathbb{C}^m$, and not the hermitian one. It is a compact complex manifold of dimension $n - m - 1$, which is either a m-dimensional compact complex torus (for $n = 2m + 1$) or a non kähler manifold (for $n > 2m + 1$).

We say that Λ_i, or simply i, is an *indispensable point* if every point z of $\mathcal{S}_\Lambda$ satisfies $z_i \neq 0$. We denote by k the number of indispensable points.

Observe that $(\mathbb{C}^*)^n$ acts by multiplication on $\mathcal{S}_\Lambda$ with an open and dense orbit, making it a toric variety. This action commutes with projectivization and with (3), making of N_Λ an equivariant compactification of an abelian Lie group, say G_Λ. A straightforward computation shows the following (Laurent Meersseman)

Proposition

Assume that

$$\operatorname{rank}_{\mathbb{C}} \begin{pmatrix} \Lambda_1 & \dots & \Lambda_{m+1} \\ 1 & \dots & 1 \end{pmatrix} = m + 1. \tag{4}$$

Then G_Λ *is isomorphic to the quotient of* $\mathbb{C}^{n-m-1}$ *by the* $\mathbb{Z}^{n-1}$ *abelian subgroup generated by* $(Id, B_\Lambda A_\Lambda^{-1})$ *where*

$$A_\Lambda = {}^t(\Lambda_2 - \Lambda_1, \dots, \Lambda_{m+1} - \Lambda_1) \tag{5}$$

and

$$B_\Lambda = {}^t(\Lambda_{m+2} - \Lambda_1, \dots, \Lambda_{n-1} - \Lambda_1). \tag{6}$$

Remark

It follows from [5, Lemma 1.1] *that*

$$\operatorname{rank}_{\mathbb{C}} \begin{pmatrix} \Lambda_1 & \dots & \Lambda_n \\ 1 & \dots & 1 \end{pmatrix} = m + 1.$$

Hence, up to a permutation, condition (4) *is always fulfilled.*

We say that N_Λ and $N_{\Lambda'}$ are *G-biholomorphic* if they are $(G_\Lambda, G_{\Lambda'})$-equivariantly biholomorphic.

The manifold N_Λ embeds in $\mathbb{P}^{n-1}$ as the C^∞ submanifold

$$\mathcal{N} = \{[z] \in \mathbb{P}^{n-1} \quad | \quad \sum_{i=1}^{n} \Lambda |z_i|^2 = 0\}. \tag{7}$$

It is crucial to notice that this embedding is not arbitrary but has a clear geometric meaning. Indeed, it is proven in that action (3) induces a foliation of $\mathcal{S}_\Lambda$; that every leaf admits a unique point closest to the origin (for the euclidean metric); and finally that (7) is the projectivization of the set of all these minima **This is a sort of non-algebraic Kempf-Ness Theorem.** So we may say that this embedding is canonical.

The maximal compact subgroup $(\mathbb{S}^1)^n \subset (\mathbb{C}^*)^n$ acts on $\mathcal{S}_\Lambda$, and thus on N_Λ. This action is clear on the smooth model (7). Notice that it reduces to a $(\mathbb{S}^1)^{n-1}$ since we projectivized everything.

The quotient of N_Λ by this action is easily seen to be a simple convex polytope of dimension $n - 2m - 1$, cf. Up to scaling, it is canonically identified to

$$K_\Lambda := \{r \in (\mathbb{R}^+)^n \quad | \quad \sum_{i=1}^{n} \Lambda r_i = 0, \sum_{i=1}^{n} r_i = 1\}. \tag{8}$$

It is important to have a description of K_Λ as a convex polytope in $\mathbb{R}^{n-2m-1}$. This can be done as follows. Take a Gale diagramm of Λ, that is a basis of solutions $(v_1, \ldots, v_n)$ over $\mathbb{R}$ of the system

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \Lambda_i x_i = 0 \\ \displaystyle\sum_{i=1}^{n} x_i = 0 \end{cases} \tag{9}$$

Take also a point ϵ in K_Λ. This gives a presentation of K_Λ as

$$\{x \in \mathbb{R}^{n-2m-1} \quad | \quad \langle x, v_i \rangle \geq -\epsilon_i \text{ for } i = 1, \ldots, n\} \tag{10}$$

This presentation is not unique. Indeed, taking into account that K_Λ is unique only up to scaling, we have

Lemma

The projection (10) *is unique up to action of the affine group of* $\mathbb{R}^{n-2m-1}$.

On the combinatorial side, K_Λ has the following property. A point $r \in K_\Lambda$ is a vertex if and only if the set I of indices i for which r_i is zero is maximal, that is has $n-2m-1$ elements. Moreover, we have

$$r \text{ is a vertex } \iff \mathcal{S}_\Lambda \cap \{z_i = 0 \text{ for } i \in I\} \neq \emptyset \iff 0 \in \mathcal{H}(\Lambda_{I^c}) \tag{11}$$

for I^c the complementary subset to I in $\{1, \ldots, n\}$. This gives a numbering of the faces of K_Λ by the corresponding set of indices of zero coordinates.

To be more precise, we have

$$\begin{aligned} &J \subset \{1, \ldots, n\} \text{ is a face of codimension Card } J \\ &\iff \mathcal{S}_\Lambda \cap \{z_i = 0 \text{ for } i \in J\} \neq \emptyset \iff 0 \in \mathcal{H}(\Lambda_{J^c}) \end{aligned} \tag{12}$$

In particular, K_Λ has $n - k$ facets. Observe moreover that the action (3) fixes $\mathcal{S}_\Lambda \cap \{z_i = 0 \text{ for } i \in J\}$, hence its quotient defines a submanifold N_J of N_Λ of codimension Card J.

Going back to G_Λ, we see that its Lie algebra is generated by the linear vector fields $z_i\partial/\partial z_i$ for $i = 1, \ldots, n$. Due to the quotient by (3), they only generate a vector space of dimension $n - m - 1$ as needed. Amongst these $n - m - 1$ linearly independent vector fields, we can find m of them which extend to $\mathcal{S}_\Lambda$ without zeros and which generates a locally free action of $\mathbb{C}^m$ onto $\mathcal{S}_\Lambda$. For example, we can take the vector fields

$$\eta_i(z) = \left\langle \text{Re } \Lambda_i, \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} \right\rangle, \tag{13}$$

We denote by $\mathcal{F}$ the foliation induced by this action. It is easy to check that $\mathcal{F}$ is independent of the choice of vector fields. Indeed, changing the vector fields just means changing the parametrization of $\mathcal{F}$, that is changing the $\mathbb{C}^m$-action by taking a different basis of $\mathbb{C}^m$.

Pull back the Fubini-Study form of $\mathbb{P}^{n-1}$ to the embedding (7). This is the *canonical Euler form* ω of Λ, as defined in. It is a representative of the Euler class of a particular $\mathbb{S}^1$-bundle associated to $\mathcal{N}_\Lambda$, hence the name. Then $\mathcal{F}$ is transversely kähler with transverse kähler form ω. For our purposes, we will not focus on ω but on the ray $\mathbb{R}^{>0}\omega$ it generates Recall that Λ *fulfills condition (K)* if (9) admits a basis of solutions with integer coordinates; and that Λ *fulfills condition (H)* if (9) does not admit any solution with integer coordinates. If condition (K) is fulfilled, then $\mathcal{F}$ is a foliation by compact complex tori and the quotient space is a projective toric orbifold, see which contains a thorough study of this case.

We just note here that, even if condition (K) is not satisfied, the foliation $\mathcal{F}$ has some compact orbits. Indeed, let I be a vertex of K_Λ. Then, by (11), 0 belongs to $\mathcal{H}(\Lambda_{I^c})$, so by [5, Lemma 1.1],

$$\operatorname{rank}_{\mathbb{C}} \begin{pmatrix} \Lambda_{i_1^c} & \dots & \Lambda_{i_{2m+1}^c} \\ 1 & \dots & 1 \end{pmatrix} = m+1. \tag{14}$$

Hence, up to performing a permutation, we may assume at the same time (4) and

$$I \cap \{1, \dots, m+1\} = \emptyset. \tag{15}$$

We have

Proposition

For each vertex I of K_Λ, the corresponding submanifold N_I is a compact complex torus of dimension m and is a leaf of $\mathcal{F}$. Moreover, assume that Λ satisfies (4) *and* (15)*. Then, letting B_I denote the matrix obtained from* (6) *by erasing the rows $\Lambda_i - \Lambda_1$ for $i \in I$, the torus N_I is isomorphic to the torus of lattice $(Id, B_I A_\Lambda^{-1})$.*

Here we use results of Moerdijk to reconstruct the LVM manifold together with the group action out of the diffeology. A description of the category of irrational fans as equivalent to doubly equivariant maps of LVM manifolds.

Here the deformation theory of equivariant LVM manifolds is explained and then together with the reconstruction theorem we conclude that this implies the existence of the moduli stack of torics.

Let Λ be an admissible configuration. We want to describe the set $\mathcal{M}_\Lambda$ of G-biholomorphism classes of LVM manifolds $N_{\Lambda'}$ such that $\mathcal{S}_{\Lambda'}$ is equal to $\mathcal{S}_\Lambda$ up to a permutation of coordinates in $\mathbb{C}^n$.

We assume that Λ satisfies (4) and

$$\Lambda_i \text{ is indispensable } \iff i \leq k \tag{16}$$

that is, the k indispensable points are the first k vectors of the configuration. In the same way, every class $[N_{\Lambda'}]$ of $\mathcal{M}_\Lambda$ can be represented by a configuration Λ' satisfying (4), (16) and

$$\mathcal{S} := \mathcal{S}_\Lambda = \mathcal{S}_{\Lambda'}. \tag{17}$$

Remark

Condition (17) *is equivalent to* K_Λ *being combinatorially equivalent to* $K_{\Lambda'}$ *with same numbering* (12)*. Observe that because of our convention* (16)*, having the same numbering implies having the same number of indispensable points.*

Now, observe that, because of (4), there exists an affine transformation T of $\mathbb{C}^m$ sending Λ onto a configuration (which we still denote by Λ) whose first $m+1$ vectors satisfies

$$\Lambda_1 = ie_1, \qquad \Lambda_2 - \Lambda_1 = e_1, \qquad \ldots, \qquad \Lambda_{m+1} - \Lambda_1 = e_m \quad (18)$$

for $(e_1, \ldots, e_m)$ the canonical basis of $\mathbb{C}^m$.

It is straightforward to check that this does not change N_Λ up to G-biholomorphism. In the same way, each class of $\mathcal{M}_\Lambda$ can be represented by an element Λ' satisfying (4), (16), (17) and (18). We call $\mathcal{S}$-*normalized configuration* such a configuration.

Let $\mathcal{T}_\Lambda$ be the set of $\mathcal{S}$-normalized configurations. This is an open and connected set in $(\mathbb{C}^m)^{n-m-1}$.

Assume now that $N_{\Lambda'}$ is G-biholomorphic to N_Λ. Then, G_Λ and $G_{\Lambda'}$ as subgroups of $\text{Aut}(N_\Lambda)$, respectively $\text{Aut}(N_{\Lambda'})$ are isomorphic Lie groups. Hence, their universal cover are isomorphic as Lie groups, that is, using the presentation given in Proposition 1, there exists a matrix M in $\text{GL}_{n-m-1}(\mathbb{C})$ which sends the lattice of G_Λ bijectively onto that of $G_{\Lambda'}$. Using notations (5) and (6), this means that there exists a matrix P in $\text{SL}_{n-1}(\mathbb{Z})$ such that

$$M(Id, B_\Lambda A_\Lambda^{-1}) = (Id, B_{\Lambda'} A_{\Lambda'}^{-1})P. \qquad (19)$$

Decomposing P as

$$P = \begin{pmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{pmatrix} \qquad (20)$$

with P_1 a square matrix of size $n-m-1$ and Q_2 a square matrix of size m, we obtain

$$MB_\Lambda A_\Lambda^{-1} = (P_1 + B_{\Lambda'} A_{\Lambda'}^{-1} Q_1) B_\Lambda A_\Lambda^{-1} = P_2 + B_{\Lambda'} A_{\Lambda'}^{-1} Q_2. \qquad (21)$$

Because of (18), this means that

$$ {}^tB_\Lambda = ({}^tP_2 + {}^tQ_2\,{}^tB_{\Lambda'})({}^tP_1 + {}^tQ_1\,{}^tB_{\Lambda'})^{-1} \tag{22} $$

that is

Proposition

Let Λ and Λ' be two $\mathcal{S}$-normalized configurations. Then N_Λ and $N_{\Lambda'}$ are G-biholomorphic if and only if Λ and Λ' satisfies (22).

Thus, $\mathcal{M}_\Lambda$ is the quotient of $\mathcal{T}_\Lambda$ by the action of $\mathrm{SL}_{n-1}(\mathbb{Z})$ described in (22). We claim

Proposition

If the number k of indispensable points is less than $m+1$, then the moduli space $\mathcal{M}(X)$ is an orbifold.

Proof From the previous description, it is enough to prove that the stabilizers of action (22) are finite. Let f be a G-biholomorphism of N_Λ. Set

$$\mathcal{S}_1 = \{w \in \mathbb{C}^{n-m-1} \quad | \quad (1,\dots,1,w) \in \mathcal{S}\}. \tag{23}$$

Observe that (23) is a covering of the quotient N_1 of $\mathcal{S} \cap \{z_1 \cdots z_{m+1} \neq 0\}$ by the action (3). Indeed, we have a commutative diagram

$$\begin{array}{ccccc} (\mathbb{C}^*)^{n-m-1} & \longrightarrow & \mathcal{S}_1 & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow & & \downarrow \\ G_\Lambda & \longrightarrow & N_1 & \longrightarrow & N_\Lambda \end{array} \tag{24}$$

where the horizontal maps are inclusions and the first two vertical ones are coverings.

Then, up to composing with a permutation of $\mathbb{C}^n$, we may assume that f sends N_1 onto itself. Because of assumption (16), the set (23) is a 2-connected open subset of $\mathbb{C}^{n-m-1}$, hence the restriction of f to N_1, say f_1, lifts to a biholomorphic map F_1 of (23). More precisely, $\mathcal{S}_1$ is equal to $\mathbb{C}^{n-m-1}$ minus a finite union of codimension 2 vector subspaces, hence by Hartogs, F_1 extends as a biholomorphism of $\mathbb{C}^{n-m-1}$.

On the other hand, the restriction of f to G_Λ preserves G_Λ and lifts as a biholomorphism $\tilde{F}$ of its universal covering $\mathbb{C}^{n-m-1}$. And we have a commutative diagramm

$$\begin{array}{ccc} \mathbb{C}^{n-m-1} & \xrightarrow{\exp(2i\pi -)} & (\mathbb{C}^*)^{n-m-1} \\ \tilde{F}\downarrow & & \downarrow F_1 \\ \mathbb{C}^{n-m-1} & \xrightarrow{\exp(2i\pi -)} & (\mathbb{C}^*)^{n-m-1} \end{array} \tag{25}$$

But, since the linear map $\tilde{F} = M$ must preserve the abelian subgroup of Proposition 1, using (18) and (19), we have

$$\tilde{F}(z + e_i) = \tilde{F}(z) + P_1 e_i := \tilde{F}(z) + \sum_{j=1}^{n-m-1} a_{ij} e_j \tag{26}$$

that is Q_1 is equal to 0. But through (25), this implies that

$$F_1(w) = \left(w_1^{a_{1j}} \cdots w_{n-m-1}^{a_{n-m-1j}}\right)_{j=1}^{n-m-1} \tag{27}$$

Now, recall that F_1 is a biholomorphism of the whole $\mathbb{C}^{n-m-1}$, so must send a coordinate hyperplane onto another one without ramifying. This shows that $P_1 = (a_{ij})$ is a matrix of permutation. Hence every stabilizer is a subgroup of the group of permutations with $n - m - 1$ elements, so is finite.

Example

Tori. Let $n = 2m + 1$, then there are $2m + 1$ idispensable points, $\mathcal{S}$ is $(\mathbb{C}^*)^n$ and N is a compact complex torus of dimension m [4, Theorem 1]. The associate polytope K is reduced to a point and $N = G$. The moduli space $\mathcal{M}$ is equal to the moduli space of compact complex tori of dimension m, which is not an orbifold for $m > 1$.

Example

Hopf surfaces. Let $n = 4$ and $m = 1$, then there are two indispensable points and $\mathcal{S}$ is $(\mathbb{C}^*)^2 \times \mathbb{C}^2 \setminus \{(0,0)\}$. A $\mathcal{S}$-admissible configuration is given by a couple complex numbers (λ_3, λ_4) belonging to

$$\{z \in \mathbb{C} \quad | \quad \Re z < 0 \text{ and } \Re z < \Im z\}. \tag{28}$$

The manifold N_Λ is equal to the diagonal Hopf surface obtained by taking the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the group generated by

$$(z, w) \longmapsto (\exp 2i\pi(\lambda_3 - \lambda_1) \cdot z, (\exp 2i\pi(\lambda_4 - \lambda_1) \cdot w) \tag{29}$$

Two points (λ_3, λ_4) and (λ'_3, λ'_4) with coordinates in (28) are equivalent if and only if their difference is in the lattice $\mathbb{Z} \oplus Z$ or if the difference of (λ_3, λ_4) by the switched (λ'_4, λ'_3) is in this lattice. The isotropy group of a point is $\mathbb{Z}_2$ for the diagonal $\lambda_3 = \lambda_4$ and is zero elsewhere. The moduli space is an orbifold. Observe that not all Hopf surfaces are obtained as LVM-manifolds, but only the linear diagonal ones. Now, they coincide with the set of Hopf surfaces that are equivariant compactifications of $(\mathbb{C}^*)^2$.

- Bosio, F and Meersseman, L., “Real quadrics in $\mathbb{C}^n$, complex manifolds and convex polytopes", *Acta Math.*, Vol. 197, pp. 53-127, 2006.
- Santiago López de Medrano and Alberto Verjovsky, *A new family of complex, compact, non-symplectic manifolds*, Bulletin of the Brazilian Mathematical Society **28** (1997), no. 2, 253–269.
- Laurent Meersseman, *Un nouveau procédé de construction géométrique de variétés compactes, complexes, non algébriques, en dimension quelconque*, PhD Thesis, Lille, 1998.
- Laurent Meersseman, *A new geometric construction of compact complex manifolds in any dimension*, Mathematische Annalen **317** (2000), 79–115.
- Laurent Meersseman and Alberto Verjovsky, *Holomorphic principal bundles over projective toric varieties*, Journal für die Reine und Angewandte Mathematik (2004), 57–96.