Unicity of constant higher order mean curvature spacelike hypersurfaces in generalized Robertson-Walker spacetimes

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To appear in Math. Proc. Cambridge Phil. Soc.

IV International Meeting on Lorentzian Geometry

Universidade de Santiago de Compostela, Spain

5-8 February 2007

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- We will denote by $-I \times_f M^n$ the $(n+1)$-dimensional product manifold $I \times M$ endowed with the Lorentzian metric

$$\langle , \rangle = -dt^2 + f^2(t) \langle , \rangle_M,$$

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- That is, $-I \times_f M^n$ is nothing but a Lorentzian warped product with Lorentzian base $(I, -dt^2)$, Riemannian fiber $(M^n, \langle , \rangle_M)$, and warping function f.
- We will refer to $-I \times_f M^n$ as a generalized Robertson-Walker (GRW) spacetime. In particular, when the Riemannian factor M^n has constant sectional curvature then $-I \times_f M^n$ is classically called a Robertson-Walker (RW) spacetime.

Spacelike hypersurfaces

- Consider a smooth immersion

$$\psi : \Sigma^n \to -I \times_f M^n$$

of an n-dimensional connected manifold Σ into a GRW spacetime, and assume that the induced metric via ψ is a Riemannian metric on Σ; that is, Σ is a spacelike hypersurface.

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- Since

$$\partial_t = (\partial/\partial_t)_{(t,x)}, \quad (t,x) \in -I \times_f M^n,$$

is a unitary timelike vector field globally defined on the ambient GRW spacetime, then there exists a unique unitary timelike normal field N globally defined on Σ which is in the same time-orientation as ∂_t, so that

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- We will refer to that normal field N as the future-pointing Gauss map of the hypersurface. Its opposite will be refered as the past-pointing Gauss map of Σ.
- The height function of Σ, $h = \pi_I \circ \psi$, is the restriction of the projection $\pi_I(t,x) = t$ to Σ.

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- Associated to the shape operator there are n algebraic invariants given by

$$S_k(p) = \sigma_k(\kappa_1(p), \dots, \kappa_n(p)), \quad 1 \leq k \leq n,$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

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- The kth -mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = (-1)^k S_k = \sigma_k(-\kappa_1, \dots, -\kappa_n),$$

for every $0 \leq k \leq n$.

Higher order mean curvatures

- In particular, when $k = 1$

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- By the choice of the sign $(-1)^k$, the mean curvature vector $\overrightarrow{H}$ is given by $\overrightarrow{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\overrightarrow{H}(p)$ is in the same time-orientation as N.

The Newton transformations and their associated differential operators

- The Newton transformations P_k are defined inductively from A by

$$P_0 = I \quad \text{and} \quad P_k = \binom{n}{k} H_k I + A \circ P_{k-1},$$

for every $k = 1 \ldots, n$, where I denotes the identity in $\mathcal{X}(\Sigma)$.

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- Then each $P_k(p)$ is a self-adjoint linear operator on the tangent space $T_p\Sigma$ which commutes with $A(p)$, and $A(p)$ and $P_k(p)$ can be simultaneously diagonalized.

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- Then each $P_k(p)$ is a self-adjoint linear operator on the tangent space $T_p\Sigma$ which commutes with $A(p)$, and $A(p)$ and $P_k(p)$ can be simultaneously diagonalized.
- Let ∇ be the Levi-Civita connection of Σ. To each P_k, we associate the second order linear differential operator

$$L_k(f) = \mathrm{trace}(P_k \circ \nabla^2 f),$$

where $\nabla^2 f$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f:

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathcal{X}(\Sigma).$$

The Newton transformations and their associated differential operators

- Observe that

$$\mathrm{div}(P_k(\nabla f)) = \langle \mathrm{div} P_k, \nabla f \rangle + L_k(f),$$

where, for an orthonormal basis $E_1, \ldots, E_n$,

$$\mathrm{div} P_k := \mathrm{trace}(\nabla P_k) = \sum_{i=1}^{n} (\nabla_{E_i} P_k)(E_i).$$

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- In particular, if P_k is divergence-free then $L_k(f) = \mathrm{div}(P_k(\nabla f))$ and L_k is a divergence form differential operator on Σ.

Ellipticity and positive definiteness

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Lemma 1

Let Σ be a spacelike hypersurface immersed into a GRW spacetime. If there exists an elliptic point of Σ, with respect to an appropriate choice of the Gauss map N, and $H_{k+1} > 0$ on Σ, for $1 \leq k \leq n-1$, then for all $1 \leq j \leq k$ the operator L_j is elliptic or, equivalently, P_j is positive definite (for that appropriate choice of the Gauss map, if j is odd).

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- The proof of Lemma 1 follows from that of Cheng-Rosenberg (An. Acad. Brasil. Ciênc., 2005) or Barbosa-Colares (Ann. Global Anal. Geom., 1997), considering that in our case, and by our sign convention in the definition of the j-th mean curvatures, we have

$$\binom{n}{j} H_j = \sigma_j(-\kappa_1, \ldots, -\kappa_n) = (-1)^j S_j.$$

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Lemma 2 (Existence of an elliptic point)

Let $\psi : \Sigma^n \to -I \times_f M^n$ be a compact spacelike hypersurface immersed into a spatially closed GRW spacetime, and assume that $f'(h)$ does not vanish on Σ (equivalently, $\psi(\Sigma)$ is contained in a slab

$$\Omega(t_1, t_2) = (t_1, t_2) \times M \subset -I \times_f M^n$$

on which f' does not vanish)

1. If $f'(h) > 0$ on Σ (equivalently, $f' > 0$ on (t_1, t_2)), then there exists an elliptic point of Σ with respect to its future-pointing Gauss map.
2. If $f'(h) < 0$ on Σ (equivalently, $f' < 0$ on (t_1, t_2)), then there exists an elliptic point of Σ with respect to its past-pointing Gauss map.

The operator L_k acting on the height function

- The vector field on $-I \times_f M^n$ given by

$$K(t,x) = f(t)(\partial/\partial t)_{(t,x)}, \quad (t,x) \in -I \times_f M^n,$$

is a non-vanishing future-pointing closed conformal vector field.

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- In fact,

$$\overline{\nabla}_Z K = f'(t) Z$$

  for every vector Z tangent to $-I \times_f M^n$ at a point (t,x), where $\overline{\nabla}$ denotes the Levi-Civita connection on $-I \times_f M^n$.

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- This closed conformal field K will be an essential tool in our computations.

Lemma 3

Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map N. Let $h = \pi_I \circ \psi$ b the height function of Σ, and let $g : I \to \mathbb{R}$ be any primitive of f. Then, for every $k = 0, \ldots, n-1$ we have

$$L_k(h) = -(\log f)'(h)(c_k H_k + \langle P_k(\nabla h), \nabla h\rangle) - \langle N, \partial_t\rangle c_k H_{k+1},$$

and

$$L_k(g(h)) = -c_k(f'(h)H_k + \langle N, K\rangle H_{k+1}),$$

where $c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}$.

The operator L_k acting on the function $\langle N, K\rangle$

Lemma 4

Let Σ^n be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map N, closed conformal vector field K and height function h. Then, for every $k = 0, \ldots, n-1$ we have

$$\begin{aligned}
L_k(\langle N, K\rangle) &= \binom{n}{k+1}\langle \nabla H_{k+1}, K\rangle + f'(h)c_k H_{k+1} \\
&+\binom{n}{k+1}\langle N, K\rangle(nH_1H_{k+1} - (n-k-1)H_{k+2}) \\
&+\frac{\langle N, K\rangle}{f^2(h)}\sum_{i=1}^{n}\mu_{i,k}K_M(N^*\wedge E_i^*)\|N^*\wedge E_i^*\|^2 \\
&-\langle N, K\rangle(\log f)''(h)\left(c_k H_k\|\nabla h\|^2 - \langle P_k\nabla h, \nabla h\rangle\right).
\end{aligned}$$

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&+ \frac{\langle N, K\rangle}{f^2(h)}\sum_{i=1}^{n}\mu_{i,k}K_M(N^*\wedge E_i^*)\|N^*\wedge E_i^*\|^2 \\
&- \langle N, K\rangle(\log f)''(h)\left(c_k H_k\|\nabla h\|^2 - \langle P_k\nabla h, \nabla h\rangle\right).
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- Here N^* and E_i^* are the projection onto the fiber M of the vector fields N and E_i, respectively, and $K_M(N^*\wedge E_i^*)$ denotes the sectional curvature in M^n of the 2-plane generated by N^* and E_i^*.

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- Obviously, every spacetime with constant sectional curvature trivially obeys NCC.
- It is not difficult to see that a GRW spacetime $-I \times_f M^n$ obeys NCC if and only if

$$\mathrm{Ric}_M \geq (n-1) \sup_I (ff'' - f'^2) \langle , \rangle_M, \qquad \text{(NCC)}$$

where Ric_M and $\langle , \rangle_M$ are respectively the Ricci and metric tensors of the Riemannian manifold M^n.

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  where Ric_M and $\langle , \rangle_M$ are respectively the Ricci and metric tensors of the Riemannian manifold M^n.
- In particular, a RW spacetime obeys NCC if and only if

$$\kappa \geq \sup_I (ff'' - f'^2),$$

  where κ denotes the constant sectional curvature of M^n.

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Theorem 1 (Alías, Romero, Sánchez, 1997)

Let $\bar{M}^{n+1}$ be a CS spacetime which is equipped with a timelike conformal vector field which is an eigenfield of the Ricci operator. If $\bar{M}^{n+1}$ obeys the null convergence condition, then every compact spacelike hypersurface with constant mean curvature must be totally umbilical.

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Theorem 2 (Montiel, 1999)

Let $-I \times_f M^n$ be a spatially closed GRW spacetime obeying the null convergence condition. Then the only compact spacelike hypersurfaces immersed into $-I \times_f M^n$ with constant mean curvature are the embedded slices $\{t\} \times M^n$, $t \in I$, unless in the case where $-I \times_f M^n$ is isometric to de Sitter spacetime in a neighborhood of Σ, which must be a round umbilical hypersphere.

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- Here, instead of the null convergence condition we need to impose on $-I \times_f M^n$ the following stronger condition

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  where K_M stands for the sectional curvature of M^n.
- We will refer to (strong NCC) as the strong null convergence condition.

Our main result

Theorem 3

Let $-I \times_f M^n$ be a spatially closed GRW spacetime obeying the strong null convergence condition, with $n \geq 3$. Assume that Σ^n is a compact spacelike hypersurface immersed into $-I \times_f M^n$ which is contained in a slab $\Omega(t_1, t_2)$ on which f' does not vanish.

1. If H_k is constant, with $2 \leq k \leq n$ then Σ is totally umbilical.
2. Moreover, Σ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere.

The latter case cannot occur if we assume that the inequality in (strong NCC) is strict.

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- In particular, the constant H_k is positive and we can consider the function $\phi = H_k^{1/k} g(h) + \langle N, K \rangle \in \mathcal{C}^\infty(\Sigma)$.
- Since H_k is constant, from Lemmas 3 and 4 we have that

$$\begin{aligned} L_{k-1}\phi &= k\binom{n}{k}(H_k - H_k^{1/k} H_{k-1}) f'(h) \\ &\quad + \binom{n}{k}\langle N, K\rangle (nH_1 H_k - (n-k)H_{k+1} - kH_k^{(k+1)/k}) \\ &\quad + \langle N, K\rangle \Theta, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Theta &= \frac{1}{f^2(h)} \sum_{i=1}^{n} \mu_{i,k-1} K_M(N^* \wedge E_i^*) \|N^* \wedge E_i^*\|^2 \\ &\quad - (\log f)''(h) \left(c_{k-1} H_{k-1} \|\nabla h\|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle \right), \end{aligned}$$

and $P_{k-1} E_i = \mu_{i,k-1} E_i$ for every $i = 1, \ldots, n$.

Sketch of the proof

- Since Σ has an elliptic point, using Garding inequalities we obtain that
$$H_k - H_k^{1/k} H_{k-1} = H_k^{1/k}(H_k^{(k-1)/k} - H_{k-1}) \leq 0 \tag{2}$$
on Σ, with equality if and only if Σ is totally umbilical, and also
$$nH_1 H_k - (n-k)H_{k+1} - kH_k^{(k+1)/k} \geq kH_k(H_1 - H_k^{1/k}) \geq 0 \tag{3}$$
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- On the other hand, Lemma 1 applies here and implies that the operator L_{k-1} is elliptic or, equivalently, P_{k-1} is positive definite.
- The positive definiteness of P_{k-1}, jointly with (strong NCC), implies also that
$$\Theta \geq \left(\frac{\alpha}{f^2(h)} - (\log f)''(h)\right)\left(c_{k-1}H_{k-1}\|\nabla h\|^2 - \langle P_{k-1}(\nabla h), \nabla h\rangle\right) \geq 0, \tag{4}$$
where $\alpha = \sup_I (ff'' - f'^2)$.

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- In particular, H_1 is a positive constant and the result follows by Theorem 2.