Spacelike surfaces with positive definite second fundamental form in 3-dimensional Lorentzian manifolds

Alfonso Romero
Departamento de Geometría y Topología
Universidad de Granada
18071-Granada

Web: http://www.ugr.es/∼aromero
E-mail: aromero@ugr.es

Santiago, February 7, 2007

Based on

J.A. Aledo, S. Haesen and A. Romero, Spacelike surfaces with positive definite second fundamental form in 3D spacetimes, *J. Geom. Phys.*, **57** (2007), 913–923.

J.A. Aledo and A. Romero, Compact spacelike surfaces in the 3-dimensional de Sitter space with non-degenerate second fundamental form, *Differ. Geom. Appl.* **19** (2003), 97–111.

The main aim:

To show how totally umbilical spacelike surfaces in 3-dimensional spacetimes can be characterized among the family of compact spacelike surfaces with positive (or negative) definite second fundamental form.

The tools:

(1) A formula relating the Gauss and mean curvature of the spacelike surface to the Gauss curvature of the second fundamental form.

(2) The classical Gauss-Bonnet theorem.

Geometrically, if a spacelike surface has positive definite second fundamental form, then the future-directed timelike geodesics orthogonal to the surface M are really spreading out near M. Thus, in particular, the volume of M does increase when it is compact. Therefore, the existence of a compact spacelike (hyper)surface with positive definite second fundamental form means that the spacetime is really expanding.

The second fundamental form with respect to a unit normal vector field of a (non-totally geodesic) totally umbilical surface is obviously definite and thus, it provides the surface with a new Riemannian metric (pointwise conformally related to the induced metric).

We will give an answer to the following natural question:

> When a spacelike surface with positive definite second fundamental form must be totally umbilical?

And we would like to point out that:

Totally umbilical spacelike surfaces in 3-dimensional spacetimes have a rich geometry. Moreover, the study of this family of surfaces lies into the conformal geometry of the spacetime, because a pointwise conformal change of the ambient metric preserves the character of being totally umbilical of a spacelike surface.

From a purely geometric point of view, 3-dimensional spacetimes have been deeply studied and clearly present a great mathematical interest. Although they are too unrealistic to give much insight into usual 4-dimensional relativistic models, 3-dimensional spacetimes are useful to explore the foundations of classical and quantum gravity[1]

[1] C. Carlip, Quantum Gravity in $2+1$ dimensions: the case of the Closed Universe, *Living Rev. Relativity* **8** (2005), 1–63, and references therein.

Preliminaries

Let $(\bar{M}, \bar{g})$ be a 3-dimensional spacetime. A smooth immersion $x : M \to \bar{M}$ of a (connected) 2-dimensional manifold M is said to be a spacelike surface if the induced metric on M via x, $x^\star(\bar{g})$, is Riemannian, which as usual we will denote by g.

If M is a spacelike surface in $\bar{M}$, the time-orientation of $\bar{M}$ allows us to define $N \in \mathfrak{X}^\perp(M)$ as the only unit timelike vector field normal to M in the same time-orientation of $\bar{M}$.

If A stands for the shape operator of M in $\bar{M}$ associated to N, we will denote the mean curvature of M by

$$H = -\frac{1}{2}\,\mathrm{trace}(A).$$ [2]

[2] The choice of the minus-sign is motivated by the fact that, in this case, the mean curvature vector field is $\vec{H} = HN$.

Of course, A satisfies the classical Gauss and Codazzi equations. But A is not a Codazzi tensor, in general. In fact, in the literature, a Codazzi tensor is a symmetric (1,1)-tensor field B which satisfies $(\nabla_X B)Y = (\nabla_Y B)X$. [3]

From the Gauss equation it follows that

$$2\,K = 2\,\bar{K} + \mathrm{trace}(A^2) - (\mathrm{trace}\,A)^2,$$

where K is the Gauss curvature of M and $\bar{K}(p)$ is the sectional curvature of each tangent plane $\mathrm{d}x_p(T_pM)$ in $\bar{M}$.

We have that $\bar{K}$ satisfies

$$\bar{K} = \frac{1}{2}\,\bar{S} + \overline{\mathrm{Ric}}(N,N),$$

where $\bar{S}$ is the scalar curvature and $\overline{\mathrm{Ric}}$ the Ricci tensor of $\bar{M}$.

[3] Which is the Codazzi equation when $\bar{R}(X,Y)N = 0$.

Therefore, using the characteristic equation for the shape operator, we obtain the following expression for the Gauss-Kronecker curvature of the surface M,

$$\det(A) = \bar{K} - K.$$

When $\det(A) > 0$, the second fundamental form II, given by

$$\mathrm{II}(X, Y) = -g(AX, Y),$$

where $X, Y \in \mathfrak{X}(M)$, determines a definite metric on M. In fact, we can suppose (up to a change of orientation) that II is positive definite, i.e., II is a Riemannian metric on M. We have

Proposition 1. On a spacelike surface M in a 3-dimensional spacetime $\bar{M}$ the following two conditions are equivalent:

(i) II is a positive definite metric on M,

(ii) The Gauss curvature K of M satisfies $K < \bar{K}$.

Remark 2. This result gives an obstruction to the existence of certain spacelike surfaces with positive definite second fundamental form in terms of the curvature of the ambient space. In fact, if $\bar{M}$ admits such a compact spacelike surface with the topology of $\mathbb{S}^2$, it follows from the Gauss-Bonnet theorem that the sectional curvature of $\bar{M}$ must be positive on some spacelike plane.

Taking into account that a compact spacelike surface M of the De Sitter spacetime $\mathbb{S}^3_1$, with sectional curvature 1, must be topologically a sphere $\mathbb{S}^2$ (and hence a non-degenerate metric on M must be definite) we get

> Corollary 3. On a compact spacelike surface M in the De Sitter spacetime $\mathbb{S}^3_1$ the following two conditions are equivalent:
>
> (i) II is a non-degenerate metric on M,
>
> (ii) The Gauss curvature K of M satisfies $K < 1$.

Now observe that if M is assumed to be complete, then the condition $K > 1$ implies that M must be compact by classical Bonnet-Myers' theorem. Therefore the following result is obtained [4]

> Corollary 4. There exists no complete spacelike surface M in the De Sitter spacetime $\mathbb{S}^3_1$ whose Gauss curvature K satisfies $K > 1$.

Remark 5. Next, let M be a 2-dimensional compact manifold, $\bar{M}$ a 3-dimendional spacetime and consider $\mathcal{E}(M, \bar{M})$ the set of spacelike immersions

$$x : M \to \bar{M}$$

with positive definite second fundamental form.

[4] Extending Proposition 4.2 in H. Li, Global rigidity theorems of hypersurfaces, *Ark. Mat.* **35** (1997), 327–351.

Suppose that $\mathcal{E}(M,\bar{M})$ is non-empty, and note that since the condition $\det(A_x) > 0$ implies that the usual area functional

$$x \mapsto \operatorname{area}(M, g_x)$$

has no critical point in $\mathcal{E}(M,\bar{M})$. Thus, it would be interesting to know if the area functional with respect to II,

$$\mathcal{F}_{\mathrm{II}}(x) = \operatorname{area}(M, \mathrm{II}_x) = \int_M \mathrm{d}\Omega_{\mathrm{II}_x} = \int_M \sqrt{\det(A_x)}\mathrm{d}\Omega_x,$$

where II_x is the second fundamental form corresponding to x, has a critical point in $\mathcal{E}(M,\bar{M})$. It turns out that this, in general, is not the case. In fact, consider $\bar{M} = \mathbb{S}^3_1$ and let $x : M \to \mathbb{S}^3_1 \subset \mathbb{L}^4$ be a compact spacelike surface in the 3-dimensional De Sitter spacetime. Then M is topologically $\mathbb{S}^2$. Consider for $t \in \mathbb{R}$ the parallel surface $x_t : \mathbb{S}^2 \to \mathbb{S}^3_1 \subset \mathbb{L}^4$, which is given by

$$x_t(p) = \overline{\exp}_{x(p)}(tN(p)) = \cosh(t)x(p) + \sinh(t)N(p),$$

where $p \in \mathbb{S}^2$, $\overline{\exp}$ denotes the exponential map in $\mathbb{S}^3_1$ and N the unit normal timelike vector field along x. It is not difficult to obtain, using[5], that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_{\mathrm{II}}(x_t) = \int_{\mathbb{S}^2} \frac{H\left[1+\det(A)\right]}{\det(A)} \mathrm{d}\Omega_{\mathrm{II}} \ .$$

Since $\det(A)$ and H are strictly positive, it follows that the functional $\mathcal{F}_{\mathrm{II}}$ has no critical point in $\mathcal{E}(\mathbb{S}^2, \mathbb{S}^3_1)$.

[5] J.A. Aledo, L.J. Alías and A. Romero, Integral formulas for compact space-like hypersurfaces in de Sitter space: Applications to the case of constant higher order mean curvature, *J. Geom. Phys.* **31** (1999), 195–208.

Now, let ∇^{II} denote the Levi-Civita connection with respect to II. The difference tensor T between the Levi-Civita connections ∇^{II} and ∇ is given by

$$T(X,Y) = {\nabla^{\mathrm{II}}}_X Y - \nabla_X Y,$$

for all $X, Y \in \mathfrak{X}(M)$. From the Koszul formula for ∇^{II}, and using the Codazzi equation, we find that

$$T(X,Y) = \frac{1}{2} A^{-1} \left\{ (\nabla_X A)Y - \left[\bar{R}(X,N)Y\right]^T \right\},$$

where $[\ ]^T$ denotes the tangent component to the surface M.

Besides the obvious symmetry $T(X,Y) = T(Y,X)$, there also holds the relation

$$\mathrm{II}(T(X,Y),Z) - \mathrm{II}(T(X,Z),Y) = \bar{g}(\bar{R}(X,N)Y,Z).$$

In the special case in which M is totally umbilical in $\bar{M}$, with $\det(A) > 0$. We have $A = \rho I$ and $\mathrm{II}(X,Y) = -\rho\, g(X,Y)$, $\rho \in C^\infty(M)$, for all $X, Y \in \mathfrak{X}(M)$. Hence

$$\nabla^{\mathrm{II}}{}_X Y - \nabla_X Y = \frac{1}{2\rho}\left\{X(\rho)Y + Y(\rho)X - g(X,Y)\nabla\rho\right\},$$

i.e. the well known formula relating the Levi-Civita connections of the pointwise conformally related metrics; and thus

$$T(X,Y) = \frac{1}{2\rho}\left\{\mathrm{II}(X,\nabla^{\mathrm{II}}\rho)Y + \mathrm{II}(Y,\nabla^{\mathrm{II}}\rho)X - \mathrm{II}(X,Y)\nabla^{\mathrm{II}}\rho\right\},$$

where we have used the fact that $\nabla^{\mathrm{II}}\rho = -\frac{1}{\rho}\nabla\rho$. In particular, we have

$$\mathrm{trace}_{\mathrm{II}}(T) = 0.$$

A formula for the Gauss curvature of the metric II

On a spacelike surface M such that II is a Riemannian metric, the Riemann curvature tensor R^{II} of II satisfies

$$R^{\mathrm{II}}(X,Y)Z = R(X,Y)Z + Q_1(X,Y)Z + Q_2(X,Y)Z,$$

with

$$Q_1(X,Y)Z = ({\nabla^{\mathrm{II}}}_Y T)(X,Z) - ({\nabla^{\mathrm{II}}}_X T)(Y,Z),$$

$$Q_2(X,Y)Z = T\big(X, T(Y,Z)\big) - T\big(Y, T(X,Z)\big),$$

and $X, Y, Z \in \mathfrak{X}(M)$. Therefore, we get

$$\mathsf{Ric}^{\mathrm{II}}(X,Y) = \mathrm{Ric}(X,Y) + \widehat{Q}_1(X,Y) + \widehat{Q}_2(X,Y),$$

where

$$\widehat{Q}_i(X,Y) = \mathrm{trace}\{V \mapsto Q_i(X,V)Y\}, \quad i = 1, 2.$$

Contracting now with respect to II we obtain a first approach to our main formula:

$$2K^{\mathrm{II}} = \mathrm{trace}_{\mathrm{II}}(\mathrm{Ric}) + \mathrm{trace}_{\mathrm{II}}(\widehat{Q}_1) + \mathrm{trace}_{\mathrm{II}}(\widehat{Q}_2),$$ [6]

where K^{II} is the Gauss curvature of the surface M with respect to the metric II.

Lemma 6. (Explaining the first right term) The trace with respect to II of the Ricci tensor Ric of M is given by

$$\mathrm{trace}_{\mathrm{II}}(\mathrm{Ric}) = -\frac{\mathrm{trace}(A)}{\det(A)}\, K.$$

[6] An alternative proof of this formula can be achieved using local computations as in L.P. Eisenhart, Riemannian Geometry, 6th edition, Princeton Univ. Press, 1966 (p. 33).

In order to express more clearly the second term, we define now the tangent vector field $\mathsf{S}(N)$ on M through the formula

$$\mathrm{II}\big(\mathsf{S}(N), X\big) = \overline{\mathrm{Ric}}(N, X),$$

for all $X \in \mathfrak{X}(M)$.

An explicit expression for this vector field can be obtained by taking the trace of the Codazzi equation, to get

$$\overline{\mathrm{Ric}}(N, X) = \mathrm{II}\big(X, \nabla^{\mathrm{II}}\mathrm{trace}(A)\big) + \mathrm{II}\big(X, A^{-1}\mathrm{div}(A)\big),$$

and therefore

$$\mathsf{S}(N) = \nabla^{\mathrm{II}}\mathrm{trace}(A) + A^{-1}\mathrm{div}(A).$$

If we use now $A\nabla^{\mathrm{II}} f = -\nabla f$, which holds true for every smooth function f, we can write

$$A\mathsf{S}(N) = \mathrm{div}\big(A - \mathrm{trace}(A)\, I\big).$$

This equation is known as the momentum constraint in the initial-value problem of General Relativity. [7]

Observe that, in the particular case of a totally umbilical spacelike surface with $A = \rho I$, there holds that

$$\mathsf{S}(N) = \nabla^{\mathrm{II}} \rho.$$

[7] For a discussion of this equation in the 3-dimensional case see L. Andersson, V. Moncrief and A.J. Tromba, On the global evolution problem in 2+1 gravity, *J. Geom. Phys.* **23** (1997), 191–205.

Lemma 7. (Explaining the second right term) The trace of the tensor $\widehat{Q}_1$ with respect to II is given by

$$\mathrm{trace}_{\mathrm{II}}(\widehat{Q}_1) = -\mathrm{div}_{\mathrm{II}}\left(\frac{A\mathsf{S}(N)}{\det(A)}\right).$$

In order to deal properly with the third term, we consider, for each $X \in \mathfrak{X}(M)$, the operator

$$T_X = \nabla^{\mathrm{II}}_X - \nabla_X$$

which satisfies

$$\mathrm{trace}_{\mathrm{II}}(T_X) = \mathrm{II}\Big(\,\mathrm{trace}_{\mathrm{II}}T + \frac{A\mathsf{S}(N)}{\det(A)},\ X\,\Big).$$

Lemma 8. (Explaining the third right term) The trace of the tensor $\widehat{Q}_2$ with respect to II is given by

$$\mathrm{trace}_{\mathrm{II}}(\widehat{Q}_2) = \|T\|^2_{\mathrm{II}} - \|\mathrm{trace}_{\mathrm{II}} T\|^2_{\mathrm{II}} - \mathrm{trace}_{\mathrm{II}}\left(T_{\frac{A\mathrm{S}(N)}{\det(A)}} \right),$$

whereby $\|T\|^2_{\mathrm{II}}$ is the squared II-length of the difference tensor T.

We can then express $\mathrm{trace}_{\mathrm{II}} T$ in terms of $\det(A)$ and $\mathsf{S}(N)$ as follows

$$\mathrm{trace}_{\mathrm{II}} T = \frac{1}{2\det(A)} \nabla^{\mathrm{II}} \det(A) - \frac{A\mathsf{S}(N)}{\det(A)}.$$

Now we come back to the previous formula

$$2K^{\mathrm{II}} = \mathrm{trace}_{\mathrm{II}}(\mathrm{Ric}) + \mathrm{trace}_{\mathrm{II}}(\widehat{Q}_1) + \mathrm{trace}_{\mathrm{II}}(\widehat{Q}_2),$$

and bearing in mind Lemmas 6, 7 and 8 above we get the following result

Theorem 9. (Main formula) Let M be a spacelike surface with Gauss-Kronecker curvature $\det(A) > 0$ in a 3-dimensional spacetime $\bar{M}$. Then, the Gauss curvature K^{II} of the metric II satisfies

$$2K^{\mathrm{II}} = -\frac{\operatorname{trace}(A)}{\det(A)}\,K - \operatorname{div}_{\mathrm{II}}\left(\frac{A\mathsf{S}(N)}{\det(A)}\right)$$

$$+\,\|T\|^2_{\mathrm{II}} - \|\operatorname{trace}_{\mathrm{II}} T\|^2_{\mathrm{II}} - \operatorname{trace}_{\mathrm{II}}\left(T_{\frac{A\mathsf{S}(N)}{\det(A)}}\right),$$

where $\|T\|^2_{\mathrm{II}}$ is the squared II-length of T.

Remark 10. In the special case that A is a Codazzi tensor, in particular for any spacelike surface in a 3-dimensional spacetime of constant sectional curvature, we have

$$\mathsf{S}(N) = 0$$

and

$$\mathrm{trace}_{\mathrm{II}}(T) = \frac{1}{2\det(A)} \nabla^{\mathrm{II}} \det(A)$$

and the previous formula reduces to

$$2K^{\mathrm{II}} = -\frac{\mathrm{trace}(A)}{\det(A)}\, K + \|T\|_{\mathrm{II}}^2 - \frac{1}{4\det(A)^2} \|\, \nabla^{\mathrm{II}} \det(A)\|_{\mathrm{II}}^2 .$$

Remark 11. We can compute the last three terms in previous general formula explicitly, and thus we obtain the following expression for K^{II} in terms of the principal curvatures $\{\lambda_1, \lambda_2\}$ and principal directions $\{e_1, e_2\}$ of the spacelike surface, [8]

$$\begin{aligned} 2K^{\mathrm{II}} &= -\frac{\operatorname{trace}(A)}{\det(A)} K - \operatorname{div}_{\mathrm{II}}\left(\frac{A\mathsf{S}(N)}{\det(A)}\right) \\ &\quad -\frac{1}{2\det(A)}\left\{ e_1\left(\log\left(\frac{\lambda_2}{\lambda_1}\right)\right)\left[e_1(\lambda_2) - \overline{\mathrm{Ric}}(N, e_1)\right]\right. \\ &\quad \left. + e_2\left(\log\left(\frac{\lambda_1}{\lambda_2}\right)\right)\left[e_2(\lambda_1) - \overline{\mathrm{Ric}}(N, e_2)\right]\right\}. \end{aligned}$$

[8] This is an extension of the formula for K^{II} in E. Cartan, Les surfaces qui admettent une seconde forme fondamentale donnée, *Bull. Sci. Math.* **67**, 8–32 (1943), 8–32, p. 18, in the case of a surface in Euclidean space and of formula (12) in T. Klotz-Milnor The curvature of $\alpha\mathrm{I} + \beta\mathrm{II} + \gamma\mathrm{III}$ on a surface in a 3-manifold of constant curvature, *Mich. Math. J.* **22** (1975), 247–255 in the case of surfaces in Riemannian spaces of constant sectional curvature.

Remark 12. In the case of M being totally umbilical, with $A = \rho I$, we have for the third term

$$\|T\|_{\mathrm{II}}^2 - \|\mathrm{trace}_{\mathrm{II}} T\|_{\mathrm{II}}^2 - \mathrm{trace}_{\mathrm{II}}\left(T_{\frac{\mathrm{AS}(N)}{\det(A)}}\right) = 0,$$

and the formula in Theorem 9 reduces to

$$K^{\mathrm{II}} = -\frac{1}{\rho} K + \frac{1}{2\rho} \Delta \log(-\rho)$$

where Δ is the Laplacian of the induced metric g. That is, we have

$$K - (-\rho) K^{\mathrm{II}} = \frac{1}{2} \Delta \log(-\rho),$$

which is the well-known relation between the Gauss curvatures of the pointwise conformally related metrics g and $\mathrm{II} = -\rho\, g$.

Remark 13. As a final remark, we would like to point out that the formula obtained in Theorem 9 may be generalized to the case of a spacelike hypersurface M with positive definite second fundamental form in an $(n+1)$-dimensional spacetime $\bar{M}$. For instance, when $\bar{M}$ is the De Sitter spacetime $\mathbb{S}_1^{n+1}$ we get

$$S^{\mathrm{II}} = -(n-1)\mathrm{trace}(A^{-1}) + (n-1)\mathrm{trace}(A)$$

$$+\|T\|_{\mathrm{II}}^2 - \frac{1}{4H_n^2}\,\|\nabla^{\mathrm{II}} H_n\|_{\mathrm{II}}^2,$$

where S^{II} is the scalar curvature of (M, II) and

$$H_n = (-1)^n \det(A)$$

the Gauss-Kronecker curvature of the spacelike hypersurface M.

Main classification results

We will apply now the previous formulas to give several characterizations of compact totally umbilical spacelike surfaces, with signed Gauss curvature, based on assumptions on the second fundamental form II.

Theorem 14. Let M be a compact spacelike surface in a 3-dimensional spacetime $\bar{M}$, with non-zero Euler-Poincaré characteristic $\chi(M)$, $\det(A) > 0$ and K signed (i.e. $K > 0$ or < 0). Then, M is totally umbilical if and only if

$$\chi(M) \int_M \left\{ \|T\|^2_{\mathrm{II}} - \|\mathrm{trace}_{\mathrm{II}} T\|^2_{\mathrm{II}} - \mathrm{trace}_{\mathrm{II}} T_{\frac{\mathrm{AS}(N)}{\det(A)}} \right\} \mathrm{d}\Omega_{\mathrm{II}} \geq 0.$$

Sketch of Proof. If M is totally umbilical, then the integrand in the statement is identically zero. Conversely, assume first that $K > 0$. From the Gauss-Bonnet theorem it follows that $\chi(M) > 0$. Then, we obtain

$$\int_M K^{\mathrm{II}}\,\mathrm{d}\Omega_{\mathrm{II}} \geq -\int_M \frac{\mathrm{trace}(A)}{2\det(A)}\,K\,\mathrm{d}\Omega_{\mathrm{II}}.$$

The Euler inequality states that $-\mathrm{trace}(A) \geq 2\sqrt{\det(A)}$, with equality holding at every point if and only if the surface is totally umbilical. Hence, since $K > 0$, we have

$$\int_M K^{\mathrm{II}}\,\mathrm{d}\Omega_{\mathrm{II}} \geq \int_M \frac{K}{\sqrt{\det(A)}}\,\mathrm{d}\Omega_{\mathrm{II}}.$$

Using the relation $\mathrm{d}\Omega_{\mathrm{II}} = \sqrt{\det(A)}\mathrm{d}\Omega$ between the area elements of M with respect to II and g respectively, and the Gauss-Bonnet theorem again, we have

$$2\pi\chi(M) = \int_M K^{\mathrm{II}}\,\mathsf{d}\Omega_{\mathrm{II}} \geq \int_M \frac{K}{\sqrt{\det(A)}}\,\mathrm{d}\Omega_{\mathrm{II}} =$$

$$= \int_M K\,\mathrm{d}\Omega = 2\pi\chi(M),$$

and thus, equality holds in the Euler inequality. The case $K < 0$ follows analogously. $\square$

In the case that the 3-dimensional spacetime is the De Sitter spacetime $\mathbb{S}^3_1$ with constant sectional curvature 1, we have of course $\bar{K} = 1$. Moreover, the inequality $\det(A) > 0$ is here equivalent to $K < 1$. On the other hand, if we assume that K is constant, then $\det(A)$ is also constant and the formula in Remark 10 reduces to

$$K^{\mathrm{II}} = -\frac{\operatorname{trace}(A)}{2\det(A)}\, K + \frac{1}{2}\, \|T\|^2_{\mathrm{II}}.$$

Since every compact spacelike surface in the 3-dimensional de Sitter spacetime is topologically a sphere, it follows from the Gauss-Bonnet theorem that, if K is constant, then $K > 0$. Thus the previous theorem gives in particular

> Corollary 15. Every compact spacelike surface of the De Sitter spacetime $\mathbb{S}^3_1$, with constant Gauss curvature $K < 1$, is a totally umbilical round sphere.

It is clear that a totally umbilical (and not totally geodesic) spacelike surface of $\mathbb{S}^3_1$ has constant K^{II}. The following result can be seen as a (non trivial) converse of this fact.

Theorem 16. Let M be a compact spacelike surface in the De Sitter space $\mathbb{S}^3_1$ with Gauss curvature $K < 1$. If the Gauss curvature K^{II} of II is constant, then M is a totally umbilical round sphere.

Sketch of Proof. Note that the constant K^{II} is determined from the Gauss-Bonnet theorem; in fact, we have

$$4\pi = \int_M K^{\mathrm{II}} \, \mathrm{d}\Omega_{\mathrm{II}} = K^{\mathrm{II}} \, \mathrm{area}(M, \mathrm{II}).$$

Since M is compact there exists $p_0 \in M$ where K attains its maximum value $K(p_0) < 1$. Now, we evaluate at p_0 the formula in Remark 10 to get

$$K^{\mathrm{II}} \geq H(p_0) \, \frac{K(p_0)}{1 - K(p_0)}.$$

Observe that the Gauss-Bonnet theorem also guarantees that this maximum value $K(p_0) > 0$, which jointly the Euler inequality yields

$$\frac{4\pi}{\mathrm{area}(M, \mathrm{II})} = K^{\mathrm{II}} \geq \frac{1}{\sqrt{1 - K(p_0)}} K(p_0) \geq \frac{1}{\sqrt{1 - K}} K$$

on M, because the function $f(t) = t/\sqrt{1-t}$ is strictly increasing for $t < 1$.

Then, from previous inequality we have

$$4\pi = \int_M K \, \mathrm{d}\Omega \leq \frac{4\pi}{\mathrm{area}(M, \mathrm{II})} \int_M \mathrm{d}\Omega_{\mathrm{II}} = 4\pi,$$

and therefore the equality

$$K^{\mathrm{II}} = \frac{K}{\sqrt{1-K}}$$

holds, that is, K is constant on M.

These results can be summarized as follows:

For a compact spacelike surface M in $\mathbb{S}_1^3$ with Gauss curvature $K < 1$, the following assertions are equivalent:

(i) K is constant.

(ii) K^{II} is constant.

(iii) M is a totally umbilical round sphere.

Remark 17. The method to obtain the formula in Theorem 9 also works for surfaces in Riemannian spaces; in particular, for a surface with positive Gauss-Kronecker curvature in the Euclidean space $\mathbb{E}^3$, the hyperbolic space $\mathbb{H}^3$ or the sphere $\mathbb{S}^3$ we have

$$2K^{\mathrm{II}} = \frac{2HK}{K-c} + \|T\|^2_{\mathrm{II}} - \frac{1}{4(K-c)^2}\|\nabla^{\mathrm{II}} K\|^2_{\mathrm{II}}.$$ [9]

and, consequently, we have the following version of Liebmann classical rigidity result[10]

> The only compact surfaces in $\mathbb{E}^3$, in $\mathbb{H}^3$ or in an open hemisphere $\mathbb{S}^3_+$ which have constant Gauss curvature are the totally umbilical round spheres.

---

[9] For the case $c = 0$, this formula was proved using local computations in R. Schneider, Closed convex hypersurfaces with second fundamental form of constant curvature, *Proc. Amer. Math. Soc.* **35** (1972), 230–233.

[10] J.A. Aledo, L.J. Alías and A. Romero, A new proof of Liebmann classical theorem for surfaces in space forms, *Rocky Mt. J. Math.* **35** (2005), 1811–1824.

Now, let us look into the assumption on T in Theorem 14.

Lemma 18. Let M be a spacelike surface with Gauss-Kronecker curvature $\det(A) > 0$ in a 3-dimensional space-time $\bar{M}$. Then for the difference tensor T we always have the inequality

$$\|T\|^2_{\mathrm{II}} \geq \frac{1}{2}\,\|\mathrm{trace}_{\mathrm{II}} T\|^2_{\mathrm{II}},$$

and equality holds if and only if

$$T(X,Y) = -\mathrm{II}(X,Y)\,\frac{A\mathsf{S}(N)}{\det(A)}.$$

We are now in position to get another consequence of Theorem 14. In fact, if the equality

$$\|T\|^2_{\mathrm{II}} = \frac{1}{2}\,\|\mathrm{trace}_{\mathrm{II}} T\|^2_{\mathrm{II}}$$

holds, it follows from previous Lemma that

$$\mathrm{trace}_{\mathrm{II}} T = -2\,\frac{A\mathsf{S}(N)}{\det(A)},$$

and moreover

$$\mathrm{trace}_{\mathrm{II}} T_{\frac{A\mathsf{S}(N)}{\det(A)}} = -\left\|\frac{A\mathsf{S}(N)}{\det(A)}\right\|_{\mathrm{II}}^2.$$

holds. Therefore, we have

$$\|T\|_{\mathrm{II}}^2 - \|\mathrm{trace}_{\mathrm{II}} T\|_{\mathrm{II}}^2 - \mathrm{trace}_{\mathrm{II}} T_{\frac{A\mathsf{S}(N)}{\det(A)}} = -3\left\|\frac{A\mathsf{S}(N)}{\det(A)}\right\|_{\mathrm{II}}^2 \leq 0.$$

and equality holds if and only if $\mathsf{S}(N) = 0$.

Corollary 19. Let M be a compact spacelike surface in a 3-dimensional spacetime $\bar{M}$. If $\det(A) > 0$,

$$\|T\|_{\mathrm{II}}^2 = \frac{1}{2} \|\mathrm{trace}_{\mathrm{II}} T\|_{\mathrm{II}}^2$$

and $K < 0$, then M is totally umbilical with $A = \rho I$, and ρ constant.

Sketch of Proof. Since $K < 0$ it follows from the Gauss-Bonnet theorem that $\chi(M) < 0$, and then, from Theorem 14, we have that M must be totally umbilical.

On the other hand, we know that every totally umbilical spacelike surface in a 3-dimensional spacetime has $\mathrm{S}(N) = 0$. But $\nabla^{\mathrm{II}} \rho = \mathrm{S}(N)$. Therefore, we have that ρ must be constant. $\square$

Finally, as we have seen before, a totally umbilical spacelike surface satisfies

$$\mathrm{trace}_{\mathrm{II}} T = 0.$$

Conversely, if this holds then

$$\|T\|_{\mathrm{II}}^2 - \|\mathrm{trace}_{\mathrm{II}} T\|_{\mathrm{II}}^2 - \mathrm{trace}_{\mathrm{II}} T_{\frac{A\mathrm{S}(N)}{\det(A)}}$$

is shown to be non-negative. Therefore, we have yet another consequence of Theorem 14,

Corollary 20. Let M be a compact spacelike surface in a 3-dimensional spacetime $\bar{M}$, with $\bar{K} > K > 0$. If

$$\mathrm{trace}_{\mathrm{II}} T = 0,$$

then M is totally umbilical.

Thank you very much

for your kind attention!