Uniqueness results for constant mean curvature spacelike hypersurfaces in Lorentzian spaces

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The Steady State Space

The de Sitter space $\mathbb{S}_1^{n+1}$

The de Sitter space is the hyperquadric defined as

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{R}_1^{n+2} : \langle x, x \rangle = 1\},$$

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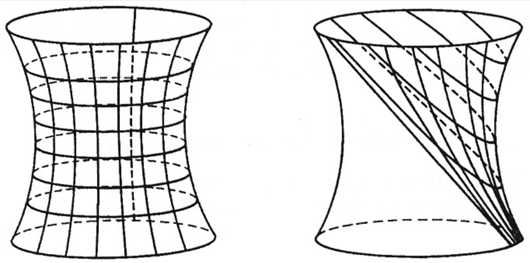
The steady state space $\mathcal{H}^{n+1}$

The steady state space is the open region of the de Sitter space

$$\mathcal{H}^{n+1} = \{x \in \mathbb{S}_1^{n+1} : \langle x, a \rangle > 0\}.$$



- $\mathcal{H}^{n+1}$ is a non-complete manifold, being only half of the de Sitter space and having as boundary the null hypersurface

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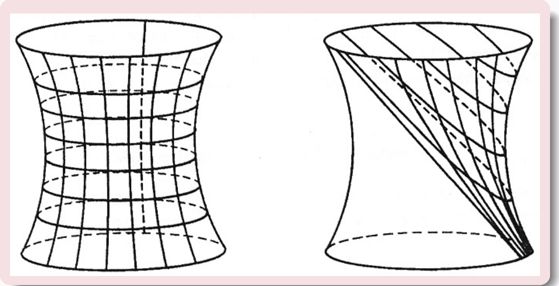

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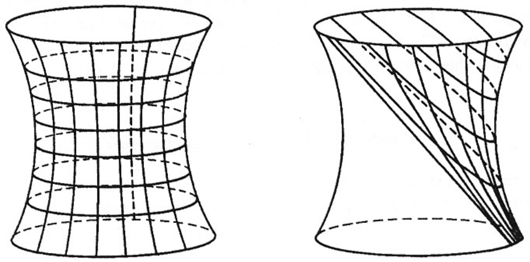
- It admits a foliation by totally umbilical spacelike hypersurfaces

$$L_\tau = \{x \in \mathbb{S}_1^{n+1} : \langle x, a \rangle = \tau\}, \quad \tau > 0$$

with constant mean curvature one with respect to $N_\tau(x) = x - \frac{1}{\tau}a$, considering $H = -\frac{1}{n}\mathrm{tr}(A)$.



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- ⋆ If Σ^n is a compact, constant mean curvature spacelike hypersurface in $\mathbb{S}_1^{n+1}$, then Σ is, necessarily, a totally umbilical spacelike hypersurface, (Montiel, 1988).
- ⋆ There exist complete, non-compact and non-totally umbilical spacelike hypersurfaces, with constant mean curvature $H^2 > 1$ in $\mathcal{H}^{n+1}$, (Montiel, 2003).

- A spacelike hypersurface is said to be contained in a slab if there exist $0 < \tau_1 < \tau_2$ such that

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- It is not difficult to obtain

$$\nabla u = a^\top, \qquad \Delta u = nH\langle N, a \rangle - nu$$

$$\|\nabla u\|^2 = \langle N, a \rangle^2 - u^2$$

where $a = a^\top - \langle N, a \rangle N + \langle a, x \rangle x$ and $\| \cdot \|$ denotes the norm of a vector field on Σ.

- In order to prove our main results, we will apply the Omori-Yau maximum principle.

Lemma 1 (Omori-Yau maximum principle)

Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. If $u \in \mathcal{C}^\infty(M)$ is bounded from above on M then there exists a sequence of points $\{p_j \in M\}$ such that

$$\lim_{j \longrightarrow \infty} u(p_j) = \sup_M u, \quad \|\nabla u(p_j)\| < 1/j \ \ \textit{and} \ \ \Delta u(p_j) < 1/j.$$

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- **Parabolicity Criterium (Ahlfors and Blanc-Fiala-Huber)**

  Any complete Riemannian surface with non-negative Gaussian curvature is parabolic.

CMC spacelike hypersurfaces in $\mathcal{H}^{n+1}$

Proposition

Let $f : \Sigma^n \to \mathcal{H}^{n+1}$ be a complete constant mean curvature spacelike hypersurface contained in a slab $\Omega(\tau_1, \tau_2)$ for some $0 < \tau_1 < \tau_2$. Then $H = 1$ necessarily. Moreover, in the 2-dimensional case, there exists $\tau^\star$, $\tau_1 \leq \tau^\star \leq \tau_2$ such that $\Sigma^2 = L_{\tau^\star}$.

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- **Proof of Proposition**

⋆ By Gauss equation for a spacelike hypersurface,

$$\mathrm{Ric}(X, X) \geq (n-1) - \frac{n^2 H^2}{4},$$

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⋆ As the function u is bounded, we can apply Lemma 1. There exists a sequence $\{p_j\} \in \Sigma^n$ such that

$$\lim_{j \to \infty} u(p_j) = \sup_\Sigma u \leq \tau_2, \quad \|\nabla u(p_j)\|^2 = (\langle N, a\rangle^2 - u^2)(p_j) < (1/j)^2$$

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- ⋆ In an analogue way, applying Lemma 1 to $-u$, $H \geq 1$.
- ⋆ Therefore, $H = 1$.
- ⋆ Assume now $n = 2$. By a result of Akutagawa (1977), Σ must be totally umbilical. The only totally umbilical spacelike hypersurfaces with constant mean curvature $H = 1$ are the leaves of the foliation L_τ.

An isometrically equivalent model

An isometric model

Consider the generalized Robertson-Walker spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^n$, that is, $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle , \rangle = -dt^2 + e^{2t}(dx_1^2 + ... + dx_n^2)$$

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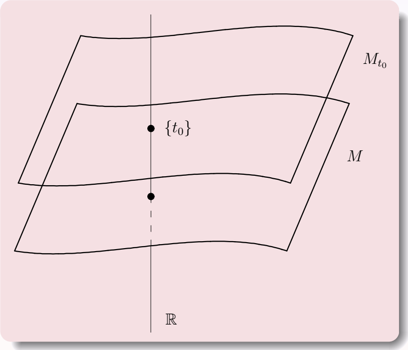

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$$\begin{array}{ccc} \mathcal{H}^{n+1} & \Leftrightarrow & -\mathbb{R} \times_{e^t} \mathbb{R}^n \\ L_{\tau_0} & \leftrightarrow & \{\log(\tau_0)\} \times \mathbb{R}^n \\ u = \tau_0 & \leftrightarrow & h = \log(\tau_0) \end{array}$$

- A spacelike hypersurface is contained in a slab if there exist two real numbers $t_1 < t_2$ such that

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Let M^n be a Riemannian manifold. If $-\mathbb{R} \times_{e^t} M^n$ admits a complete spacelike hypersurface $f : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$, contained in a slab $\overline{\Omega}(t_1, t_2)$ for some $t_1 < t_2$, then M is necessarily complete and the projection $\Pi = \pi_M \circ f : \Sigma \to M$ is a covering map.

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- The following result generalizes Proposition,

CMC spacelike hypersurfaces in $-\mathbb{R} \times_{e^t} M^n$

Theorem

Let M^n be a (necessarily complete) Riemannian manifold with non negative sectional curvature K_M, that is $K_M(\Pi) \geq 0$ for every tangent plane $\Pi \subset TM$. Let $f : \Sigma^n \to -\mathbb{R} \times_{e^t} M^n$ be a complete constant mean curvature spacelike hypersurface contained in a slab $\overline{\Omega}(t_1, t_2)$ for some $t_1 < t_2$. Then $H = 1$ necessarily. Moreover, when $n = 2$, Σ is a slice.

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- **Proof of Theorem**

⋆ Let $\{E_1, ...E_n\}$ be an orthonormal frame on $T\Sigma$, and $X \in T\Sigma$, $\|X\| = 1$. The Ricci curvature tensor of Σ^n can be written in terms of the sectional curvature of M^n as

$$\mathrm{Ric}(X, X) \geq e^{2h} \sum_{i=1}^{n} K_M(X^* \wedge E_i^*) Q_M(X^* \wedge E_i^*) - \frac{n^2 H^2}{4} + n - 1$$

where $Q_M(X \wedge Y) = \langle X, X \rangle_M \langle Y, Y \rangle_M - \langle X, Y \rangle_M^2$

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- ⋆ In an analogue way, applying the Omori-Yau maximum principle to $-h$, $H \geq 1$. Therefore, $H = 1$.

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⋆ Assume now $n = 2$. The sectional curvature in $-\mathbb{R} \times_{e^t} M^2$, $\overline{K}$, is written in terms of the Gaussian curvature of M along Σ, κ_M, as

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- By the parabolicity of Σ, h is constant, being Σ a slice.

Remark

In Theorem, we have proved something stronger: Let $f : \Sigma^n \rightarrow M^n \times_{e^t} \mathbb{R}_1$ be a complete spacelike hypersurface such that the height function is bounded from above on Σ, then $\inf_\Sigma H \leq 1$. When the height function is bounded from below, $\sup_\Sigma H \geq 1$.

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Let M^2 be a Riemannian surface with non negative Gaussian curvature. The only complete spacelike surfaces $f : \Sigma^2 \to -\mathbb{R} \times_{e^t} M^2$ with constant mean curvature $H \leq 1$ bounded from below are the slices $\{t^*\} \times M^2$.

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Corollary 2

Let M^2 be a (necessarily complete) Riemannian surface with non negative Gaussian curvature. The only complete spacelike surfaces $f : \Sigma^2 \to -\mathbb{R} \times_{e^t} M^2$ with constant mean curvature $H \geq 1$ bounded from above are the slices $\{t^*\} \times M^2$.

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Thanks!!