Unicity of constant higher order mean curvature spacelike hypersurfaces in generalized Robertson-Walker spacetimes

A. Gervasio Colares
Universidade Federal do Ceará, Brazil

Joint work with Luis J. Alías

IV International Meeting on Lorentzian Geometry
Universidade de Santiago de Compostela, Spain

5-8 February 2007
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We will denote by $-I \times_f M^n$ the $(n+1)$-dimensional product manifold $I \times M$ endowed with the Lorentzian metric 

$$\langle , \rangle = -dt^2 + f^2(t)\langle , \rangle_M,$$

where $f > 0$ is a positive smooth function on $I$, and $\langle , \rangle_M$ stands for the Riemannian metric on $M^n$. 

That is, $-I \times_f M^n$ is nothing but a Lorentzian warped product with Lorentzian base $(I, -dt^2)$, Riemannian fiber $(M^n, \langle , \rangle_M)$, and warping function $f$.

We will refer to $-I \times_f M^n$ as a generalized Robertson-Walker (GRW) spacetime. In particular, when the Riemannian factor $M^n$ has constant sectional curvature then $-I \times_f M^n$ is classically called a Robertson-Walker (RW) spacetime.
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Consider a smooth immersion

$$\psi : \Sigma^n \to -I \times_f M^n$$

of an $n$-dimensional connected manifold $\Sigma$ into a GRW spacetime, and assume that the induced metric via $\psi$ is a Riemannian metric on $\Sigma$; that is, $\Sigma$ is a spacelike hypersurface.

Since $\partial_t = \left(\partial/\partial t\right)$, $(t, x) \in -I \times_f M^n$, is a unitary timelike vector field globally defined on the ambient GRW spacetime, then there exists a unique unitary timelike normal field $N$ globally defined on $\Sigma$ which is in the same time-orientation as $\partial_t$, so that

$$\langle N, \partial_t \rangle \leq -1 < 0$$
on $\Sigma$.

We will refer to that normal field $N$ as the future-pointing Gauss map of the hypersurface. Its opposite will be referred as the past-pointing Gauss map of $\Sigma$.

The height function of $\Sigma$, $h = \pi_I \circ \psi$, is the restriction of the projection $\pi_I(t, x) = t$ to $\Sigma$. 
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Unicity of spacelike hypersurfaces in GRW spacetimes
Let $A$ be the shape operator of $\Sigma$ with respect to either the future or the past-pointing Gauss map $N$. 

Associated to the shape operator there are $n$ algebraic invariants given by

$$S_k(p) = \sigma_k(\kappa_1(p),...,\kappa_n(p)),$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(x_1,...,x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1}...x_{i_k}.$$

The $k$th mean curvature $H_k$ of the hypersurface is then defined by

$$H_k = (-1)^k S_k = \sigma_k(-\kappa_1,...,-\kappa_n),$$

for every $0 \leq k \leq n$. 

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Unicity of spacelike hypersurfaces in GRW spacetimes
Higher order mean curvatures

- Let $A$ be the shape operator of $\Sigma$ with respect to either the future or the past-pointing Gauss map $N$.
- As is well known, $A$ defines a self-adjoint linear operator on each tangent space $T_p\Sigma$, and its eigenvalues

$$\kappa_1(p), \ldots, \kappa_n(p)$$

are the principal curvatures of the hypersurface.
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$$S_k(p) = \sigma_k(\kappa_1(p), \ldots, \kappa_n(p)), \quad 1 \leq k \leq n,$$

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$$\binom{n}{k} H_k = (-1)^k S_k = \sigma_k(-\kappa_1, \ldots, -\kappa_n),$$

for every $0 \leq k \leq n$. 
In particular, when $k = 1$

$$H_1 = -\frac{1}{n} \sum_{i=1}^{n} \kappa_i = -\frac{1}{n} \text{trace}(A) = H$$

is the mean curvature of $\Sigma$, which is the main extrinsic curvature of the hypersurface.
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By the choice of the sign $(-1)^k$, the mean curvature vector $\vec{H}$ is given by $\vec{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\vec{H}(p)$ is in the same time-orientation as $N$. 
The Newton transformations and their associated differential operators

The Newton transformations $P_k$ are defined inductively from $A$ by

$$P_0 = I \quad \text{and} \quad P_k = \binom{n}{k} H_k I + A \circ P_{k-1},$$

for every $k = 1 \ldots, n$, where $I$ denotes the identity in $\mathcal{X}(\Sigma)$. 
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- Then each $P_k(p)$ is a self-adjoint linear operator on the tangent space $T_p \Sigma$ which commutes with $A(p)$, and $A(p)$ and $P_k(p)$ can be simultaneously diagonalized.
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- Then each $P_k(p)$ is a self-adjoint linear operator on the tangent space $T_p\Sigma$ which commutes with $A(p)$, and $A(p)$ and $P_k(p)$ can be simultaneously diagonalized.

- Let $\nabla$ be the Levi-Civita connection of $\Sigma$. To each $P_k$, we associate the second order linear differential operator
  
  $$
  L_k(f) = \text{trace}(P_k \circ \nabla^2 f),
  $$

  where $\nabla^2 f$ denotes the self-adjoint linear operator metrically equivalent to the hessian of $f$:  

  $$
  \langle \nabla^2 f(X), Y \rangle = \langle \nabla_X (\nabla f), Y \rangle, \quad X, Y \in \mathcal{X}(\Sigma).
  $$
Observe that

$$\text{div}(P_k(\nabla f)) = \langle \text{div} P_k, \nabla f \rangle + L_k(f),$$

where, for an orthonormal basis $E_1, \ldots, E_n$,

$$\text{div} P_k := \text{trace}(\nabla P_k) = \sum_{i=1}^{n} (\nabla_{E_i} P_k)(E_i).$$
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\]

In particular, if \( P_k \) is divergence-free then \( L_k(f) = \text{div}(P_k(\nabla f)) \) and \( L_k \) is a divergence form differential operator on \( \Sigma \).
By an elliptic point in a spacelike hypersurface we mean a point of $\Sigma$ where all the principal curvatures are negative, with respect to an appropriate choice of the Gauss map $N$.
Ellipticity and positive definiteness

- By an elliptic point in a spacelike hypersurface we mean a point of $\Sigma$ where all the principal curvatures are negative, with respect to an appropriate choice of the Gauss map $N$.

**Lemma 1**

Let $\Sigma$ be a spacelike hypersurface immersed into a GRW spacetime. If there exists an elliptic point of $\Sigma$, with respect to an appropriate choice of the Gauss map $N$, and $H_{k+1} > 0$ on $\Sigma$, for $1 \leq k \leq n - 1$, then for all $1 \leq j \leq k$ the operator $L_j$ is elliptic or, equivalently, $P_j$ is positive definite (for that appropriate choice of the Gauss map, if $j$ is odd).
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The proof of Lemma 1 follows from that of Cheng-Rosenberg (An. Acad. Brasil. Ciênc., 2005) or Barbosa-Colares (Ann. Global Anal. Geom., 1997), considering that in our case, and by our sign convention in the definition of the $j$-th mean curvatures, we have

$$\binom{n}{j} H_j = \sigma_j(-\kappa_1, \ldots, -\kappa_n) = (-1)^j S_j.$$
In order to apply Lemma 1, it is convenient to have some geometric condition which implies the existence of an elliptic point.
Ellipticity and positive definiteness

In order to apply Lemma 1, it is convenient to have some geometric condition which implies the existence of an elliptic point.

Lemma 2 (Existence of an elliptic point)

Let $\psi : \Sigma^n \to -I \times_f M^n$ be a compact spacelike hypersurface immersed into a spatially closed GRW spacetime, and assume that $f'(h)$ does not vanish on $\Sigma$ (equivalently, $\psi(\Sigma)$ is contained in a slab

$$\Omega(t_1, t_2) = (t_1, t_2) \times M \subset -I \times_f M^n$$

on which $f'$ does not vanish)

1. If $f'(h) > 0$ on $\Sigma$ (equivalently, $f' > 0$ on $(t_1, t_2)$), then there exists an elliptic point of $\Sigma$ with respect to its future-pointing Gauss map.

2. If $f'(h) < 0$ on $\Sigma$ (equivalently, $f' < 0$ on $(t_1, t_2)$), then there exists an elliptic point of $\Sigma$ with respect to its past-pointing Gauss map.
The operator $L_k$ acting on the height function

The vector field on $-I \times f M^n$ given by

$$K(t, x) = f(t)(\partial/\partial t)(t, x), \quad (t, x) \in -I \times f M^n,$$

is a non-vanishing future-pointing closed conformal vector field.
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  is a non-vanishing future-pointing closed conformal vector field.
- In fact,
  $$\nabla_Z K = f'(t)Z$$
  for every vector $Z$ tangent to $-I \times_f M^n$ at a point $(t, x)$, where $\nabla$ denotes the Levi-Civita connection on $-I \times_f M^n$. 

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The operator $L_k$ acting on the height function

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- This closed conformal field $K$ will be an essential tool in our computations.
The operator $L_k$ acting on the height function

**Lemma 3**

Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map $N$. Let $h = \pi_I \circ \psi$ be the height function of $\Sigma$, and let $g : I \to \mathbb{R}$ be any primitive of $f$. Then, for every $k = 0, \ldots, n - 1$ we have

$$L_k(h) = -(\log f)'(h)(c_k H_k + \langle P_k(\nabla h), \nabla h \rangle) - \langle N, \partial_t \rangle c_k H_{k+1},$$

and

$$L_k(g(h)) = -c_k (f'(h)H_k + \langle N, K \rangle H_{k+1}),$$

where $c_k = (n - k) \binom{n}{k} = (k + 1) \binom{n}{k+1}$. 

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Unicity of spacelike hypersurfaces in GRW spacetimes
The operator $L_k$ acting on the function $\langle N, K \rangle$

**Lemma 4**

Let $\Sigma^n$ be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map $N$, closed conformal vector field $K$ and height function $h$. Then, for every $k = 0, \ldots, n - 1$ we have

$$L_k(\langle N, K \rangle) = \binom{n}{k+1} \langle \nabla H_{k+1}, K \rangle + f'(h) c_k H_{k+1}$$
$$+ \binom{n}{k+1} \langle N, K \rangle (nH_1 H_{k+1} - (n - k - 1) H_{k+2})$$
$$+ \frac{\langle N, K \rangle}{f^2(h)} \sum_{i=1}^{n} \mu_{i,k} K_M(N^* \wedge E_i^*) \| N^* \wedge E_i^* \|^2$$
$$- \langle N, K \rangle \left( \frac{1}{f(h)} \right)' \left( \frac{1}{f(h)} \right) \left( c_k H_k \| \nabla h \|^2 - \langle P_k \nabla h, \nabla h \rangle \right) .$$

Here $N^*$ and $E_i^*$ are the projection onto the fiber $M$ of the vector fields $N$ and $E_i$, respectively, and $K_M(N^* \wedge E_i^*)$ denotes the sectional curvature in $M$ of the 2-plane generated by $N^*$ and $E_i^*$. 
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$$L_k(\langle N, K \rangle) = \left( \begin{pmatrix} n \\ k+1 \end{pmatrix} \langle \nabla H_{k+1}, K \rangle + f'(h)c_kH_{k+1} \right. + \left( \begin{pmatrix} n \\ k+1 \end{pmatrix} \langle N, K \rangle(nH_1H_{k+1} - (n - k - 1)H_{k+2}) + \frac{\langle N, K \rangle}{f^2(h)} \sum_{i=1}^{n} \mu_{i,k} K_M(N^* \wedge E_i^*) \left\| N^* \wedge E_i^* \right\|^2 - \langle N, K \rangle (\log f)''(h) \left( c_kH_k \left\| \nabla h \right\|^2 - \langle P_k \nabla h, \nabla h \rangle \right) \right.$$

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Unicity of spacelike hypersurfaces in GRW spacetimes
Null convergence condition

- Recall that a spacetime obeys the null convergence condition (NCC) if its Ricci curvature is non-negative on null (or lightlike) directions.
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- Obviously, every spacetime with constant sectional curvature trivially obeys NCC.
Null convergence condition

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- Obviously, every spacetime with constant sectional curvature trivially obeys NCC.
- It is not difficult to see that a GRW spacetime $-I \times f M^n$ obeys NCC if and only if

$$
\text{Ric}_M \geq (n - 1) \sup_I (ff'' - f'^2) \langle , \rangle_M,
$$

(NCC)

where $\text{Ric}_M$ and $\langle , \rangle_M$ are respectively the Ricci and metric tensors of the Riemannian manifold $M^n$. 

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Unicity of spacelike hypersurfaces in GRW spacetimes
Null convergence condition

- Recall that a spacetime obeys the null convergence condition (NCC) if its Ricci curvature is non-negative on null (or lightlike) directions.
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- It is not difficult to see that a GRW spacetime $-I \times_f M^n$ obeys NCC if and only if

$$\text{Ric}_M \geq (n - 1) \sup_I (ff'' - f'^2)\langle \cdot, \cdot \rangle_M, \quad \text{(NCC)}$$

where $\text{Ric}_M$ and $\langle \cdot, \cdot \rangle_M$ are respectively the Ricci and metric tensors of the Riemannian manifold $M^n$.

- In particular, a RW spacetime obeys NCC if and only if

$$\kappa \geq \sup_I (ff'' - f'^2),$$

where $\kappa$ denotes the constant sectional curvature of $M^n$. 
Conformally stationary (CS) spacetimes are time-orientable spacetimes which are equipped with a globally defined timelike conformal vector field.

Theorem 1 (Alías, Romero, Sánchez, 1997)

Let $\bar{M}^{n+1}$ be a CS spacetime which is equipped with a timelike conformal vector field which is an eigenfield of the Ricci operator. If $\bar{M}^{n+1}$ obeys the null convergence condition, then every compact spacelike hypersurface with constant mean curvature must be totally umbilical.

Theorem 2 (Montiel, 1999)

Let $-I \times f\mathcal{M}^n$ be a spatially closed GRW spacetime obeying the null convergence condition. Then the only compact spacelike hypersurfaces immersed into $-I \times f\mathcal{M}^n$ with constant mean curvature are the embedded slices $\{t\} \times \mathcal{M}^n$, $t \in I$, unless in the case where $-I \times f\mathcal{M}^n$ is isometric to de Sitter spacetime in a neighborhood of $\Sigma$, which must be a round umbilical hypersphere.
Conformally stationary (CS) spacetimes are time-orientable spacetimes which are equipped with a globally defined timelike conformal vector field.

Obviously, every GRW spacetime is a CS spacetime.
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**Theorem 2 (Montiel, 1999)**

Let $-I \times_f M^n$ be a spatially closed GRW spacetime obeying the null convergence condition. Then the only compact spacelike hypersurfaces immersed into $-I \times_f M^n$ with constant mean curvature are the embedded slices $\{t\} \times M^n$, $t \in I$, unless in the case where $-I \times_f M^n$ is isometric to de Sitter spacetime in a neighborhood of $\Sigma$, which must be a round umbilical hypersphere.
Our main result

Our objective now is to extend the results above to the case of hypersurfaces with constant higher order mean curvature.

Here, instead of the null convergence condition we need to impose on $-\mathbf{I} \times \mathbf{f}$ the following stronger condition

$$K_M \geq \sup_{\mathbf{I}}(\mathbf{f}f'' - \mathbf{f}'^2),$$

(strong NCC)

where $K_M$ stands for the sectional curvature of $M^n$.

We will refer to (strong NCC) as the strong null convergence condition.
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Here, instead of the null convergence condition we need to impose on $-I \times_f M^n$ the following stronger condition

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where $K_M$ stands for the sectional curvature of $M^n$.

We will refer to (strong NCC) as the strong null convergence condition.
Our main result

Theorem 3

Let $-I \times_f M^n$ be a spatially closed GRW spacetime obeying the strong null convergence condition, with $n \geq 3$. Assume that $\Sigma^n$ is a compact spacelike hypersurface immersed into $-I \times_f M^n$ which is contained in a slab $\Omega(t_1, t_2)$ on which $f'$ does not vanish.

1. If $H_k$ is constant, with $2 \leq k \leq n$ then $\Sigma$ is totally umbilical.

2. Moreover, $\Sigma$ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$ has positive constant sectional curvature and $\Sigma$ is a round umbilical hypersphere.

The latter case cannot occur if we assume that the inequality in (strong NCC) is strict.
Sketch of the proof

- We may assume without loss of generality that $f'(h) > 0$ on $\Sigma$, and choose on $\Sigma$ the future-pointing Gauss map $N$. 

We know from Lemma 2 that there exists a point $p_0 \in \Sigma$ where all the principal curvatures (with the chosen orientation) are negative. In particular, the constant $H_k$ is positive and we can consider the function $\phi = H_1 / k g(h) + \langle N, K \rangle \in C^\infty(\Sigma)$.

Since $H_k$ is constant, from Lemmas 3 and 4 we have that
\[
L_k - \frac{1}{k} \phi = k (n_k) (H_k - \frac{1}{k} H_1) f'(h) \quad (1)
\]
\[
+ (n_k) \langle N, K \rangle (n H_1 H_k - (n - k) H_k + 1 - k H_1 / k) + \langle N, K \rangle \Theta,
\]
where $\Theta = 1^2 f(h) \sum_{i=1}^n \mu_i, k - 1 K M(N^\ast \wedge E_i^\ast) \|N^\ast \wedge E_i^\ast\|^2 - (\log f)'(h) (c_k - 1 \|\nabla h\|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle)$, and $P_{k-1} E_i = \mu_i, k - 1 E_i$ for every $i = 1, \ldots, n$. A. Gervasio Colares

Unicity of spacelike hypersurfaces in GRW spacetimes
Sketch of the proof

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\begin{align*}
\phi &= H_k^{1/k} \, g(h) + \langle N, K \rangle \in C^\infty(\Sigma) \\
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Sketch of the proof

- We may assume without loss of generality that \( f'(h) > 0 \) on \( \Sigma \), and choose on \( \Sigma \) the future-pointing Gauss map \( N \).
- We know from Lemma 2 that there exists a point \( p_0 \in \Sigma \) where all the principal curvatures (with the chosen orientation) are negative.
- In particular, the constant \( H_k \) is positive and we can consider the function \( \phi = H_k^{1/k} g(h) + \langle N, K \rangle \in C^\infty(\Sigma) \).
- Since \( H_k \) is constant, from Lemmas 3 and 4 we have that

\[
L_{k-1}\phi = k \binom{n}{k} (H_k - H_k^{1/k} H_{k-1}) f'(h)
\]

\[
+ \binom{n}{k} \langle N, K \rangle (nH_1 H_k - (n - k) H_{k+1} - k H_k^{(k+1)/k})
\]

\[
+ \langle N, K \rangle \Theta,
\]

where

\[
\Theta = \frac{1}{f^2(h)} \sum_{i=1}^{n} \mu_{i,k-1} K_M(\nabla^* \wedge E_i^*) \| \nabla^* \wedge E_i^* \|^2
\]

\[
- (\log f)''(h) \left( c_{k-1} H_{k-1} \| \nabla h \|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle \right),
\]

and \( P_{k-1} E_i = \mu_{i,k-1} E_i \) for every \( i = 1, \ldots, n \).
Since $\Sigma$ has an elliptic point, using Garding inequalities we obtain that
\[ H_k - H_k^{1/k} H_{k-1} = H_k^{1/k} (H_k^{(k-1)/k} - H_{k-1}) \leq 0 \] (2)
on $\Sigma$, with equality if and only if $\Sigma$ is totally umbilical, and also
\[ nH_1 H_k - (n - k)H_{k+1} - kH_k^{(k+1)/k} \geq kH_k (H_1 - H_k^{1/k}) \geq 0 \] (3)
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A. Gervasio Colares

Unicity of spacelike hypersurfaces in GRW spacetimes
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- On the other hand, Lemma 1 applies here and implies that the operator $L_{k-1}$ is elliptic or, equivalently, $P_{k-1}$ is positive definite.

- The positive definiteness of $P_{k-1}$, jointly with (strong NCC), implies also that
  \[ \Theta \geq \left( \frac{\alpha}{f^2(h)} - (\log f)''(h) \right) \left( c_{k-1} H_{k-1} \| \nabla h \|^2 - \langle P_{k-1}(\nabla h), \nabla h \rangle \right) \geq 0, \]  
  where $\alpha = \sup_I (ff'' - f'^2)$.
Summing up, using (2), (3) and (4), and taking into account that $f'(h) > 0$ and $\langle N, K \rangle < 0$, we obtain from (1) that $L_{k-1}\phi \leq 0$ on $\Sigma$. Since $L_{k-1}$ is an elliptic operator on the Riemannian manifold $\Sigma$, which is compact, we have, by the maximum principle, that $\phi$ must be constant. Hence, $L_{k-1}\phi = 0$ and the three terms in (1) vanish on $\Sigma$. In particular, (2) is an equality and $\Sigma$ is a totally umbilical hypersurface. Since $H_k$ is a positive constant and $\Sigma$ is totally umbilical, we have that all the higher order mean curvatures are constant. In particular, $H_1$ is a positive constant and the result follows by Theorem 2.
Sketch of the proof

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Summing up, using (2), (3) and (4), and taking into account that \( f'(h) > 0 \) and \( \langle N, K \rangle < 0 \), we obtain from (1) that \( L_{k-1}\phi \leq 0 \) on \( \Sigma \).

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