LOOKING FOR A NATURAL NOTION OF SUPERSPACE†

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Definition. A **representation** of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, satisfying,

$$\rho([x,y]) = [\rho(x), \rho(y)]$$

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Why Lie superalgebras?

Roughly: To bring symmetric maps in,....

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$$\Longrightarrow \quad \Gamma := [\![\cdot \, , \, \cdot]\!]|_{\mathfrak{g}_1 \times \mathfrak{g}_1} \quad \text{is symmetric}$$

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$$\Rightarrow \quad [\![x + u, y + v]\!] = [x, y] + \rho(x)(v) - \rho(y)u + \Gamma(u, v)$$

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Proposition. This is the case when $\rho = \mathrm{ad}$.

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Within this context, one writes $\mathfrak{g} = \mathfrak{gl}(V_0|V_1)$ to distinguish it from $\mathfrak{gl}(V_0 \oplus V_1)$.

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$$\mathfrak{g} = \{\,\text{Linear maps } S : V \to V\,\} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with

$$\mathfrak{g}_0 = \{S : V \to V \mid S(V_\alpha) \subset V_\alpha\,\} \simeq \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

$$\mathfrak{g}_1 = \{S : V \to V \mid S(V_\alpha) \subset V_{\alpha+1}\,\} \simeq \left\{ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \right\}$$

$$[\![S, T]\!] = S \circ T - (-1)^{|S|\,|T|} T \circ S$$

Within this context, one writes $\mathfrak{g} = \mathfrak{gl}(V_0|V_1)$ to distinguish it from $\mathfrak{gl}(V_0 \oplus V_1)$.

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Thus obtain a Lie superalgebra $\mathfrak{g} = (\mathbb{R}\,\Delta) \oplus (\mathbb{R}\,\mathrm{d} \oplus \mathbb{R}\,\delta)$ embedded in $\mathfrak{gl}\,(\Omega(M)_0|\Omega(M)_1)$, where

$$\Omega(M)_0 = \oplus_{k \geq 0} \Omega^{2k}(M), \qquad \text{and} \qquad \Omega(M)_1 = \oplus_{k \geq 0} \Omega^{2k+1}(M)$$

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which together $[\![\mathrm{d}, \mathcal{L}_X]\!] = 0$ and $[\![\mathrm{d}, \mathrm{d}]\!] = 0$, yield more Lie superalgebras embedded in $\mathfrak{gl}\left(\Omega(M)_0 | \Omega(M)_1\right)$.

A geometric difference between Lie algebras and Lie superalgebras.

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Let $V = V_0 \oplus V_1$ be a *complex* supervector space, and let

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To keep track of their type of symmetry, write

$$B_\alpha(v,w) = \varepsilon_\alpha\, B_\alpha(w,v)\,, \qquad \varepsilon_\alpha = \begin{cases} +1 & \text{if } B_\alpha \text{ symmetric} \\ -1 & \text{if } B_\alpha \text{ skew-symmetric} \end{cases}$$

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Define a nondegenerate bilinear form B on $V = V_0 \oplus V_1$:

$$B(v_0 + v_1, w_0 + w_1) := B_0(v_0, w_0) + B_1(v_1, w_1)$$

We might now consider the **Lie algebra**

$$\mathfrak{g}_B(V_0 \oplus V_1) = \{S \in \mathfrak{gl}(V_0 \oplus V_1) \mid B(S(u), v) + B(u, S(v)) = 0\}$$

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$$S \leftrightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \text{so that,} \quad S \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} \alpha(v_0) + \beta(v_1) \\ \gamma(v_0) + \delta(v_1) \end{pmatrix}.$$

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Besides, $B_1(w, \gamma(u)) = -B_0(\beta(w), u)$ is satisfied if and only if, either $\gamma \equiv 0 \equiv \beta$, or else $\varepsilon_{B_0} \varepsilon_{B_1} = 1$.

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathfrak{g}_B(V_0 | V_1) \quad \Longleftrightarrow \quad \begin{cases} \alpha \in \mathfrak{g}_{B_0}(V_0), \\ \delta \in \mathfrak{g}_{B_1}(V_1), \\ B_1(w, \gamma(u)) = +B_0(\beta(w), u). \end{cases}$$

Besides, $B_1(w, \gamma(u)) = +B_0(\beta(w), u)$ is satisfied if and only if, either $\gamma \equiv 0 \equiv \beta$, or else $\varepsilon_{B_0} \varepsilon_{B_1} = -1$.

Remark. Let $v_0 \in V_0$, and $v_1 \in V_1$.
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$$\left[\begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta' \\ \gamma' & 0 \end{pmatrix} \right] \in \mathfrak{g}_{B_0}(V_0) \oplus \mathfrak{g}_{B_1}(V_1) \quad \Longleftrightarrow \quad \varepsilon_{B_0} \varepsilon_{B_1} = 1$$

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Moral. The Lie algebra $\mathfrak{g}_B(V_0 \oplus V_1)$ likes $\varepsilon_{B_0}\varepsilon_{B_1} = 1$ to describe the linear transformations that preserve the nondegenerate bilinear form

$$B(v_0 + v_1, w_0 + w_1) := B_0(v_0, w_0) + B_1(v_1, w_1)$$

whereas the Lie superalgebra $\mathfrak{g}_B(V_0|V_1)$ likes $\varepsilon_{B_0}\varepsilon_{B_1} = -1$.

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$$[x, \Gamma(u, v)] = \Gamma(\rho(x)u, v) + \Gamma(u, \rho(x)v) \tag{J1}$$

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$$\text{(J2)} \quad \rho\big(\Gamma(u,v)\big)(w) + \rho\big(\Gamma(w,u)\big)(v) + \rho\big(\Gamma(v,w)\big)(u) = 0$$

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Proposition. This is the case when $\rho = \mathrm{ad}$.

Observe that: A subgroup of the group $\mathrm{GL}(\mathfrak{g}_0) \times \mathrm{GL}(\mathfrak{g}_1)$ **acts** on the set of such triples, producing isomorphic Lie superalgebras on each orbit:

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$$\begin{aligned}
[\,\cdot\,,\,\cdot\,]' &= T\,[\,T^{-1}(\,\cdot\,)\,,\,T^{-1}(\,\cdot\,)\,] \\
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When $[\,\cdot\,,\,\cdot\,]$ is kept fixed in $\mathfrak{g}_0$, and $\rho = \mathrm{ad}$,

$$G = \{(T, S) \mid [\mathrm{ad}(\,\cdot\,), T \circ S^{-1}] = 0\} \subset \mathrm{Aut}(\mathfrak{g}_0) \times \mathrm{GL}(\mathfrak{g}_0)$$

since $\mathrm{ad}(T^{-1}(x)) = T^{-1} \circ \mathrm{ad}(x) \circ T$.

Write $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0) = \{\Gamma\text{'s satisfying (J1)}\}$.

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To classify the different Lie superalgebras on $\mathfrak{g}_0$ with $[\,\cdot\,,\,\cdot\,]$ fixed and $\rho = \mathrm{ad}$, amounts to parametrize the orbits in $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$ under the left G-action

$$\Gamma \quad \mapsto \quad (T, S) \cdot \Gamma = T\Big(\Gamma\big(S^{-1}(\,\cdot\,),\, S^{-1}(\,\cdot\,)\big)\Big)$$

Application. Spacetime is conveniently identified —locally, at least— with the Lie algebra $\mathfrak{u}_2 = \mathrm{Lie}\,(\mathrm{U}(2))$. In fact,

$$x \in \mathfrak{u}_2 \;\Leftrightarrow\; x = i \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}; \quad \det(x) = x_1^2 + x_2^2 + x_3^2 - x_0^2$$

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Project: Apply the classification to $\mathfrak{g}_0 = \mathfrak{u}_2$ to understand what the **natural** possibilities for super-spacetimes might be.

Strategy: Note first that $\mathfrak{u}_2$ is a **real subalgebra** of $\mathfrak{gl}_2(\mathbb{C})$. Thus, apply the classification scheme first to $\mathfrak{gl}_2(\mathbb{C})$.

Consider the following basis of $\mathfrak{gl}_2(\mathbb{C})$:

$$x_0 := I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad x_1 := H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$x_2 := E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x_3 := F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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and the following basis of $\mathfrak{u}_2$:

$$w_0 := iI = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad w_3 := iH = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$w_2 := E - F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad w_1 := i(E+F) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Proposition. *The space* $\mathrm{Sym}_{\mathrm{ad}}(\mathfrak{gl}_2(\mathbb{C}))$ *depends on three complex parameters* (λ, μ, ν) *in such a way that* $\Gamma : \mathfrak{gl}_2(\mathbb{C}) \times \mathfrak{gl}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C})$ *is given by,*

$$
\begin{array}{llll}
\Gamma(x_0, x_0) = \lambda x_0 & & & \\
\Gamma(x_0, x_1) = \mu x_1 & \Gamma(x_1, x_1) = 2\nu x_0 & & \\
\Gamma(x_0, x_2) = \mu x_2 & \Gamma(x_1, x_2) = 0 & \Gamma(x_2, x_2) = 0 & \\
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Notation. Let us denote by $\mathfrak{gl}_2(\lambda, \mu, \nu)$ the $\mathbb{F}$-Lie superalgebra (with $\mathbb{F}$ either $\mathbb{C}$ or $\mathbb{R}$) $\mathfrak{gl}_2 \oplus \mathfrak{gl}_2$ defined by the parameter values (λ, μ, ν).

Theorem. $\mathfrak{gl}_2(\lambda,\mu,\nu) \simeq \mathfrak{gl}_2(\lambda',\mu',\nu')$ *are isomorphic if and only there are nonzero constants* a*,* b *and* c *in the ground field* $\mathbb{F}$*, such that,*

$$\lambda' = \lambda\,\frac{1}{ab^2}, \qquad \mu' = \mu\,\frac{1}{abc}, \qquad \nu' = \nu\,\frac{a}{c^2}.$$

In particular, there are exactly eight different isomorphism classes of such Lie superalgebras when the ground field is $\mathbb{C}$*, whereas there are ten when the ground field is* $\mathbb{R}$*, since the sign of the product* $\lambda\nu$ *must be preserved.*

For the real Lie superalgebras $\mathfrak{u}_2(\lambda, \mu, \nu)$ whose underlying supervector space is $\mathfrak{u}_2 \oplus \mathfrak{u}_2$, use the basis $w_0 = iI$, $w_3 = iH$, $w_2 = E - F$ and $w_1 = i(E + F)$, to get:

$$\Gamma(w_0, w_0) = i\lambda w_0$$

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Therefore, λ, μ and ν have to be restricted, from taking arbitrary complex values in $\mathfrak{gl}_2(\mathbb{C}; \lambda, \mu, \nu)$, to take only purely imaginary values on $\mathfrak{u}_2(\lambda, \mu, \nu)$.

Classif. Thm. 3-dim'l Lie Superalgs

Notation: Write, $\mathfrak{g}_0 = \mathrm{Span}\{e_1, e_2, e_3\}$, and

$$\begin{aligned}
[e_1, e_2] &= ae_2 + ce_3\,, \\
[e_1, e_3] &= be_2 + de_3\,, \quad \Rightarrow \quad \mathrm{ad}(e_1)|_{\mathfrak{g}_0'} \;\leftrightarrow\; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
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\end{aligned}$$

Prop. Let $\mathfrak{g}_0 = \mathfrak{g}_1$ be 3-dimensional, with $\rho = \mathrm{ad}$. Set $\mathfrak{g}_0' = [\mathfrak{g}_0, \mathfrak{g}_0]$. Then,

$\dim \mathfrak{g}_0'$	3	2	1	0
$\dim \mathrm{Sym}_{\mathrm{ad}}(\mathfrak{g}_0)$	0	1	5	18

$\mathfrak{g}_0$	$\dim \mathfrak{g}_0'$	A	Constraints
$\mathfrak{p}(\mathbb{F})$	2	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
$\mathfrak{q}_\lambda(\mathbb{F})$	2	$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$	$0 < \|\lambda\| \leq 1$
$\mathfrak{q}_\lambda^1(\mathbb{R})$	2	$\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$	$\lambda \in \mathbb{R}$
$\mathfrak{q}_0(\mathbb{F})$	1	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	
$\mathfrak{h}(\mathbb{F})$	1	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	
$\mathfrak{a}(\mathbb{F})$	0	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	

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6. $\exists$ 10 (resp., 18) based on $\mathfrak{a}(\mathbb{C})$ (resp., $\mathfrak{a}(\mathbb{R})$), plus a family depending on four complex parameters (resp., two families depending on two real parameters each, plus three families depending on four real parameters each).

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Sketch of The Proofs

Lemma. *Let* $\mathbb{F}$ *be either* $\mathbb{R}$ *or* $\mathbb{C}$*, and let* $\dot{\mathbb{F}} = \mathbb{F} - \{0\}$*. The set of pairs* $(T, S) \in \mathrm{Aut}(\mathfrak{gl}_n) \times \mathrm{GL}(\mathfrak{gl}_n)$ *such that* $[T(x), S(y)] = S([x, y])$*, with the group structure inherited from the direct product* $\mathrm{Aut}(\mathfrak{gl}_n) \times \mathrm{GL}(\mathfrak{gl}_n)$*, is isomorphic to the direct product group* $\dot{\mathbb{F}} \times \dot{\mathbb{F}} \times \dot{\mathbb{F}} \times \mathrm{Aut}(\mathfrak{sl}_n)$*.*

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Proof. Use the fact $\mathfrak{g} = \mathfrak{gl}_n = \langle I_n \rangle \oplus \mathfrak{sl}_n$, to see that:

(1) $T \in \mathrm{Aut}(\mathfrak{gl}_n)$ if and only if $\exists a \in \dot{\mathbb{F}}$ and $t \in \mathrm{Aut}(\mathfrak{sl}_n)$

$$T = \begin{pmatrix} a & 0 \\ 0 & t \end{pmatrix}$$

(2) For any $x \in \mathfrak{gl}_n$,

$$\mathrm{ad}(x) = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{ad}(x)|_{\mathfrak{sl}_n} \end{pmatrix}$$

(3) $T^{-1} \circ S$ commutes with $\mathrm{ad}(x)$ if and only if

$$T^{-1} \circ S = \begin{pmatrix} b & 0 \\ 0 & u \end{pmatrix}$$

for some $b \in \dot{\mathbb{F}}$, and $u \in \mathrm{GL}(\mathfrak{sl}_n)$ such that $u \circ (\,\mathrm{ad}(x)|_{\mathfrak{sl}_n}\,) = (\,\mathrm{ad}(x)|_{\mathfrak{sl}_n}\,) \circ u$.

Since $u \circ (\,\mathrm{ad}(x)|_{\mathfrak{sl}_n}\,) = (\,\mathrm{ad}(x)|_{\mathfrak{sl}_n}\,) \circ u$ is to hold true for any $x \in \mathfrak{sl}_n$, $u = c\,\mathrm{Id}_{\mathfrak{sl}_n}$ for some $c \in \dot{\mathbb{F}}$. Whence,

$$S = \begin{pmatrix} a & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & c\,\mathrm{Id}_{\mathfrak{sl}_n} \end{pmatrix}$$

and we obtain the desired correspondence via $(T, S) \leftrightarrow (a, b, c, t)$. The statement about the group structure follows easily. $\square$

Now, the structure of the vector space $\mathrm{Hom}_{\mathrm{ad}}(S^2(\mathfrak{g}_1), \mathfrak{g}_0)$ of ad-equivariant maps when $\mathfrak{g}_0 = \mathfrak{g}_1 = \mathfrak{gl}_n$ can be done with the help of Schur's Lemma once we know the ad-invariant subspaces that appear in the decomposition of $S^2(\mathfrak{g}_1)$. Since $\mathfrak{gl}_n = \langle I_n \rangle \oplus \mathfrak{sl}_n$,

$$
\begin{aligned}
S^2(\langle I_n \rangle \oplus \mathfrak{sl}_n) &= (S^0(\langle I_n \rangle) \otimes S^2(\mathfrak{sl}_n)) \oplus (S^1(\langle I_n \rangle) \otimes S^1(\mathfrak{sl}_n)) \oplus (S^2(\langle \\
&\simeq S^2(\mathfrak{sl}_n) \oplus \mathfrak{sl}_n \oplus \langle I_n \rangle
\end{aligned}
$$

It is well known, however, that (see [9], P. 300)

$$
S^2(\mathfrak{sl}_n) = \begin{cases} \langle I_n \rangle \oplus V_5 & \text{if } n = 2 \\ \langle I_n \rangle \oplus \mathfrak{sl}_3 \oplus V_{27} & \text{if } n = 3 \\ \langle I_n \rangle \oplus \mathfrak{sl}_n \oplus V_{n_1} \oplus V_{n_2} & \text{if } n \geq 4 \end{cases}
$$

where V_j is an $\mathfrak{sl}_n$-irreducible subspace of dimension j, $n_1 = \frac{n+3}{n-1}\binom{n}{2}\binom{n}{2}$, and $n_2 = \frac{n-3}{n-1}\binom{n}{2}\binom{n+1}{2}$. In other words, we may rewrite it symbolically as,

$$S^2(\mathfrak{sl}_n) = \langle\, I_n \rangle \oplus (1-\delta_{2n})\mathfrak{sl}_n \oplus W$$

where δ_{2n} is the Kronecker symbol, and W does not contain any ad-invariant subspace isomorphic neither to $\langle\, I_n\rangle$, nor to $\mathfrak{sl}_n$. Therefore, using Schur's Lemma we conclude that,

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{ad}}(S^2(\langle\, I_n\rangle \oplus \mathfrak{sl}_n), \langle\, I_n\rangle \oplus \mathfrak{sl}_n) = \\
&\quad = \mathrm{Hom}_{\mathrm{ad}}(\langle\, I_n\rangle \oplus (1-\delta_{2n})\mathfrak{sl}_n \oplus W \oplus \mathfrak{sl}_n \oplus \langle\, I_n\rangle, \langle\, I_n\rangle \oplus \mathfrak{sl}_n) \\
&\quad = \mathrm{Hom}_{\mathrm{ad}}(\langle\, I_n\rangle, \langle\, I_n\rangle) \oplus \mathrm{Hom}_{\mathrm{ad}}(\langle\, I_n\rangle, \langle\, I_n\rangle) \\
&\qquad \oplus (1-\delta_{2n})\ \mathrm{Hom}_{\mathrm{ad}}(\mathfrak{sl}_n, \mathfrak{sl}_n) \oplus \mathrm{Hom}_{\mathrm{ad}}(\mathfrak{sl}_n, \mathfrak{sl}_n) \\
&\quad = \lambda\ \mathrm{Id}_{\langle\, I_n\rangle} \oplus \mu\ \mathrm{Id}_{\mathfrak{sl}_n} \oplus \nu\ \mathrm{Id}_{\langle\, I_n\rangle} \oplus (1-\delta_{2n})\,\epsilon\ \mathrm{Id}_{\mathfrak{sl}_n}\,, \quad \lambda, \mu, \nu, \epsilon \in \mathbb{C}
\end{aligned}$$

which, after relabeling the generators, can be rewritten as

$$\mathrm{Hom}_{\mathrm{ad}}(S^2(\langle\, I_n\rangle \oplus \mathfrak{sl}_n), \langle\, I_n\rangle \oplus \mathfrak{sl}_n) = \lambda\, \mathbf{e}_{\lambda} \oplus \mu\, \mathbf{e}_{\mu} \oplus \nu\, \mathbf{e}_{\nu} \oplus (1-\delta_{2n})\, \epsilon\, \mathbf{e}_{\epsilon}$$

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