Stationary surfaces in $\mathbb{L}^3$

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Let $\mathbb{L}^3$ be the 3-dimensional Lorentz-Minkowski space, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $\langle , \rangle = dx_1^2 + d_2^2 - dx_3^2$, where $x = (x_1, x_2, x_3)$ are the canonical coordinates in $\mathbb{R}^3$. We are interested in the following variational problem. Let Π be a spacelike plane and denote by Π^+ one of the two halfspaces at which Π divides $\mathbb{L}^3$. Let S be a compact spacelike surface whose boundary ∂S lies in Π and $\text{int}(S) \subset \Pi^+$ and let us denote Ω the bounded domain by ∂S in Π. We consider all perturbations in such way that S is adhered to Π ($\partial S \subset \Pi$) and $\text{int}(S)$ remains in Π^+. Under the effect of a potential energy that, up constants, measures at each point the distance to Π, the energy of the system involves the surface area of S, the area of the region of contact of S with Π and the energy defined by Y:

$$E = |S| - \cosh\beta|\Omega| + \int_S Y \ dS.$$

We are interested in those configurations in a state of equilibrium, that is, the energy of the system is critical under any perturbation that do not change the volume enclosed by $S \cup \Omega$. According to the principle of virtual works, the equilibrium of the system is achieved if the following two conditions hold: i) the mean curvature H of S is a linear function on the distance to Π: $H(x) = \kappa x_3(x) + \lambda$, κ is a constant and λ is a Lagrange multiplier arising from the volume constraint and ; ii) the hyperbolic angle β with which S and Π intersect along ∂S is constant. In such case, we shall say that S is a *stationary surface*. In absence of the potential Y, S is a spacelike surface with constant mean curvature. Constant mean curvature spacelike surfaces are interesting in relativity theory [7] and some results have been obtained in the context of our variational problem [1, 2]. As first result, we obtain:

Theorem 1 *Given a stationary surface in $\mathbb{L}^3$ whose mean curvature depends only on the time coordinate, $H(x) = H(x_3)$, there is a vertical straight-line L orthogonal to Π about which M is rotational symmetric. Moreover, S is topologically a disc and the intersection of S with a plane orthogonal to L is a circle whose center lies on the axis L.*

Therefore, a stationary surface is known by its profile curve, $x_3 = u(r)$, that is, $S = \{r\cos\theta, r\sin\theta, u(r)); r \in [0, R), \theta \in \mathbb{R}\}$. The Euler-Lagrange writes now in terms of u as

$$\frac{1}{r}\frac{d}{dr}\left(\frac{ru'(r)}{\sqrt{1-u'(r)^2}}\right) = \kappa\, u(r), \qquad 0 \leq r < R.$$

From now, and after a change of coordinates, we assume rotational symmetry and that the potential is proportional to the time coordinate. Thus, the mean curvature of S satisfies $H(x) = \kappa x_3$. In this communication, our aim is twofold:

1. An understanding of the qualitative properties of the shape of a stationary surface with respect to the sign of the constant κ (if $\kappa > 0$, we say that S is a sessile surface and if $\kappa < 0$, S is a pendent surface).
2. The control of the size of a stationary surface, as for example, about the measure of the height and the volume of a stationary surface in relation with the initial data.

In relation with 1), we prove that *a sessile stationary surface in Minkowski space $\mathbb{L}^3$ is asymptotic with the light cone of $\mathbb{L}^3$ at infinity.* Also, we discuss the problem of existence and uniqueness. As consequence,

Theorem 2 *Given κ, β and Π, there exists a (sessile or pendent) stationary surface supported on Π and where β is the angle of contact between the surface and Π. The surface is unique up isometries of the ambient space $\mathbb{L}^3$.*

With respect to the objective 2), we prove:

Theorem 3 *Given $\kappa > 0$, any sessile stationary surface S resting on Π on a disc of radius R, the height $q = u(R) - u(0)$ of S satisfies:*

$$q < R\frac{\cosh\beta - 1}{\sinh\beta}.$$

In the same way, we prove

Theorem 4 *Let S be a sessile stationary surface resting on Π on a disc of radius R and let β be the contact angle along its boundary. Then S satisfies*

$$\frac{2\sinh\beta}{R\kappa} + \frac{R}{\sinh\beta} + \frac{2R}{3}\frac{1-\cosh^3\beta}{\sinh^3\beta} < u(0) < \frac{2\sinh\beta}{R\kappa}.$$

$$u(R) < \frac{2\sinh\beta}{R\kappa} + R\frac{\cosh\beta}{\sinh\beta} + \frac{2R}{3}\frac{1-\cosh^3\beta}{\sinh^3\beta}.$$

Finally, we point up also an existence result in terms of the volume:

Theorem 5 *For each constants κ, β and a positive real number V, there exists a unique (sessile or pendent) stationary surface resting on a spacelike plane Π enclosing a volume V and that makes a constant angle β along its boundary.*

With respect to the techniques, we use the classical theory of ordinary differential equations in proving results on existence; on the other hand, and with respect to the second objective, we will compare our stationary surfaces with known barriers: rotational spacelike surfaces with constant mean curvature, that is, hyperbolic discs. The fact that the mean curvature equation is of elliptic type allows us to apply the classical maximum principle and so, to establish relations between the size of our stationary surfaces with the one of appropriate hyperbolic discs.

If we compare our study of stationary surfaces in $\mathbb{L}^3$ with the analysis given in Euclidean 3-space for liquid drops [3], we can see analogies, mainly the techniques, and differences. With respect to these ones, we point out that the spacelike condition on the surface imposes strong geometric restrictions for its possible configurations. It is worthwhile to bring out some of them:

1. Stationary surfaces in $\mathbb{L}^3$ can extend to be graphs in the whole plane. This means that the associated Euler equation can solved be entire solutions. This is not the case in Euclidean setting.
2. Sessile stationary surfaces in $\mathbb{L}^3$ have not the phenomenon of meniscus. Moreover and by fixing the constant κ, the ambient space $\mathbb{L}^3$ can be foliated by sessile stationary surfaces.
3. Pendent stationary surfaces in $\mathbb{L}^3$ do not present vertical points. In particular, prescribing any volume, there exists a pendent stationary surface enclosing that volume.

Part of these results of this communication can be found in [4, 5, 6].

References

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