Initial data sets for the Schwarzschild spacetime

Alfonso García-Parrado Gómez-Lobo and Juan A. Valiente Kroon

Santiago de Compostela 6^{th} February 2007

Overview

- A local invariant characterization of Schwarzschild geometry.
- Schwarzschild initial data: necessary conditions.
- Schwarzschild initial data: sufficient conditions.
- Example: time-symmetric initial data.

References

[1] A. García-Parrado & J. A. Valiente Kroon, *Initial data sets for the Schwarzschild spacetime* Phys. Rev. D **75**, 024027 (2007).

[2] J. J. Ferrando & J. A. Sáez, *An intrinsic characterization of the Schwarzschild metric*, Class. Quantum Grav. **15**, 1323 (1998).

Background

Definition 1 *Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth connected Lorentzian manifold and $(\mathcal{S}, h_{ij})$ a Riemannian manifold. An isometric embedding is an embedding $\phi : \mathcal{S} \to \mathcal{M}$ such that $\phi^* g_{\mu\nu} = h_{ij}$.*

Remarks

- In our setting $\mathcal{M}$ will be a *globally hyperbolic* spacetime and $\mathcal{S}$ a three dimensional Riemannian manifold.
- We will use abstract indexes to denote tensors. The signature convention for the spacetime is $(-, +, +, +)$. Greek indexes $\mu, \nu, \rho, \ldots$ will refer to tensors on $\mathcal{M}$ and Latin indexes $a, b, i, j, \ldots$ to tensors defined in $\mathcal{S}$.

An important question is to know when a given Riemannian manifold $(\mathcal{S}, h_{ij})$ can be isometrically embedded in a spacetime $(\mathcal{M}, g_{\mu\nu})$ which solves Einstein equations in such a way that the image $\phi(\mathcal{S})$ under the isometric embedding ϕ is a *Cauchy hypersurface* of $\mathcal{M}$. When such embedding exists then $\phi(\mathcal{S})$ is called *initial data hypersurface* and $\mathcal{M}$ the *initial data development.*

Theorem 1 *A Riemannian manifold $(\mathcal{S}, h_{ij})$ can be isometrically embedded in a vacuum spacetime $(\mathcal{M}, g_{\mu\nu})$ if and only if there exists a smooth symmetric tensor field K_{ij} defined on $\mathcal{S}$ fulfilling the following conditions (vacuum constraints)*

$$r + K^2 - K^{ij} K_{ij} = 0, \tag{1}$$

$$D^j K_{ij} - D_i K = 0, \tag{2}$$

where D_i is the Levi-Civita connection of h_{ij} (so $D_i h_{jk} = 0$), r is the scalar curvature of the Riemann tensor constructed from D_i and $K \equiv K^i{}_i$.

When the conditions of this theorem hold the triad $(\mathcal{S}, h_{ij}, K_{ij})$ is called *vacuum initial data set.*

A local characterization of Schwarzschild geometry

- $\star$-product of tensors which are 2-fold forms (antisymmetric in pairs of indexes)

$$(U\star V)_{\mu\nu\lambda\rho} \equiv \frac{1}{2} U_{\mu\nu\kappa\pi} V^{\kappa\pi}{}_{\lambda\rho} \,,$$

- Let $C_{\mu\nu\lambda\rho}$ denote the Weyl tensor of the metric $g_{\mu\nu}$ and $C^*_{\mu\nu\lambda\rho}$ its dual. We define

$$\mathrm{tr}(C\star C\star C) \equiv \frac{1}{2}(C\star C\star C)^{\mu\nu}{}_{\mu\nu},\ \rho \equiv \left(\frac{1}{12}\mathrm{tr}(C\star C\star C)\right)^{1/3},\ \alpha \equiv \frac{1}{9\rho^2} g^{\mu\nu}\nabla_\mu\rho\nabla_\nu\rho + 2\rho,$$

$$S_{\mu\nu\lambda\sigma} \equiv \frac{1}{3\rho}\left(C_{\mu\nu\lambda\sigma} + \rho(g_{\mu\nu}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\sigma})\right),\ P_{\mu\nu} \equiv C^*_{\lambda\mu\sigma\nu}\nabla^\lambda\rho\nabla^\sigma\rho,\ Q_{\mu\nu} \equiv S_{\lambda\mu\sigma\nu}\nabla^\lambda\rho\nabla^\sigma\rho.$$

Theorem 2 (Ferrando & Sáez, 1998) *Necessary and sufficient conditions for a vacuum spacetime $(\mathcal{M}, g_{\mu\nu})$ to be locally isometric to the Schwarzschild spacetime are*

$$\rho \neq 0,\ (S\star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda} = 0,\ P_{\mu\nu} = 0,\ Q_{\mu\lambda}Q^{\lambda}{}_{\nu} = \frac{9\rho^2(\alpha^2 - 2\rho)}{\alpha^9} Q_{\mu\nu},\ Q^{\mu}{}_{\mu} = \frac{9\rho^2(\alpha^2-2\rho)}{\alpha^9},\ \alpha > 0,$$

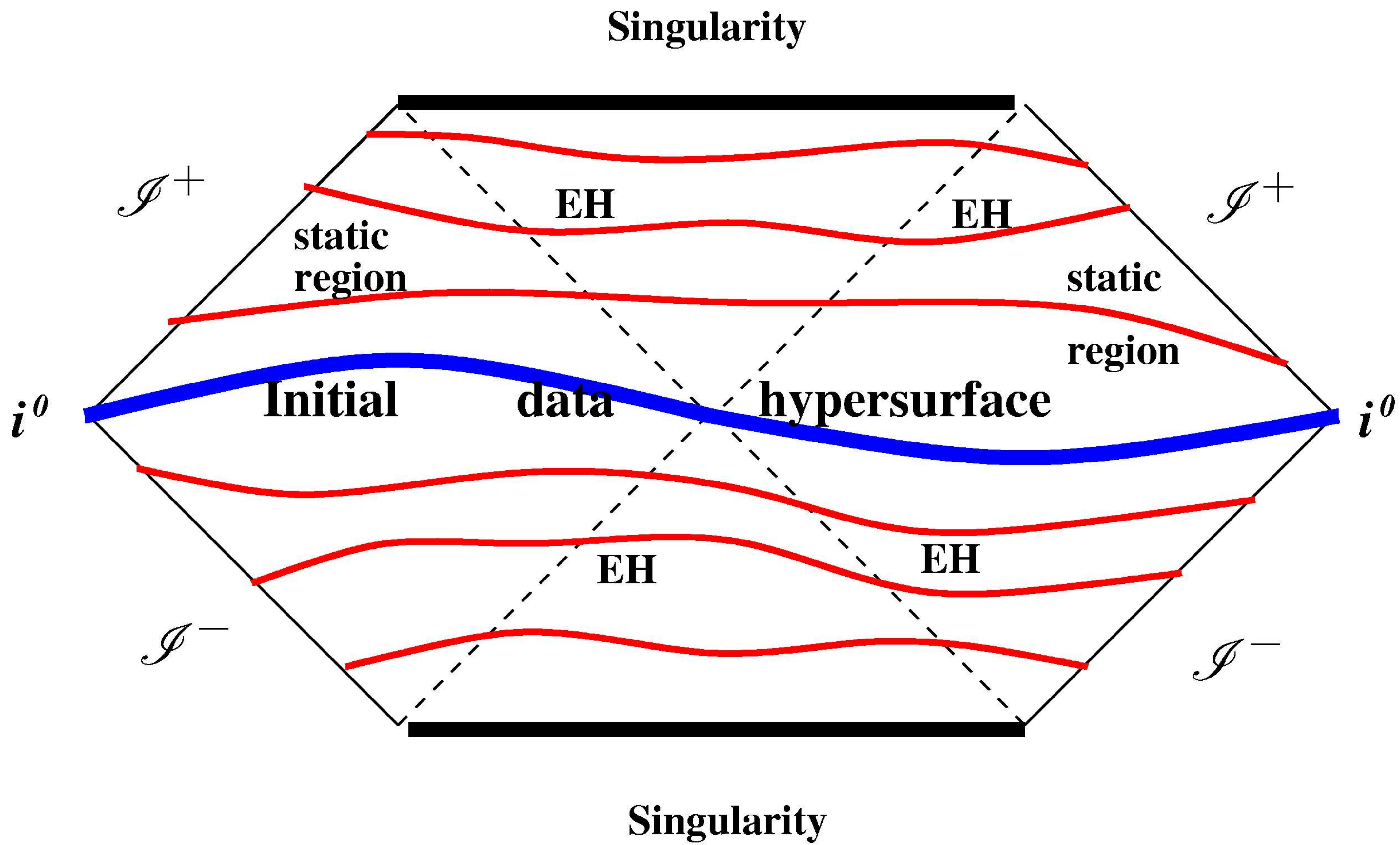
Moreover, the Schwarzschild mass m is given invariantly by

$$m = \frac{\rho}{\alpha^{3/2}},$$

Remarks: the formulation of theorem 2 given here is slightly different from that of [2]. The algebraic conditions of theorem 2 on $Q_{\mu\nu}$ are equivalent to

$$Q_{\mu\nu} = -9\alpha\rho^2 K_\mu K_\nu,$$

for some vector K^μ. The subset of $\mathcal{M}$ in which K^μ is a **timelike Killing vector** are the familiar static regions.



Singularity
$\mathscr{I}^+$
EH
EH
$\mathscr{I}^+$
static region
static region
i^0
Initial data hypersurface
i^0
EH
EH
$\mathscr{I}^-$
$\mathscr{I}^-$
Singularity

Orthogonal decompositions

Theorem 3 *For any globally hyperbolic spacetime $\mathcal{M}$ there exists a foliation whose leaves are smooth Cauchy hypersurfaces. Moreover, given a smooth Cauchy hypersurface $S \subset \mathcal{M}$ a foliation can be found whose leaves include S.*

This result enables us to define on such spacetime a unit timelike vector field n^μ, which is orthogonal to all the leaves of a given foliation of $\mathcal{M}$. From n^μ we may define other useful objects.

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu,\ \ h_{\mu\lambda}h^{\lambda}{}_{\nu} = h_{\mu\nu},\ \ h^{\mu}{}_{\mu} = 3,\ \ K_{\mu\nu} \equiv -\frac{1}{2}\mathcal{L}_n h_{\mu\nu},\ \ K_{\mu\nu} = K_{\nu\mu}.$$

The tensors $h_{\mu\nu}$ and $K_{\mu\nu}$ play the role of the first and second fundamental form for of any of the foliation leaves. Also they allow us to define the orthogonal decomposition of any other tensor. Examples:

orthogonal decomposition of spacetime metric: $g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu,$

spacetime volume element: $\eta_{\mu\nu\sigma\lambda} = n_\lambda \varepsilon_{\mu\nu\sigma} - n_\sigma \varepsilon_{\mu\nu\lambda} + n_\nu \varepsilon_{\mu\sigma\lambda} - n_\mu \varepsilon_{\nu\sigma\lambda},\ \ \varepsilon_{\mu\sigma\lambda} \equiv \eta_{\nu\mu\sigma\lambda} n^\nu.$

Orthogonal decomposition of Weyl tensor and its dual

$$C_{\mu\nu\lambda\sigma} = 2\Big(l_{\mu[\lambda}E_{\sigma]\nu} - l_{\nu[\lambda}E_{\sigma]\mu} - n_{[\lambda}B_{\sigma]\tau}\varepsilon^{\tau}{}_{\mu\nu} - n_{[\mu}B_{\nu]\tau}\varepsilon^{\tau}{}_{\lambda\sigma} \Big),$$

$$C^{*}_{\mu\nu\lambda\sigma} = 2\Big(l_{\mu[\lambda}B_{\sigma]\nu} - l_{\nu[\lambda}B_{\sigma]\mu} + n_{[\lambda}E_{\sigma]\tau}\varepsilon^{\tau}{}_{\mu\nu} + n_{[\mu}E_{\nu]\tau}\varepsilon^{\tau}{}_{\lambda\sigma} \Big),$$

where $E_{\tau\sigma}$ and $B_{\tau\sigma}$ are resp. the electric and magnetic parts of Weyl tensor

$$E_{\tau\sigma} \equiv C_{\tau\nu\sigma\lambda} n^\nu n^\lambda, \quad B_{\tau\sigma} \equiv C^{*}_{\tau\nu\sigma\lambda} n^\nu n^\lambda, \quad l_{\mu\nu} \equiv h_{\mu\nu} + n_\mu n_\nu.$$

We also need to find the orthogonal decomposition of objects involving the covariant derivatives. To that end we define the operator D_μ by

$$D_\mu T^{\alpha_1\dots\alpha_p}_{\quad\beta_1\dots\beta_q} \equiv h^{\alpha_1}_{\ \ \rho_1}\dots h^{\alpha_p}_{\ \ \rho_p} h^{\sigma_1}_{\ \ \beta_1}\dots h^{\sigma_q}_{\ \ \beta_q} h^{\lambda}_{\ \ \mu}\nabla_\lambda T^{\rho_1\dots\rho_p}_{\quad\sigma_1\dots\sigma_q},\ \ p,q\in\mathbb{N}, \tag{4}$$

where $T^{\alpha_1\dots\alpha_p}_{\quad\beta_1\dots\beta_q}$ is any tensor field in $\mathcal{M}$. Important properties of D_μ are

$$D_\mu h_{\nu\sigma} = 0,\ \ D_\mu n_\nu = -K_{\mu\nu}.$$

Also, if $\phi : \mathcal{S} \to \mathcal{M}$ is an isometric embedding such that $\phi(\mathcal{S})$ is one of the leaves of the foliation of $\mathcal{M}$ then for any covariant tensor $\boldsymbol{T}$ on $\mathcal{M}$ we have

$$\phi^*(D_\mu \boldsymbol{T}) = D_i(\phi^*\boldsymbol{T}),$$

where D_i is the Levi-Civita connection of the Riemannian metric $h_{ij} = \phi^*(h_{\mu\nu})$.

Orthogonal decomposition of the second Bianchi identity

$$\nabla_\mu C_{\nu\sigma\lambda\pi} + \nabla_\nu C_{\sigma\mu\lambda\pi} + \nabla_\sigma C_{\mu\nu\lambda\pi} = 0,$$

evolution equations: ($A^\mu \equiv n^\sigma\nabla_\sigma n^\mu$):

$$\mathcal{L}_n E_{\mu\nu} = A_\lambda(B_{\nu\sigma}\varepsilon_\mu^{\ \ \sigma\lambda} + B_{\mu\sigma}\varepsilon_\nu^{\ \ \sigma\lambda}) - 3E_\nu^{\ \ \sigma}K_{\mu\sigma} - 2E_\mu^{\ \ \sigma}K_{\nu\sigma} + 2E_{\mu\nu}K^\sigma_{\ \ \sigma} + E^{\sigma\lambda}K_{\sigma\lambda}h_{\mu\nu} + \varepsilon_\mu^{\ \ \sigma\lambda}(D_\lambda B_{\nu\sigma})$$
$$\mathcal{L}_n B_{\mu\nu} = -A_\lambda(E_{\nu\sigma}\varepsilon_\mu^{\ \ \sigma\lambda} + E_{\mu\sigma}\varepsilon_\nu^{\ \ \sigma\lambda}) - 3B_\nu^{\ \ \sigma}K_{\mu\sigma} - 2B_\mu^{\ \ \sigma}K_{\nu\sigma} + 2B_{\mu\nu}K^\sigma_{\ \ \sigma} + B^{\sigma\lambda}K_{\sigma\lambda}h_{\mu\nu} - \varepsilon_\mu^{\ \ \sigma\lambda}(D_\lambda E_{\nu\sigma}),$$

constraints:

$$D^\pi E_{\pi\mu} + B_{\pi\sigma}K^\pi_{\ \ \lambda}\epsilon^{\sigma\lambda}_{\ \ \ \ \mu} = 0,\ \ D^\pi B_{\pi\mu} - E_{\pi\sigma}K^\pi_{\ \ \lambda}\epsilon^{\sigma\lambda}_{\ \ \ \ \mu} = 0,$$

Orthogonal decomposition of Ferrando and Sáez conditions

Suppose $\phi : \mathcal{S} \rightarrow \mathcal{M}$ is an isometric embedding of a Riemannian manifold in Schwarzschild geometry. If we choose a foliation containing $\phi(\mathcal{S})$ as one of its leaves, perform the orthogonal decomposition of Ferrando and Sáez characterization with respect to this foliation and pull back the resulting conditions to $\mathcal{S}$ we can obtain a set of **necessary** conditions for the existence of the isometric embedding ϕ.

Theorem 4 (Necessary conditions for the existence of an isometric embedding in Schwarzschild) *Define the quantities*

$$E_{ij} \equiv r_{ij} + KK_{ij} - K_{ik}K^k{}_j,\ B_{ij} \equiv \epsilon_i{}^{kl} D_k K_{lj},\ \rho = \left(\frac{1}{2}B_i{}^j B^{il} E_{jl} - \frac{1}{6}E_{ij}E_l{}^j E^{il}\right)^{1/3},$$

$$P \equiv -\frac{1}{2}E^{ij}K_{ij} - \rho K - \frac{1}{6\rho}\epsilon^{jk}{}_i\left(E^{il}D_k B_{lj} + B^{il}D_k E_{lj}\right),\ P_i \equiv D_i\rho,\ \tilde{E}_i \equiv E_{ij}P^j,\ \tilde{B}_i \equiv B_{ij}P^j,$$

$$\gamma^2 \equiv P_iP^i,\ \Omega \equiv \tilde{E}_iP^i,\ \alpha \equiv \frac{\gamma^2 - P^2}{9\rho^2} + 2\rho,\ Y^2 = \frac{\gamma^2\rho - \Omega}{27\alpha\rho^3},\ 27\alpha\rho^3 YY_i = \epsilon_i{}^{jk}P_j\tilde{B}_k - P\tilde{E}_i + \rho PP_i,$$

Necessary conditions for a pair (h_{ij}, K_{ij}), satisfying the Einstein vacuum constraints, equations (1) and (2), on a manifold $\mathcal{S}$ to be Schwarzschild initial data are

$$\rho \neq 0,\ \alpha > 0,\ B_{ij} = -\frac{1}{\rho}(B_i{}^k E_{kj} + B_j{}^k E_{ki}),\ E_{ij} = \frac{1}{\rho}(B_i{}^k B_{kj} - E_i{}^k E_{kj}) + 2\rho h_{ij},$$

$$\tilde{B}_jP^j = 0,\ \tilde{E}_k\epsilon^k{}_{li}P^l - P\tilde{B}_i = 0,\ (P^2+\gamma^2)B_{ij} + 2PE_{k(i}\epsilon^k{}_{j)l}P^l - 2P_{(i}\tilde{B}_{j)} = 0,$$

$$2B_{l(i}\epsilon_{j)}{}^{lk}P_kP - 2P_{(i}\tilde{E}_{j)} + E_{ji}(P^2+\gamma^2) - \rho P_jP_i + h_{ji}((-P^2+\gamma^2)\rho + \Omega) = -27\alpha\rho^3 Y_jY_i.$$

Sufficient conditions for the existence of the embedding

Theorem 5 *Let* $(\mathcal{S}, h_{ij}, K_{ij})$ *be a vacuum initial data set fulfilling the conditions of theorem 4 and denote by* $\mathcal{D}(\phi(\mathcal{S}))$ *its maximal development. If an open set* $\mathcal{N} \subset \mathcal{D}(\phi(\mathcal{S}))$ *contains* $\phi(\mathcal{S})$ *and admits a Killing vector,* ξ^μ*, nowhere tangent to* $\phi(\mathcal{S})$*, then* $(\mathcal{D}(\phi(\mathcal{S})), g)$ *is isometric to a patch of the Schwarzschild spacetime.*

Proof : (sketch) if a Killing vector ξ^μ exists on $\mathcal{N}$ then we have

$$\mathcal{L}_\xi((S \star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda}) = 0,\ \mathcal{L}_\xi P_{\mu\nu} = 0,\ \mathcal{L}_\xi \left(Q_{\mu\lambda} Q^\lambda{}_\nu - \frac{9\rho^2(\alpha^2 - 2\rho)}{\alpha^9} Q_{\mu\nu} \right) = 0,\ \mathcal{L}_\xi \left(Q^\mu{}_\mu - \frac{9\rho^2(\alpha^2 - 2\rho)}{\alpha^9} \right) = 0$$

Also if conditions of theorem 4 hold in $\mathcal{S}$ then

$$((S{\star}S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda})|_{\phi(\mathcal{S})} = 0,\ P_{\mu\nu}|_{\phi(\mathcal{S})} = 0,\ \left(Q_{\mu\lambda} Q^\lambda{}_\nu - \frac{9\rho^2(\alpha^2 - 2\rho)}{\alpha^9} Q_{\mu\nu} \right)\Big|_{\phi(\mathcal{S})} = 0,\ \left(Q^\mu{}_\mu - \frac{9\rho^2(\alpha^2 - 2\rho)}{\alpha^9} \right)\Big|_{\phi(\mathcal{S})} = 0$$

Previous two equations are a Cauchy initial value problem for a system of first order linear PDE equations. If ξ^μ is nowhere tangent to $\phi(\mathcal{S})$ then this hypersurface is not a characteristic of the PDE system of equations. The uniqueness of the system solution entails

$$(S \star S)_{\mu\nu\sigma\lambda} - S_{\mu\nu\sigma\lambda} = 0,\ P_{\mu\nu} = 0,\ Q_{\mu\lambda} Q^\lambda{}_\nu - \frac{9\rho^2(\alpha^2 - 2\rho)}{\alpha^9} Q_{\mu\nu} = 0,\ Q^\mu{}_\mu - \frac{9\rho^2(\alpha^2 - 2\rho)}{\alpha^9} = 0.$$

on an open subset $\mathcal{U} \subset \mathcal{N}$ containing $\phi(\mathcal{S})$. Conditions of theorem 4 also imply

$$\rho|_{\phi(\mathcal{S})} \neq 0,\ \alpha|_{\phi(\mathcal{S})} > 0 \Rightarrow \rho \neq 0,\ \alpha > 0,\ \text{on } \mathcal{U}^* \subset \mathcal{N} \text{ containing } \phi(\mathcal{S}).$$

Conditions of theorem 2 hold in $\mathcal{U} \cap \mathcal{U}^* \subset \mathcal{N}$ and hence the maximal development $(\mathcal{D}(\phi(\mathcal{S})), g)$ must be a subset of Schwarzschild. □

Killing initial data

When does the development of an initial data set possess a Killing vector? Next theorem gives an answer to this question.

Theorem 6 (Coll, 1977) *Let $(\mathcal{S}, h_{ij}, K_{ij})$ be a vacuum initial data set and supposse that there exists a pair (Y, Y^j) of tensor fields on $\mathcal{S}$ solving the following system of PDE's (r_{ij} Ricci tensor of D_i)*

$$D_{(i}Y_{j)} - YK_{ij} = 0, \tag{5}$$

$$D_iD_jY - \mathcal{L}_{Y^l}K_{ij} = Y(r_{ij} + KK_{ij} - 2K_{il}K^l{}_j). \tag{6}$$

Then the development of $(\mathcal{S}, h_{ij}, K_{ij})$ possesses a Killing vector field. Furthermore, the number of linearly independent Killing vector fields is in correspondence with the number of linearly independent pairs (Y, Y^j) solving the above linear PDE system.

Remarks:

1. If we denote by ξ^μ the Killing vector referred to by previous theorem and consider a foliation of the development containing $\phi(\mathcal{S})$ then its orthogonal decomposition with respect to the normal vector to the leaves of the foliation reads
$$\xi^\mu = Yn^\mu + Y^\mu,$$
and the pull-back under the isometric embedding $\phi : \mathcal{S} \to \mathcal{M}$ of Y and Y^μ is precisely the pair (Y, Y^j).
$Y \neq 0 \iff \xi^\mu$ is not tangent to $\phi(\mathcal{S})$.

If we append conditions (5) and (6) to the set of conditions of theorem (4) we obtain a set of necessary and sufficient conditions which ensure that an initial data set is a Schwarzschild initial data set.

Example 0.1 (Time-symmetric embeddings) In this example we explain how to use our techniques to find isometric embeddings in Schwarzschild geometry with the property that the second fundamental form of the embedded hypersurface is zero. This entails $K_{ij} = 0$ and hence

$$E_{ij} = r_{ij}, \quad B_{ij} = 0, \quad P = 0, \tag{7}$$

Conditions of theorem 4 yield

$$E_{ij} = \rho \left(h_{ij} - \frac{3}{\gamma^2} P_i P_j \right), \tag{8}$$

and conditions of theorem 6 are reduced to

$$D_i D_j Y = Y E_{ij}. \tag{9}$$

We try the ansatz

$$Y = \frac{\gamma}{\rho^{4/3}}, \; Y^j = 0. \tag{10}$$

which, when replaced in (9), yields

$$-28\gamma P_i P_j + 9E_{ij}\gamma\rho^2 + 12\rho(P_j D_i\gamma + P_i D_j\gamma) + 12\gamma\rho D_j P_i - 9\rho^2 D_j D_i\gamma = 0. \tag{11}$$

If we prove that this equation is an identity for our initial data set then the development will be a patch of Schwarzschild. Consider next initial data such that

$$h_{ij}dx^i dx^j = dx^2 + \frac{1}{F^2(x,y,z)}(dy^2 + dz^2), \; F(x,y,z) > 0, \; F(x,y,z) \in C^\infty(\mathcal{S}). \tag{12}$$

The coordinate x is adapted to P_i in such a way that $P_i dx^i = \gamma dx$. For which function F is the initial data a time-symmetric Schwarzschild initial data set ?

Set up an orthonormal frame to carry out the calculations.

$$P^i\partial_i = \gamma(e_1)^i\partial_i = \partial_x,\ (e_2)^i\partial_i = F\partial_y,\ (e_3)^i\partial_i = F\partial_z. \tag{13}$$

In this frame we have

$$E_{ij} = -2\rho(e^1)_i(e^1)_j + \rho(e^2)_i(e^2)_j + \rho(e^3)_i(e^3)_j,\ \rho = \rho(x),\ \gamma = \gamma(x),\ Y = \rho^{-4/3}\gamma = \rho^{-4/3}\rho',\ ' \equiv d/dx \tag{14}$$

In this frame $E_{ij} = r_{ij}$ and $r = 0$ (hamiltonian constraint) yield

$$-\frac{2(\partial_x F)^2}{F^2} + \frac{\partial_x^2 F}{F} = -\rho,\ -\frac{\partial_y F\partial_x F}{F} + \partial^2_{xy}F = 0,\ -\frac{\partial_z F\partial_x F}{F} + \partial^2_{xz}F = 0, \tag{15}$$

$$-(\partial_z F)^2 + F\partial_z^2 F - (\partial_y F)^2 + F\partial_y^2 F - \frac{3(\partial_x F)^2}{F^2} + \frac{\partial_x^2 F}{F} = \rho, \tag{16}$$

$$-(\partial_y F)^2 - (\partial_z F)^2 + F(\partial_y^2 F + \partial_z^2 F) + \frac{2\partial_x^2 F}{F} - \frac{5(\partial_x F)^2}{F^2} = 0. \tag{17}$$

In our adapted frame (11) takes the form

$$(e^2)_i(e^2)_j\left(9\gamma\rho^3 - \frac{12\gamma^2\rho\partial_x F}{F} + \frac{9\alpha'\rho^2\partial_x F}{F}\right) + (e^3)_i(e^3)_j\left(9\gamma\rho^3 - \frac{12\gamma^2\rho\partial_x F}{F} + \frac{9\alpha'\rho^2\partial_x F}{F}\right) +$$
$$+(e^1)_i(e^1)_j(-28\gamma^3 - 18\gamma\rho^3 + 36\rho\gamma\gamma' - 9\rho^2\gamma'') = 0. \tag{18}$$

This is an identity if (14), (15), (16) and (17) hold. The other conditions of theorem 4 (except $\alpha > 0$ and $\rho \neq 0$) can be reduced to the differential equations

$$F(x,y,z) = \Phi(x)G(y,z),\ \frac{5\Phi'^2}{\Phi^4} - \frac{2\Phi''}{\Phi^3} = k,\ -(\partial_y G)^2 - (\partial_z G)^2 + G(\partial_y^2 G + \partial_z^2 G) = k, \tag{19}$$

where k is a constant. The standard time-symmetric hypersurface of Schwarzschild is included here for $k = -1$.

Conclusions

In this work we have developed necessary conditions for an initial data set to be Schwarzschild initial data. These necessary conditions turn out to be sufficient if a Killing vector exists in the development. Some points have been left open.

1. One of the most interesting questions coming up from our work is whether conditions of theorem 4 are in fact sufficient to characterize Schwarzschild initial data. A definitive answer to this question would be obtained if it were shown that conditions of theorem 4 imply that equations (5) and (6) have nontrivial solutions. If this is not possible one could try to combine conditions of theorem 4 with equations (5) and (6) in order to obtain a minimal set of necessary and sufficient conditions.

2. The work done here for the case of Schwarzschild spacetime could in principle be carried over to other equally relevant exact solutions of Einstein field equations such as a globally hyperbolic region of Kerr geometry. We would need first to write explicit invariant characterizations similar to that of Ferrando and Sáez for Schwarzschild and then perform the orthogonal decomposition. The most difficult point is, as already happens in the case of Schwarzschild, to obtain conditions ensuring the "propagation" of the conditions coming from the orthogonal decomposition.

The calculations of this work have been performed with the freely available MATHEMATICA packages "*xTensor*" and "*xCoba*"

see: `http://metric.iem.csic.es/Martin-Garcia/xAct/`