On the geometry of three-dimensional homogeneous Lorentzian manifolds

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Abstract

A pseudo-Riemannian manifold (M, g) is *homogeneous* provided that, for any points $p, q \in M$, there is an isometry ϕ such that $\phi(p) = q$. Using the notion of *homogeneous pseudo-Riemannian structure* introduced in [8], we proved that a three-dimensional connected, simply connected and complete homogeneous Riemannian manifold is either symmetric or a Lie group endowed of a left-invariant Lorentzian metric [1]. Together with the results on three-dimensional Lorentzian Lie groups obtained by Cordero and Parker [6] and Rahmani [10], this leads to the classification of three-dimensional homogeneous Lorentzian manifolds.

The result above is also the starting point to characterize and classify some classes of three-dimensional homogeneous Lorentzian manifolds, having a special geometric meaning. It is interesting to compare such results in the Lorentzian case with their Riemannian analogues.

Naturally reductive and g.o. Lorentzian spaces are both related to the notion of homogeneous geodesic. A (connected) homogeneous pseudo-Riemannian manifold (M, g) can be identified with $(K/H, g)$, where H is the isotropy group at a fixed point o of M. A geodesic Γ through the origin $o \in M = K/H$ is called *homogeneous* if it is the orbit of a 1-parameter subgroup.

Homogeneous geodesics have been investigated by many authors. In the framework of Lorentzian geometry, they aquire a new interest. In fact, homogeneous Lorentzian spaces for which all null (that is, lightlike) geodesics are homogeneous, are candidates for constructing solutions to the 11-dimensional supergravity, which preserve more than 24 of the available 32 supersymmetries[7]. Together with R.A. Marinosci, we studied the set of homogeneous geodesics of three-dimensional Lorentzian Lie groups ([3],[4]). This permits to determine all three-dimensional g.o. and naturally reductive Lorentzian spaces.

Einstein-like metrics, introduced by A. Gray in [9], are defined through conditions on the Ricci tensor. Several papers have been devoted to Einstein-like metrics of Riemannian manifolds. In Lorentzian geometry, Einstein-like metrics have been studied in three-dimensional Lorentzian manifolds admitting a parallel null vector field [5].

We completely classified Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds [2]. As in the Riemannian case, the Ricci tensor being cyclic-parallel (respectively, a Codazzi tensor) is related to natural reductivity (respectively, symmetry). However, differently from the Riemannian case, some exceptional examples arise.

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