MULTIPLY WARPED PRODUCTS: GENERALIZED KASNER SPACE-TIMES

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IV International Meeting on Lorentzian Geometry

Santiago de Compostela, SPAIN

February 8, 2007

Abstract. We will study expressions that relate the Ricci (respectively, scalar) curvature of a multiply warped product with the Ricci (respectively, scalar) curvatures of its base and fibers as well as warping functions. Then we will introduce and consider a kind of generalization of Kasner space-times called as the generalized Kasner space-time which has the metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \sum_{i=1}^{k} \varphi^{2p_i} \mathrm{d}x_i^2.$$

Moreover, we state necessary and sufficient conditions for a multiply generalized Robertson-Walker space-time to be Einstein (respectively, with constant scalar curvature) generalized Kasner space-times of dimension 4.

TABLE OF CONTENTS

1. PRELIMINARIES

1.1. Basic Definitions

Definition 1.1 Let (B, g_B) and (F_i, g_{F_i}) be *pseudo-Riemannian* manifolds and also let $b_i \colon B \to (0, \infty)$ be smooth functions for any $i \in \{1, 2, \cdots, m\}$. The *multiply warped product* is the *product manifold* $M = B \times F_1 \times F_2 \times \cdots \times F_m$ furnished with the metric tensor $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \cdots \oplus b_m^2 g_{F_m}$ defined by

$$g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \cdots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m})$$

Definition 1.2 The multiply warped product (M, g) is a *Lorentzian multiply warped product* if (F_i, g_{F_i}) are all *Riemannian* for any $i \in \{1, 2, \cdots, m\}$ and either (B, g_B) is *Lorentzian* or else (B, g_B) is a one-dimensional manifold with a *negative definite* metric $-\mathrm{d}t^2$.

Definition 1.3 If B is an open connected interval I of the form $I = (t_1, t_2)$ equipped with the negative definite metric $g_B = -\mathrm{d}t^2$, where $-\infty \leq t_1 < t_2 \leq \infty$ and (F_i, g_{F_i}) is Riemannian for any $i \in \{1, 2, \cdots, m\}$ then the Lorentzian multiply warped product (M, g) is called a multiply generalized Robertson-Walker space-time.

Definition 1.4 A generalized Kasner space-time (M, g) is a Lorentzian multiply warped product of the form $M = I \times_{\varphi^{p_1}} F_1 \times \cdots \times_{\varphi^{p_m}} F_m$ with the metric $g = -\mathrm{d}t^2 \oplus \varphi^{2p_1} g_{F_1} \oplus \cdots \oplus \varphi^{2p_m} g_{F_m}$ where $\varphi\colon I \to (0, \infty)$ is smooth and $p_i \in \mathbb{R}$, for any $i \in \{1, \cdots, m\}$ and also $I = (t_1, t_2)$ with $-\infty \leq t_1 < t_2 \leq \infty$.

1.2. Examples

- Schwarzschild Space-time: If $r < 2m$, then

$$\mathrm{d}s^2 = -\left(\frac{2m}{r} - 1\right)^{-1} \mathrm{d}r^2 + \left(\frac{2m}{r} - 1\right) \mathrm{d}t^2 + r^2 \mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$ on $\mathbb{S}^2$.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + b_1^2(t)\mathrm{d}r^2 + b_2^2(t)\mathrm{d}\Omega^2$$

where

$$b_1(t) = \sqrt{\frac{2m}{F^{-1}(t)} - 1} \quad \text{and} \quad b_2(t) = F^{-1}(t) \quad \text{also}$$

$$t = F(r) = 2m \arccos\left(\sqrt{\frac{2m - r}{2m}}\right) - \sqrt{r(2m - r)} \quad \text{such that}$$

$$\lim_{r \to 2m} F(r) = m\pi \quad \text{and} \quad \lim_{r \to 0} F(r) = 0.$$

- Kasner Space-time: If $t > 0$, then

$$g = -\mathrm{d}t^2 \oplus t^{2p_1}\mathrm{d}x^2 \oplus t^{2p_2}\mathrm{d}y^2 \oplus t^{2p_3}\mathrm{d}z^2$$

where $p_1 + p_2 + p_3 = (p_1)^2 + (p_2)^2 + (p_3)^2 = 1$.

(a) $-1/3 \leq p_1 < 0 < p_2 \leq p_3 < 1$ by excluding the case of two p_i's zero and one p_i equal to 1.

(b) the *Kasner* space-time is *globally hyperbolic*

(c) By [5], the *Kasner* space-time is future-directed time-like and future-directed null geodesic complete but it is past-directed time-like and past-directed null geodesic incomplete. Moreover, it is also space-like geodesic incomplete.

(d) The Kasner space-time is Einstein with $\lambda = 0$ (i.e., Ricci-flat).

- Static Bañados-Teitelboim-Zanelli (BTZ) Space-time: If $r < r_H$, then

$$\mathrm{d}s^2 = -N^{-2}\mathrm{d}r^2 + N^2\mathrm{d}t^2 + r^2\mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ on $\mathbb{S}^2$ and

$$N^2 = m - \frac{r^2}{l^2} \quad \text{with} \quad r_H = l\sqrt{m}.$$

2. MULTIPLY GENERALIZED ROBERTSON-WALKER SPACE-TIMES

Let $M = I \times_{b_1} F_1 \times \cdots \times_{b_m} F_m$ be a multiply GRW space-time with the metric $g = -\mathrm{d}t^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$. Then

Theorem 2.1 If $\partial/\partial t \in \mathfrak{X}(I)$ and $v_i \in \mathfrak{X}(F_i)$, for any $i \in \{1, \cdots, m\}$. If $v = \sum_{i=1}^m v_i \in \mathfrak{X}(F)$, then

$$\begin{aligned}\mathrm{Ric}\Big(\frac{\partial}{\partial t} + v, \frac{\partial}{\partial t} + v\Big) \;&=\; \sum_{i=1}^m \Big(\mathrm{Ric}_{F_i}(v_i, v_i) + \Big(b_i b_i'' + (s_i - 1)(b_i')^2 \\ &\quad + \; b_i b_i' \sum_{k=1, k\neq i}^m s_k \frac{b_k'}{b_k}\Big) g_{F_i}(v_i, v_i) - s_i \frac{b_i''}{b_i}\Big)\end{aligned}$$

Theorem 2.2 If τ denotes the scalar curvature, then

$$\tau = 2\sum_{i=1}^m s_i \frac{b_i''}{b_i} + \sum_{i=1}^m \frac{\tau_{F_i}}{b_i^2} + \sum_{i=1}^m s_i(s_i - 1)\frac{(b_i')^2}{b_i^2} + \sum_{i=1}^m \sum_{k=1, k\neq i}^m s_k s_i \frac{b_i' b_k'}{b_i b_k}$$

Remark 2.3 Let (M, g) be an arbitrary n-dimensional pseudo-Riemannian manifold.

- If (M, g) is Einstein and $n \geq 3$, then λ is constant and $\lambda = \tau/n$ where τ is the constant scalar curvature of (M, g).

- If (M, g) is Einstein and $n = 2$ then λ is not necessarily constant.
- If (M, g) has constant sectional curvature k, then (M, g) is Einstein with $\lambda = k(n-1)$ and has constant scalar curvature $\tau = n(n-1)k$
- (M, g) is Einstein with Ricci curvature λ and $n = 3$ then (M, g) is a space of constant (sectional) curvature $\mathrm{K} = \lambda/2$.
- If (M, g) is a Lorentzian manifold then (M, g) is Einstein if and only if $\mathrm{Ric}(v, v) = 0$ for any null vector field v on M.

Theorem 2.4 The space-time (M, g) is Einstein with Ricci curvature λ if and only if the following conditions are satisfied for any $i \in \{1, \cdots, m\}$

- each fiber (F_i, g_{F_i}) is Einstein with Ricci curvature λ_{F_i} for any $i \in \{1, \cdots, m\}$
- $\displaystyle\sum_{i=1}^{m} s_i \frac{b_i''}{b_i} = \lambda$ and
- $\displaystyle\lambda_{F_i} + b_i b_i'' + (s_i - 1)(b_i')^2 + b_i b_i' \sum_{k=1, k\neq i}^{m} s_k \frac{b_k'}{b_k} = \lambda b_i^2$

Proposition 2.5 If the space-time (M, g) has constant scalar curvature τ, then each fiber (F_i, g_{F_i}) has constant scalar curvature τ_{F_i}, for any $i \in \{1, \cdots, m\}$.

Let (M, g) be an n-dimensional pseudo-Riemannian manifold.

Lemma 2.6 For any $t \in \mathbb{R}$ and $v \in C^\infty_{>0}(B)$. Then

(i) $\operatorname{grad}_g v^t = t v^{t-1} \operatorname{grad}_g v$

(ii) $\dfrac{\Delta_g v^t}{v^t} = t\left[(t-1)\dfrac{\|\operatorname{grad}_g v\|_g^2}{v^2} + \dfrac{\Delta_g v}{v}\right]$

Lemma 2.7 Let L_g be a differential operator on $C^\infty_{>0}(M)$ defined by

$$L_g v = \sum_{i=1}^{k} r_i \frac{\Delta_g v^{a_i}}{v^{a_i}}$$

where $r_i, a_i \in \mathbb{R}$ and $\zeta := \sum_{i=1}^{k} r_i a_i$ also $\eta := \sum_{i=1}^{k} r_i a_i^2$. Then,

(i) $L_g v = (\eta - \zeta)\dfrac{\|\operatorname{grad}_g v\|_g^2}{v^2} + \zeta \dfrac{\Delta_g v}{v}$

(ii) If $\zeta \neq 0$ and $\eta \neq 0$, for $\alpha = \dfrac{\zeta}{\eta}$ and $\beta = \dfrac{\zeta^2}{\eta}$ then we have

$$L_g v = \beta \frac{\Delta_g v^{\frac{1}{\alpha}}}{v^{\frac{1}{\alpha}}}.$$

3. GENERALIZED KASNER SPACE-TIMES

Let (M, g) be a generalized Kasner space-time of the form $M = I \times_{\varphi^{p_1}} F_1 \times \cdots \times_{\varphi^{p_m}} F_m$ with the metric $g = -\mathrm{d}t^2 \oplus \varphi^{2p_1} g_{F_1} \oplus \cdots \oplus \varphi^{2p_m} g_{F_m}$.

Notation 3.1 For a generalized Kasner space-time,

$$(\zeta;\eta) \qquad\qquad \zeta := \sum_{l=1}^{m} s_l p_l \quad \text{and} \quad \eta := \sum_{l=1}^{m} s_l p_l^2.$$

Theorem 3.2 The space-time (M, g) is Einstein with Ricci curvature λ if and only if

(i) each fiber (F_i, g_{F_i}) is Einstein with Ricci curvature λ_{F_i} for any $i \in \{1, \cdots, m\}$,

(ii) $\lambda = \sum_{l=1}^{m} s_l \frac{(\varphi^{p_l})''}{\varphi^{p_l}} = (\eta - \zeta)\frac{(\varphi')^2}{\varphi^2} + \zeta\frac{\varphi''}{\varphi}$ and

(iii) $\frac{\lambda_{F_i}}{\varphi^{2p_i}} + p_i \left[(\zeta - 1)\frac{(\varphi')^2}{\varphi^2} + \frac{\varphi''}{\varphi} \right] = \lambda.$

Remark 3.3 If we assume that $\zeta \neq 0$, then $\eta \neq 0$. Hence, (iii) is equivalent to

$$\frac{\lambda_{F_i}}{\varphi^{2p_i}} + \frac{p_i}{\zeta}\frac{(\varphi^{\zeta})''}{\varphi^{\zeta}} = \lambda,$$

and (ii) is equivalent to

$$\lambda = \frac{\zeta^2}{\eta}\frac{(\varphi^{\frac{\eta}{\zeta}})''}{\varphi^{\frac{\eta}{\zeta}}}.$$

Theorem 3.4 The space-time (M, g) has constant scalar curvature τ if and only if

(i) each fiber (F_i, g_{F_i}) has constant scalar curvature τ_{F_i} for any $i \in \{1, \cdots, m\}$ and

(ii) $\tau = 2\zeta\frac{\varphi''}{\varphi} + [(\zeta - 2)\zeta + \eta]\frac{(\varphi')^2}{\varphi^2} + \sum_{i=1}^{m} \frac{\tau_{F_i}}{\varphi^{2p_i}}$

Remark 3.5 If $\zeta \neq 0$, then (ii) is equivalent to

$$\tau = \frac{4\zeta^2}{\zeta^2 + \eta}\frac{(\varphi^{\frac{\zeta^2+\eta}{2\zeta}})''}{\varphi^{\frac{\zeta^2+\eta}{2\zeta}}} + \sum_{i=1}^{m} \frac{\tau_{F_i}}{\varphi^{2p_i}}$$

Remark 3.6 In order for a classical Kasner space-time to be Einstein or to have a constant scalar curvature is $p_1 = p_2 = p_3 = 0$.

4. APPLICATIONS

4.1. 4-Dimensional Generalized Kasner Space-Times

Definition 4.1 Let (M, g) be a generalized Kasner space-time of the form $M = I \times_{\varphi^{p_1}} F_1 \times \cdots \times_{\varphi^{p_m}} F_m$ with the metric $g = -\mathrm{d}t^2 \oplus \varphi^{2p_1} g_{F_1} \oplus \cdots \oplus \varphi^{2p_m} g_{F_m}$.

- (M, g) is said to be of *Type (I)* if $m = 1$ and $\dim(F) = 3$.
- (M, g) is said to be of *Type (II)* if $m = 2$ and $\dim(F_1) = 1$ and $\dim(F_2) = 2$.
- (M, g) is said to be of *Type (III)* if $m = 3$ and $\dim(F_1) = 1$, $\dim(F_2) = 1$ and $\dim(F_3) = 1$.

Classification of Einstein Type (I) generalized Kasner space-times:

Alías L. J., Romero A. and Sánchez M., *Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems*, Tohôku Math. J. **49**, (1997), 337-345.

Classification of Type (I) generalized Kasner space-times with constant scalar curvature:

Ehrlich P. E., Jung Y.-T. and Kim S.-B., *Constant scalar curvatures on warped product manifolds*, Tsukuba J. Math., **20** (1), (1996), 239-256.

Classification of Einstein Type (II) generalized Kasner space-times:

Here, $\zeta = p_1 + 2p_2$ and $\eta = p_1^2 + 2p_2^2$.

$$\begin{cases} (\eta - \zeta)\dfrac{(\varphi')^2}{\varphi^2} + \zeta\dfrac{\varphi''}{\varphi} = \lambda \\ p_1\left[(\zeta - 1)\dfrac{(\varphi')^2}{\varphi^2} + \dfrac{\varphi''}{\varphi}\right] = \lambda \\ \dfrac{\lambda_{F_2}}{\varphi^{2p_2}} + p_2\left[(\zeta - 1)\dfrac{(\varphi')^2}{\varphi^2} + \dfrac{\varphi''}{\varphi}\right] = \lambda. \end{cases}$$

ζ	η	$\frac{\eta}{\zeta^2}$	λ	λ_{F_2}	p_1	p_2	metric	φ
0	0	-	0	0	0	0	$-\mathrm{d}t^2 + g_{F_1} + g_{F_2}$	-
0	$\frac{3}{2}p_1^2 \neq 0$	-	0	0	$\neq 0$	$-\frac{1}{2}p_1$	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{-p_1} g_{F_2}$	$\varphi_0 = cte > 0$
0	$\frac{3}{2}p_1^2 \neq 0$	-	$-$	$\neq 0$	$\neq 0$	$-\frac{1}{2}p_1$	no metric	-
$\neq 0$	ζ^2	1	0	0	$\neq 0$	$0, -2p_1$	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_2} g_{F_2}$	$(\varphi^{\zeta}; 0)$
$\neq 0$	ζ^2	1	$\neq 0$	λ	$\neq 0$	0	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + g_{F_2}$	$(\varphi^{\zeta}; \lambda)$
$\neq 0$	$\neq 0$	$\neq 1$	0	0	p_1	$\neq 0$	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{2p_2} g_{F_2}$	$\varphi_0 = cte > 0$
$\neq 0$	$\neq 0$	$\neq 1$	0	< 0	0	$\neq 0$	$-\mathrm{d}t^2 + g_{F_1} + \varphi^{2p_2} g_{F_2}$	$(\varphi^{\frac{\eta}{\zeta}}; 0)$
$\neq 0$	$\neq 0$	$\neq 1$	> 0	0	p_2	$\neq 0$	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_1} g_{F_2}$	$(\varphi^{\zeta}; 3\lambda; *)$
$\neq 0$	$\neq 0$	$\neq 1$	$\neq 0$	$\neq 0$	p_1	$\neq 0$	no metric	-

Classification of Type (II) generalized Kasner space-times with constant scalar curvature:

Here, $\zeta = p_1 + 2p_2$ and $\eta = p_1^2 + 2p_2^2$.

$$\tau = 2\zeta \frac{\varphi''}{\varphi} + [(\zeta - 2)\zeta + \eta] \frac{(\varphi')^2}{\varphi^2} + \frac{\tau_{F_2}}{\varphi^{2p_2}}.$$

ζ	η	$\frac{\eta}{\zeta^2}$	τ_{F_2}	p_1	p_2	φ eq.
0	0	-	τ_{F_2}	0	0	$\tau = \tau_{F_2}$
0	$\frac{3}{2}p_1^2$	-	0	$\neq 0$	$-\frac{1}{2}p_1$	$\tau = \eta\frac{(\varphi')^2}{\varphi^2}$
0	$\frac{3}{2}p_1^2$	-	$\neq 0$	$\neq 0$	$-\frac{1}{2}p_1$	$\tau = \eta\frac{(\varphi')^2}{\varphi^2} + \frac{\tau_{F_2}}{\varphi^{2p_2}}$
$\zeta \neq 0$	ζ^2	1	0	$\neq 0$	0	$-2u'' = -\tau u; \quad u = \varphi^\zeta$
$\zeta \neq 0$	ζ^2	1	0	$\neq 0$	$-2p_1$	$-2u'' = -\tau u; \quad u = \varphi^\zeta$
$\zeta \neq 0$	ζ^2	1	$\neq 0$	$\neq 0$	0	$-2u'' = -(\tau - \tau_{F_2}); \quad u = \varphi^\zeta$
$\zeta \neq 0$	ζ^2	1	$\neq 0$	$\neq 0$	$-2p_1$	$-2u'' = -\tau u + \tau_{F_2}u^{-\frac{1}{3}}; u = \varphi^\zeta$
$\zeta \neq 0$	$\eta \neq 0$	$\neq 1$	0	p_1	$\neq 0$	$u = (\varphi^\zeta)^{\frac{1+\frac{\eta}{\zeta^2}}{2}}$
$\zeta \neq 0$	$\eta \neq 0$	$\neq 1, \frac{1}{3}$	$\neq 0$	p_1	$\neq 0$	$u = (\varphi^\zeta)^{\frac{1+\frac{\eta}{\zeta^2}}{2}}$
$\zeta \neq 0$	$\frac{\zeta^2}{3}$	$\frac{1}{3}$	$\neq 0$	$\frac{\zeta}{3}$	$\frac{\zeta}{3}$	$-3u'' = -\tau u + \tau_{F_2}; \quad u = \varphi^{\frac{2}{3}\zeta}$

Classification of Einstein Type (III) generalized Kasner space-times:

Here, $\zeta = p_1 + p_2 + p_3$ and $\eta = p_1^2 + p_2^2 + p_3^2$.

$$\begin{cases} (\eta - \zeta)\dfrac{(\varphi')^2}{\varphi^2} + \zeta\dfrac{\varphi''}{\varphi} = \lambda \\ p_1\left[(\zeta - 1)\dfrac{(\varphi')^2}{\varphi^2} + \dfrac{\varphi''}{\varphi}\right] = \lambda \\ p_2\left[(\zeta - 1)\dfrac{(\varphi')^2}{\varphi^2} + \dfrac{\varphi''}{\varphi}\right] = \lambda \\ p_3\left[(\zeta - 1)\dfrac{(\varphi')^2}{\varphi^2} + \dfrac{\varphi''}{\varphi}\right] = \lambda. \end{cases}$$

ζ	η	$\frac{\eta}{\zeta^2}$	λ	p_1	p_2	p_3	metric	φ
0	0	-	0	0	0	0	$-\mathrm{d}t^2 + g_{F_1} + g_{F_2} + g_{F_3}$	-
0	$\neq 0$	-	0	p_1	p_2	p_3	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{2p_2} g_{F_2} + \varphi_0^{2p_3} g_{F_3}$	$\varphi_0 = cte > 0$
$\neq 0$	ζ^2	1	0	p_1	p_2	p_3	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_2} g_{F_2} + \varphi^{2p_3} g_{F_3}$	$(\varphi^{\zeta}; 0)$
$\neq 0$	$\neq 0$	$\neq 1$	0	p_1	p_2	p_3	$-\mathrm{d}t^2 + \varphi_0^{2p_1} g_{F_1} + \varphi_0^{2p_2} g_{F_2} + \varphi_0^{2p_3} g_{F_3}$	$\varphi_0 = cte > 0$
$\neq 0$	$\neq 0$	$\neq 1$	> 0	p_1	p_1	p_1	$-\mathrm{d}t^2 + \varphi^{2p_1} g_{F_1} + \varphi^{2p_1} g_{F_2} + \varphi^{2p_1} g_{F_3}$	$(\varphi^{\zeta}; 3\lambda; *)$

Classification of Type (III) generalized Kasner space-times with constant scalar curvature:

Here, $\zeta = p_1 + p_2 + p_3$ and $\eta = p_1^2 + p_2^2 + p_3^2$.

If $\zeta \neq 0$ and $u = \varphi^{\frac{\zeta^2+\eta}{2\zeta}}$ then

$$-\frac{4\zeta^2}{\zeta^2+\eta}u'' = -\tau u$$

$$u(t) = \begin{cases} \mathcal{A}e^{i\sqrt{-\tau\frac{\zeta^2+\eta}{4\zeta^2}}\,t} + \mathcal{B}e^{-i\sqrt{-\tau\frac{\zeta^2+\eta}{4\zeta^2}}\,t} & \text{if } \tau < 0, \\ \mathcal{A}t + \mathcal{B} & \text{if } \tau = 0, \\ \mathcal{A}e^{\sqrt{\tau\frac{\zeta^2+\eta}{4\zeta^2}}\,t} + \mathcal{B}e^{-\sqrt{\tau\frac{\zeta^2+\eta}{4\zeta^2}}\,t} & \text{if } \tau > 0, \end{cases}$$

with constants $\mathcal{A}$ and $\mathcal{B}$ such that $u > 0$.

If $\zeta = 0$ and $u = \varphi^{\frac{\zeta^2+\eta}{2\zeta}}$ then

$$\begin{cases} \tau = \eta\dfrac{(\varphi')^2}{\varphi^2}, \\ \varphi > 0. \end{cases}$$

Since $\eta > 0$, the latter is equivalent to

$$\begin{cases} \left(\varphi\frac{\sqrt{\tau}}{\sqrt{\eta}}+\varphi'\right)\left(\varphi\frac{\sqrt{\tau}}{\sqrt{\eta}}-\varphi'\right)=0, \\ \varphi>0. \end{cases}$$

Solutions of the equation above are given as,

$$\varphi(t)=Ce^{\pm\frac{\sqrt{\tau}}{\sqrt{\eta}}t},$$

where C is a positive constant.

4.2. BTZ Black Holes

All the cases considered in [Hong S.-T., Choi J. and Park Y.-J., *(2+1) BTZ Black hole and multiply warped product spacetimes*, Gen. Relativity Gravitation, **35**, (2003), 2105-2116] i.e.,

$$\mathrm{d}s^2=N^2\mathrm{d}t^2-N^{-2}\mathrm{d}r^2+r^2\mathrm{d}\phi^2$$

can be expressed as a $(2+1)$ multiply generalized Robertson-Walker space-time, i.e.,

$$\mathrm{d}s^2=-\mathrm{d}t^2+b_1^2(t)\mathrm{d}x^2+b_2^2(t)\mathrm{d}\phi^2$$

by considering the corresponding square lapse function N^2 where

$$\begin{cases} b_1(t) = N(F^{-1}(t)) \\ b_2(t) = F^{-1}(t), \end{cases}$$

with

$$F(r) = \int_a^r \frac{1}{N(\mu)} \mathrm{d}\mu$$

- The space-time is Einstein with Ricci curvature λ if and only if the square lapse function N^2 satisfies

$$N^2(r) = \frac{\lambda}{2} r^2 + c,$$

with a suitable constant c.

- The space-time has constant scalar curvature $\tau = \lambda$ if and only if the square lapse function N^2 has the form

$$N^2(r) = -c_1 \frac{1}{r} + \frac{\lambda}{6} r^2 + c_2,$$

with suitable constants c_1 and c_2.

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