Geodesics in standard stationary spacetimes and Lagrangian systems

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Standard stationary spacetime: $(L, \langle\cdot,\cdot\rangle_L)$ Lorentzian manifold, $L = M \times \mathbb{R}$ and

$$\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle + 2\langle\delta(x),\cdot\rangle dt - \beta(x)dt^2\,,$$

$(M, \langle\cdot,\cdot\rangle)$ finite dimensional, connected Riem. manifold, δ smooth vector field on M and $\beta : M \to \mathbb{R}$, smooth strictly positive function.

If $\delta \equiv 0$, then L is called standard static.

Hypothesis:

- $(M, \langle\cdot,\cdot\rangle)$ complete, at least C^3 Riemannian manifold.

Conservation laws: if a curve $z = (x,t) : I \to L$, $I \subset \mathbb{R}$ interval, is a geodesic, then $E \in \mathbb{R}$ and $K \in \mathbb{R}$ exist such that

$$E = \frac{1}{2}\langle \dot{z}, \dot{z}\rangle_L \qquad K = -\langle \partial t, \dot{z}\rangle_L = \beta(x)\dot{t} - \langle \delta(x), \dot{x}\rangle.$$

Variational approach for Lorentzian geodesics:

every geodesic $z : [a,b] \to L$ is a critical point of the (strongly indefinite) functional

$$f(z) = \int_a^b \langle \dot{z}, \dot{z}\rangle_L ds$$

defined on a manifold of curves.

- V. Benci, D. Fortunato, F. Giannoni (Ann. Inst. H. Poincaré Anal. Non Linéaire, 1991) and others: reduction to the study of a Riemannian functional.

- M. Sánchez, (Nonlinear Anal., 1999) in the standard static case: relation between geodesics and Lagrangian systems.

Theorem. Let $(L, \langle \cdot, \cdot \rangle_L)$ be a st. static sp. A curve $z = (x, t) : I \to L$, $I \subset \mathbb{R}$ interval, is a geodesic such that $\beta(x)\dot{t} = K$, if and only if x solves

$$D_s \dot{x} + \frac{1}{2} K^2 \nabla V(x) = 0 \quad V = -\frac{1}{\beta}$$

(D_s covariant derivative and ∇ gradient with respect to $\langle \cdot, \cdot \rangle$). Moreover

$$\frac{1}{2}\langle \dot{z}(s), \dot{z}(s) \rangle_L = \frac{1}{2}\langle \dot{x}(s), \dot{x}(s) \rangle + \frac{1}{2} K^2 V(x(s)) \quad \forall s \in I.$$

Results in A.V.G. (J. Differential Equations, 2007):

- a correspondence between geodesics and Lagrangian systems holds also in the standard stationary case if
  1. M is endowed by a new Riemannian metric (depending on the coefficients δ and β);
  2. a term representing the action of a magnetic field is added in the previous differential equation.

New (complete) metric on M: set $\langle \cdot, \cdot \rangle_1$ by

$$\langle u, v \rangle_1 = \langle u, v \rangle + \frac{1}{\beta(x)} \langle \delta(x), u \rangle \langle \delta(x), v \rangle \quad x \in M, u, v \in T_x M.$$

For any $v \in T_x M$, it is

$$\langle v, v \rangle_1 = \langle (I + P(x))[v], v \rangle \quad P(x)[v] = \frac{1}{\beta(x)} \langle \delta(x), v \rangle \delta(x),$$

I identity, $P(x) : T_x M \to T_x M$ linear, self–adjoint, positive operator.

For any $x \in M$, set

$$V(x) = -\frac{1}{\beta(x)} \qquad A(x) = \frac{\delta(x)}{\beta(x) + \langle \delta(x), \delta(x) \rangle}.$$

Let F^1 be the curl of A:

$$F^1(X, Y) = \langle D^1_X A, Y \rangle_1 - \langle X, D^1_Y A \rangle_1$$

(X, Y vector fields on M, D^1 Levi–Civita connection of $(M, \langle \cdot, \cdot \rangle_1)$).

New equation:

$$D_s^1 \dot{x} + \frac{1}{2} K^2 \nabla^1 V(x) = K \widehat{F}^1(x)[\dot{x}] \tag{1}$$

∇^1 gradient with resp. to $\langle \cdot, \cdot \rangle_1$, $\widehat{F}^1 : TM \to TM$ linear map $\langle \cdot, \cdot \rangle_1$– associated to F^1 ($F^1(x)[u,v] = \langle \widehat{F}^1(x)[u], v\rangle_1$, $x \in M$, $u, v \in T_x M$).

Theorem. Let $(L, \langle \cdot, \cdot \rangle_L)$ be a st. stationary sp., $I \subset \mathbb{R}$ an interval.

If $z = (x,t) : I \to L$ is a geodesic such that $\beta(x)\dot{t} - \langle \delta(x), \dot{x}\rangle = K$ for some $K \in \mathbb{R}$, then $x : I \to \mathbb{R}$ solves (1).

Every geodesic $z = (x,t) : I \to L$ for $\langle \cdot, \cdot \rangle_L$ can be obtained by a sol. $x : I \to M$ of (1) for some $K \in \mathbb{R}$, t by $\dot{t} = (K + \langle \delta(x), \dot{x}\rangle) / \beta(x)$.

Moreover

$$\frac{1}{2}\langle \dot{z}, \dot{z}\rangle_L = \frac{1}{2}\langle \dot{x}, \dot{x}\rangle_1 + \frac{1}{2} K^2 V(x) \quad \text{on } I.$$

The proof follows from the computation of the geodesic equations on L.

A geodesic $z = (x, t) : [a, b] \to L$ is a critical point of

$$\begin{aligned} f(z) &= \int_a^b \langle \dot{z}, \dot{z} \rangle_L ds \\ &= \int_a^b \left(\langle \dot{x}, \dot{x} \rangle_1 - \beta(x)(\dot{t} - \langle A(x), \dot{x} \rangle_1)^2 \right) ds. \end{aligned}$$

Differentiating f gives

$$\begin{cases} \dfrac{d}{ds}(\beta(x)(\dot{t} - \langle A(x), \dot{x} \rangle_1)) = 0 \\ D_s^1 \dot{x} + \dfrac{1}{2}(\dot{t} - \langle A(x), \dot{x} \rangle_1)^2 \nabla^1 \beta(x) = \beta(x)(\dot{t} - \langle A(x), \dot{x} \rangle_1) \widehat{F}^1(x)[\dot{x}]. \end{cases}$$

Applications: geodesics with fixed E (in a suitable real interval) and $K = \sqrt{2}$ (as in the static case, normalization of the coefficient of $\nabla^1 V$).

Boundary conditions: geod. from a point to a line or periodic trajectories.

- sol. $x : [0, a] \to M$ of

$$D_s^1 \dot{x} + \nabla^1 V(x) = \sqrt{2} \widehat{F}^1(x)[\dot{x}]$$

  joining two fixed points $x_0, x_1 \in M$ correspond to geod. $z = (x, t) : [0, a] \to L$ joining a point $z_0 = (x_0, t_0)$ to a line $(x_1, s) \subset L$, where

$$t(s) = t_0 + \int_0^s \frac{\sqrt{2} + \langle \delta(x), \dot{x} \rangle}{\beta(x)} \, d\tau.$$

Case studied by R. Bartolo, D. Fortunato, F. Giannoni, A. Masiello, P. Piccione, M. Sánchez and others.

Theorem. Let $(L, \langle \cdot, \cdot \rangle_L)$ be a st. stationary sp. If

- $\overline{A} \in \mathbb{R}$ exists such that

$$\sup_{x \in M} |A(x)|_1 = \sup_{x \in M} \frac{|\delta(x)|}{\sqrt{\beta(x)(\beta(x) + \langle \delta(x), \delta(x) \rangle)}} = \overline{A}.$$

Then, for any $E \in \mathbb{R}$ with

$$E > \overline{\beta} + \overline{A}^2 \quad \text{where} \quad \overline{\beta} = \sup_{x \in M} \left(-\frac{1}{\beta(x)} \right)$$

and for any $x_0, x_1 \in M$, $x_0 \neq x_1$, $t_0 \in \mathbb{R}$, a geodesic $z = (y, t) : [0, a] \to L$ exists joining the point $(x_0, t_0) \in L$ to the line $(x_1, s) \subset L$, such that

$$\frac{1}{2} \langle \dot{z}, \dot{z} \rangle_L = E \quad \text{and} \quad \beta(y)\dot{t} - \langle \delta(y), \dot{y} \rangle = \sqrt{2}.$$

Multiplicity:

- if M is not contractible in itself, a sequence (z_m), $z_m = (y_m, t_m) : [0, a_m] \to M$ of geodesics exists. Their arrival times $t_m(a_m)$ verify

$$\lim_{m \to +\infty} t_m(a_m) = +\infty$$

when β is bounded from above and, denoted by

$$\overline{A}_1 = \sup_{x \in M} \frac{|\delta(x)|}{\sqrt{\beta(x) + \langle \delta(x), \delta(x) \rangle}} \qquad N = \sup_{x \in M} \beta(x),$$

when

- $\overline{A}_1 < \sqrt{EN + 1}$ for any possible $E \leq 0$;
- $\overline{A}_1 < 1/\sqrt{EN + 1}$ for any $E > 0$.

The boundedness condition is satisfied when $\beta \geq \nu$, for some $\nu > 0$ or $|\delta|/\beta$ is bounded.

$E > \overline{\beta} + \overline{A}^2$: If β is bounded from above then $\overline{\beta} < 0$ (thus in some cases negative E are allowed). If $\overline{\beta} \geq 0$, unbounded interval of strictly positive E.

The theorem contains, as particular cases, some of the th. in

- D. Fortunato, F. Giannoni, A. Masiello, J. Geom. Phys., 1995 (case $E = 0$ on st. stationary sp.);

- R. Bartolo, A.V.G., M. Sánchez, Differential Geom. Appl., 2002, (st. static sp., $E > \overline{\beta}$).

Periodic trajectories:

Periodic trajectory of universal period T and proper period $a > 0$: a geodesic $z = (x, t) : [0, a] \to L$ such that

$$x(a) = x(0) \quad \dot{x}(a) = \dot{x}(0) \qquad\qquad t(a) = t(0) + T \quad \dot{t}(a) = \dot{t}(0).$$

- periodic sol. (i.e. $x : [0, a] \to M$ such that $x(0) = x(a)$, $\dot{x}(0) = \dot{x}(a)$) of

$$D_s^1 \dot{x} + \nabla^1 V(x) = \sqrt{2} \widehat{F}^1(x)[\dot{x}]$$

  give rise to periodic trajectories $z = (x, t) : [0, a] \to L$.

Set $t(0) = 0$, thus

$$T = t(a) = \int_0^a \frac{\sqrt{2} + \langle \delta(x), \dot{x} \rangle}{\beta(x)} \, ds.$$

Geometrically distinct periodic trajectories: if they have different ranges. Taking $t(0) = 0$ avoids obtaining trajectories having the same spatial components and with temporal components differing by a constant.

Theorem. Let $(L, \langle\cdot,\cdot\rangle_L)$ be a st. stationary sp. with compact M. If

- M is not contractible in itself and its fundamental group $\pi_1(M)$ is finite or it has infinitely many coniugacy classes.

Then, for any $E > \overline{\beta} + \overline{A}^2$, one non-trivial t–periodic trajectory $z = (y, t) : [0, a] \to L$ exists, such that

$$\frac{1}{2}\langle \dot{z}, \dot{z}\rangle_L = E \quad \text{and} \quad \beta(y)\dot{t} - \langle \delta(y), \dot{y}\rangle = \sqrt{2}.$$

Fixing E and the value $K = \sqrt{2}$ gives a multiplicity result: periodic traj. for different values of E are geometrically distinct.

M non-compact: further hyp. at infinity (existence of a convex at infinity function as in V. Benci, D. Fortunato, Proc. of "Variational Methods", 1988).

The theorem extends some results in

- A. Candela, Ann. Mat. Pura Appl., 1996, ($E = 0$, st. stationary sp.).

In the non–compact case, weaker assumptions than

- A. Masiello, Nonlinear Anal., 1992 (st. stationary sp., traj. with fixed T).

Proof: find fixed energy solutions of

$$D_s^1 \dot{x} + \nabla^1 V(x) = \sqrt{2}\widehat{F}^1(x)[\dot{x}].$$

Let $(M, \langle\cdot,\cdot\rangle)$ be a complete Riem. manifold, $V : M \to \mathbb{R}$ a smooth function, B smooth vector field on M, F curl of B.

$$D_s \dot{x} + \nabla V(x) = \widehat{F}(x)[\dot{x}] \tag{2}$$

where $\widehat{F} : TM \to TM$ is the linear map $\langle\cdot,\cdot\rangle$–associated to F.

Fixed $E_x \in \mathbb{R}$, find solutions $x : I \to M$, $I \subset \mathbb{R}$ interval, of (2) s.t.

$$E = \frac{1}{2}\langle \dot{x}(s), \dot{x}(s)\rangle + V(x(s)) \quad \forall s \in I.$$

If $F \equiv 0$, classical principle: solutions of (2) with fixed energy E, are, up to reparametrizations, geodesics with respect to a Jacobi metric

$$\langle \cdot, \cdot \rangle_E = (E - V(x))\langle \cdot, \cdot \rangle$$

(Riemannian in a neighbourhood of $x \in M$, if $E > V(x)$).

If $F \not\equiv 0$,

- $V \in C^1(M, \mathbb{R})$ is bounded from above;
- B is bounded.

$$\overline{V} = \sup_{x \in M} V(x) \qquad \overline{B} = \sup_{x \in M} |B(x)|$$

Proposition. Let $E > \overline{V}$. If $x \in C^2([0,1], M)$ is a non–constant solution of

$$(E - V(x))D_s^E \dot{x} = \sqrt{\frac{1}{2}\langle \dot{x}, \dot{x} \rangle_E}\, \widehat{F}(x)[\dot{x}] \qquad (3)$$

(D_s^E covariant derivative with respect to $\langle \cdot, \cdot \rangle_E$), then $a > 0$ and a reparametrization $y \in C^2([0,a], M)$ of x exist, solving

$$D_s \dot{x} + \nabla V(x) = \widehat{F}(x)[\dot{x}]$$

and having energy E.

Eq. (3) is invariant by affine rep. $a_1 s + a_2$ if $a_1 \geq 0$.

Eq. (3) has a variational structure: its associated functional is

$$G(x) = \sqrt{2 \int_0^1 \langle \dot{x}, \dot{x} \rangle_E ds} + \int_0^1 \langle B(x), \dot{x} \rangle ds$$

x varying in a suitable manifold of curves.

For some fixed x_0, x_1 in M, $x_0 \neq x_1$, consider

$$\Omega^1(x_0, x_1, M) = \left\{ x \in H^1([0,1], M) \mid x(0) = x_0,\ x(1) = x_1 \right\}$$

$$\Lambda^1(M) = \left\{ x \in H^1([0,1], M) \mid x(0) = x(1) \right\}.$$

Critical points of $G(x)$, $x \in \Omega^1(x_0, x_1, M)$ exist, under the inequality

$$E > \overline{V} + \frac{\overline{B}^2}{2}.$$

(G has minumum. If M is not contractible, multiplicity of critical points.)

If $x \in \Lambda^1(M)$, the minumum of G is 0: reinforce topological assumptions about M (thus $\Lambda^1(M)$ has subsets of arbitrarily large Ljusternik–Schnirelman category).

Work in progress:

if M is not complete and has boundary ∂M,

- extend these results (under a suitable notion of convexity for ∂M, with respect to the field A);
- study the convexity of $\partial L = \partial M \times \mathbb{R}$ using metric $\langle \cdot, \cdot \rangle_1$.