Wave Equations on Lorentzian Manifolds and Quantization

Christian Bär

joint work with

Nicolas Ginoux and Frank Pfäffle

Institut für Mathematik
Universität Potsdam

Santiago de Compostela, February 2007

Outline

Wave Operators

Throughout let M denote a timeoriented Lorentzian manifold.
Let $E \to M$ be a vector bundle.
Denote the smooth sections in E by $C^\infty(M,E)$.

Definition

A wave operator or normally hyperbolic operator is a linear differential operator $P : C^\infty(M,E) \to C^\infty(M,E)$ of second order which looks locally like

$$P = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x^j} + B(x)$$

Wave Operators; Examples

- d'Alembert operator (functions)

$$P = \Box$$

- Klein-Gordon operator (functions)

$$P = \Box + m^2 \text{or } P = \Box + m^2 + \kappa \cdot \mathrm{scal}$$

- Wave operator in electro-dynamics (1-forms)

$$P = d\delta + \delta d$$

- Square of Dirac operator (spinors)

$$P = D^2$$

Cauchy Problem

Let M be globally hyperbolic and let $S \subset M$ be a smooth spacelike Cauchy hypersurface. Let ν be the future directed timelike unit normal field along S.

Theorem

For each $u_0, u_1 \in C_c^\infty(S, E)$ and for each $f \in C_c^\infty(M, E)$ there exists a unique $u \in C^\infty(M, E)$ satisfying

$$\begin{cases} Pu = f, & \text{on } M \\ u|_S = u_0, & \text{along } S \\ \nabla_\nu u = u_1, & \text{along } S \end{cases}$$

Cauchy Problem

Well-posedness

The solution u depends continuously on the data f, u_0, and u_1.

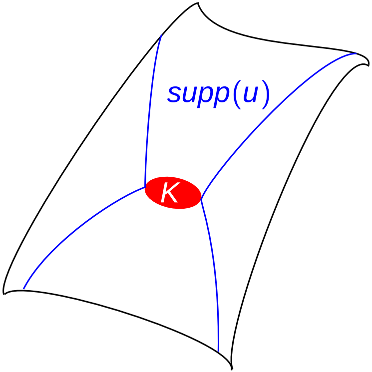
Finite propagation speed

Moreover,
$\mathrm{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$
where $K =$
$\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

Cauchy Problem

Well-posedness

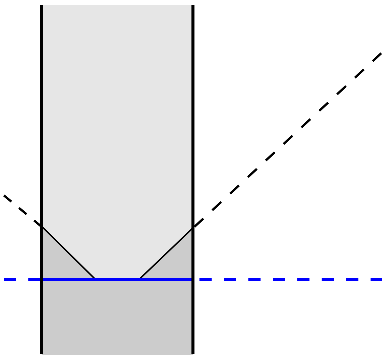
The solution u depends continuously on the data f, u_0, and u_1.



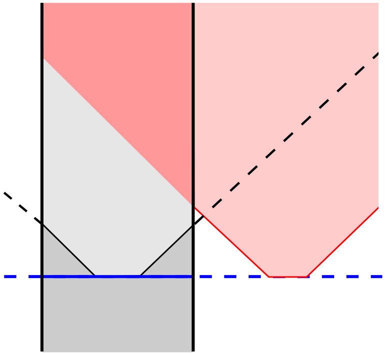
Finite propagation speed

Moreover,
$\mathrm{supp}(u) \subset J_+^M(K) \cup J_-^M(K)$
where $K =$
$\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)$.

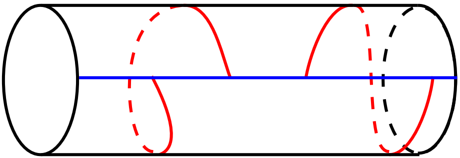
Cauchy Problem; What Can Go Wrong



Cauchy Problem; What Can Go Wrong



Cauchy Problem; What Can Go Wrong



Green's Operators

Definition

A linear operator $G : C_c^\infty(M,E) \to C^\infty(M,E)$ is called a **Green's operator** for P if

$$P \circ G = G \circ P = \mathrm{id}_{C_c^\infty(M,E)}$$

Definition

A Green's operator G is called **advanced** or **retarded** resp. if

$$\mathrm{supp}(G(u)) \subset J_+(\mathrm{supp}(u)) \text{ or } J_-(\mathrm{supp}(u))$$

resp. for any $u \in C_c^\infty(M,E)$.

Green's Operators

Definition

A linear operator $G : C_c^\infty(M,E) \to C^\infty(M,E)$ is called a Green's operator for P if

$$P \circ G = G \circ P = \mathrm{id}_{C_c^\infty(M,E)}$$

Definition

A Green's operator G is called advanced or retarded resp. if

$$\mathrm{supp}(G(u)) \subset J_+(\mathrm{supp}(u)) \text{ or } J_-(\mathrm{supp}(u))$$

resp. for any $u \in C_c^\infty(M,E)$.

Green's Operators

Theorem

Let P be a wave operator over a globally hyperbolic manifold M.

Then there exist unique advanced and retarded Green's operators for P.
These Green's operators are continuous.
The sequence of linear maps

$$0 \to C_c^\infty(M,E) \xrightarrow{P} C_c^\infty(M,E) \xrightarrow{G_+ - G_-} C_{sc}^\infty(M,E) \xrightarrow{P} C_{sc}^\infty(M,E)$$

is exact.

Green's Operators

Theorem

Let P be a wave operator over a globally hyperbolic manifold M.

Then there exist unique advanced and retarded Green's operators for P.
These Green's operators are continuous.
The sequence of linear maps

$$0 \to C_c^\infty(M,E) \xrightarrow{P} C_c^\infty(M,E) \xrightarrow{G_+ - G_-} C_{sc}^\infty(M,E) \xrightarrow{P} C_{sc}^\infty(M,E)$$

is exact.

Green's Operators

Theorem

Let P be a wave operator over a globally hyperbolic manifold M.

Then there exist unique advanced and retarded Green's operators for P.
These Green's operators are continuous.
The sequence of linear maps

$$0 \to C_c^\infty(M,E) \xrightarrow{P} C_c^\infty(M,E) \xrightarrow{G_+ - G_-} C_{sc}^\infty(M,E) \xrightarrow{P} C_{sc}^\infty(M,E)$$

is exact.

Outline

Fock Space

H complex Hilbert space, $\bigodot^n H$ completion of $\bigodot^n_{\mathrm{alg}} H$

(Bosonic or symmetric) Fock space $\mathfrak{F}(H)$ is the completion of

$$\mathfrak{F}_{\mathrm{alg}}(H) := \bigoplus_{n=0}^{\infty} \bigodot^n H.$$

Fix $v \in H$. Define the creation operator

$$a^*(v) v_1 \odot \ldots \odot v_n := v \odot v_1 \odot \ldots \odot v_n$$

and the annihilation operator

$$a(v)(w_0 \odot \cdots \odot w_n) := \sum_{k=0}^{n} (v, w_k) w_0 \odot \cdots \odot \hat{w}_k \odot \cdots \odot w_n$$

Fock Space

H complex Hilbert space, $\bigodot^n H$ completion of $\bigodot^n_{\mathrm{alg}} H$

(Bosonic or symmetric) Fock space $\mathfrak{F}(H)$ is the completion of

$$\mathfrak{F}_{\mathrm{alg}}(H) := \bigoplus_{n=0}^{\infty} \bigodot^n H.$$

Fix $v \in H$. Define the creation operator

$$a^*(v)v_1 \odot \ldots \odot v_n := v \odot v_1 \odot \ldots \odot v_n$$

and the annihilation operator

$$a(v)(w_0 \odot \cdots \odot w_n) := \sum_{k=0}^{n} (v, w_k) w_0 \odot \cdots \odot \hat{w}_k \odot \cdots \odot w_n$$

Canonical Commutator Relations

Canonical commutator relations:

$$[a(v), a(w)] = [a^*(v), a^*(w)] = 0,$$

$$[a(v), a^*(w)] = (v, w) \cdot \mathrm{id}.$$

Definition

Segal operator:

$$\theta(v) := \frac{1}{\sqrt{2}}(a(v) + a^*(v))$$

The Segal operator on $\mathfrak{F}_{\mathrm{alg}}(H)$ is essentially self-adjoint in $\mathfrak{F}(H)$.

$$[\theta(v), \theta(w)] = i \cdot \Im\mathfrak{m}(v, w)$$

Canonical Commutator Relations

Canonical commutator relations:

$$[a(v), a(w)] = [a^*(v), a^*(w)] = 0,$$

$$[a(v), a^*(w)] = (v, w) \cdot \mathrm{id}.$$

Definition

Segal operator:

$$\theta(v) := \frac{1}{\sqrt{2}}(a(v) + a^*(v))$$

The Segal operator on $\mathfrak{F}_{\mathrm{alg}}(H)$ is essentially self-adjoint in $\mathfrak{F}(H)$.

$$[\theta(v), \theta(w)] = i \cdot \mathfrak{Im}(v, w)$$

Geometric Setup

- Globally hyperbolic Lorentzian manifold M
- Real vector bundle $E \to M$ with non-degenerate metric
- Formally selfadjoint wave operator P on E

Definition

A twist structure of spin $k/2$ on E is a smooth section $Q \in C^\infty(M, \mathrm{Hom}(\bigodot^k TM, \mathrm{End}(E)))$ such that:

- $\langle Q(X_1 \odot \cdots \odot X_k)e, f\rangle = \langle e, Q(X_1 \odot \cdots \odot X_k)f\rangle$
- If X is future directed timelike, then the bilinear form $\langle\cdot,\cdot\rangle_X$ defined by
$$\langle f, g\rangle_X := \langle Q_X f, g\rangle$$
is positive definite where $Q_X = Q(X \odot \cdots \odot X)$.

Geometric Setup

- Globally hyperbolic Lorentzian manifold M
- Real vector bundle $E \to M$ with non-degenerate metric
- Formally selfadjoint wave operator P on E

Definition

A twist structure of spin $k/2$ on E is a smooth section $Q \in C^\infty(M, \mathrm{Hom}(\bigodot^k TM, \mathrm{End}(E)))$ such that:

- $\langle Q(X_1 \odot \cdots \odot X_k)e, f\rangle = \langle e, Q(X_1 \odot \cdots \odot X_k)f\rangle$
- If X is future directed timelike, then the bilinear form $\langle \cdot, \cdot \rangle_X$ defined by
$$\langle f, g\rangle_X := \langle Q_X f, g\rangle$$
is positive definite where $Q_X = Q(X \odot \cdots \odot X)$.

Examples

Example

If the metric on E is positive definite, one can choose $k = 0$ and $Q = \mathrm{id}$

Example

For spinor bundle E let $k = 1$ and $Q(X)$ be Clifford multiplication by X

Example

For $E = \Lambda^q T^*M$ let $k = 2$ and

$$Q(X \odot Y)\alpha := X^\flat \wedge \iota_Y \alpha + Y^\flat \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$

Examples

Example

If the metric on E is positive definite, one can choose $k = 0$ and $Q = \mathrm{id}$

Example

For spinor bundle E let $k = 1$ and $Q(X)$ be Clifford multiplication by X

Example

For $E = \Lambda^q T^*M$ let $k = 2$ and

$$Q(X \odot Y)\alpha := X^\flat \wedge \iota_Y \alpha + Y^\flat \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$

Examples

Example

If the metric on E is positive definite, one can choose $k = 0$ and $Q = \mathrm{id}$

Example

For spinor bundle E let $k = 1$ and $Q(X)$ be Clifford multiplication by X

Example

For $E = \Lambda^q T^*M$ let $k = 2$ and

$$Q(X \odot Y)\alpha := X^\flat \wedge \iota_Y \alpha + Y^\flat \wedge \iota_X \alpha - \langle X, Y \rangle \cdot \alpha$$

Geometric Setup

- Globally hyperbolic Lorentzian manifold M
- Real vector bundle $E \to M$ with non-degenerate metric
- Formally selfadjoint wave operator P on E
- Twist structure Q
- Cauchy hypersurface $S \subset M$

We get real Hilbert space $L^2(S, E^*)$ where

$$(u, v)_S := \int_S \langle u, v \rangle_\nu \, dA = \int_S \langle Q_\nu^* u, v \rangle \, dA$$

Geometric Setup

- Globally hyperbolic Lorentzian manifold M
- Real vector bundle $E \to M$ with non-degenerate metric
- Formally selfadjoint wave operator P on E
- Twist structure Q
- Cauchy hypersurface $S \subset M$

We get real Hilbert space $L^2(S, E^*)$ where

$$(u, v)_S := \int_S \langle u, v \rangle_\nu \, dA = \int_S \langle Q_\nu^* u, v \rangle \, dA$$

Quantum Field

- Apply Fock space construction to $H_S := L^2(S, E^*) \otimes_{\mathbb{R}} \mathbb{C}$
- Get Segal field θ

Definition

Quantum field: For $f \in C_c^\infty(S, E^*)$ put

$$\Phi_S(f) := \theta(\underbrace{i(G_+^* - G_-^*)f|_S - (Q_\nu^*)^{-1}\nabla_\nu((G_+^* - G_-^*)f)}_{\in H_S}).$$

Haag-Kastler Axioms

Theorem

- $C_c^\infty(M, E^*) \to \mathfrak{F}(H_S), \quad f \mapsto \Phi_S(f)\omega$, *is continuous for any* $\omega \in \mathfrak{F}_{\mathrm{alg}}(H_S)$
- $P\Phi_S = 0$ *in the distributional sense*
- $[\Phi_S(f), \Phi_S(g)] = 0$ *if the supports of* f *and* g *are causally independent.*
- *The linear span of the vectors* $\Phi_S(f_1) \cdots \Phi_S(f_n)\Omega$ *is dense in* $\mathfrak{F}(H_S)$ *where* $\Omega = 1 \in \bigodot^0 H_S = \mathbb{C}$ *is the vacuum vector.*

Haag-Kastler Axioms

Theorem

- $C_c^\infty(M, E^*) \to \mathfrak{F}(H_S), \quad f \mapsto \Phi_S(f)\omega,$ *is continuous for any* $\omega \in \mathfrak{F}_{\mathrm{alg}}(H_S)$
- $P\Phi_S = 0$ *in the distributional sense*
- $[\Phi_S(f), \Phi_S(g)] = 0$ *if the supports of* f *and* g *are causally independent.*
- *The linear span of the vectors* $\Phi_S(f_1)\cdots\Phi_S(f_n)\Omega$ *is dense in* $\mathfrak{F}(H_S)$ *where* $\Omega = 1 \in \bigodot^0 H_S = \mathbb{C}$ *is the vacuum vector.*

Haag-Kastler Axioms

Theorem

- $C_c^\infty(M, E^*) \to \mathfrak{F}(H_S), \quad f \mapsto \Phi_S(f)\omega$, *is continuous for any* $\omega \in \mathfrak{F}_{\mathrm{alg}}(H_S)$
- $P\Phi_S = 0$ *in the distributional sense*
- $[\Phi_S(f), \Phi_S(g)] = 0$ *if the supports of* f *and* g *are causally independent.*
- *The linear span of the vectors* $\Phi_S(f_1) \cdots \Phi_S(f_n)\Omega$ *is dense in* $\mathfrak{F}(H_S)$ *where* $\Omega = 1 \in \bigodot^0 H_S = \mathbb{C}$ *is the vacuum vector.*

Haag-Kastler Axioms

Theorem

- $C_c^\infty(M, E^*) \to \mathfrak{F}(H_S), \quad f \mapsto \Phi_S(f)\omega$, *is continuous for any* $\omega \in \mathfrak{F}_{\mathrm{alg}}(H_S)$
- $P\Phi_S = 0$ *in the distributional sense*
- $[\Phi_S(f), \Phi_S(g)] = 0$ *if the supports of* f *and* g *are causally independent.*
- *The linear span of the vectors* $\Phi_S(f_1) \cdots \Phi_S(f_n)\Omega$ *is dense in* $\mathfrak{F}(H_S)$ *where* $\Omega = 1 \in \bigodot^0 H_S = \mathbb{C}$ *is the* *vacuum vector*.

Problems

Problems:

- Construction depends on choice of Cauchy hypersurface
- Microlocal spectrum condition is violated

Algebraic quantum field theory:

- Forget Fock space (and particles)
- Regard observables (operators) as primary objects
- To each (reasonable) spacetime region associate an algebra of observables

Problems

Problems:

- Construction depends on choice of Cauchy hypersurface
- Microlocal spectrum condition is violated

Algebraic quantum field theory:

- Forget Fock space (and particles)
- Regard observables (operators) as primary objects
- To each (reasonable) spacetime region associate an algebra of observables

CCR-algebras

Let (V, ω) be a symplectic vector space.

Definition

A CCR-algebra of (V, ω) consists of a C^*-algebra A with unit and a map $W : V \to A$ such that for all $\phi, \psi \in V$ we have

- $W(0) = 1$
- $W(-\phi) = W(\phi)^*$
- $W(\phi) \cdot W(\psi) = e^{-i\omega(\phi,\psi)/2} W(\phi + \psi)$
- A is generated by the $W(\phi)$

Theorem

To each symplectic vector space there exists a CCR-algebra, unique up to $*$-isomorphism.

CCR-algebras

Let (V,ω) be a symplectic vector space.

Definition

A CCR-algebra of (V,ω) consists of a C^*-algebra A with unit and a map $W : V \to A$ such that for all $\phi,\psi \in V$ we have

- $W(0) = 1$
- $W(-\phi) = W(\phi)^*$
- $W(\phi)\cdot W(\psi) = e^{-i\omega(\phi,\psi)/2}\,W(\phi+\psi)$
- A is generated by the $W(\phi)$

Theorem

To each symplectic vector space there exists a CCR-algebra, unique up to $*$-isomorphism.

Construction of the Symplectic Vector Space

Let M be globally hyperbolic, let P a formally self-adjoint wave operator acting on sections in E.
Let G_+ and G_- be the Green's operators of P.

$$\tilde{\omega}(\phi, \psi) := \int_M \langle (G_+ - G_-)\phi, \psi \rangle \, dVol$$

defines a degenerate symplectic form on $C_c^\infty(M, E)$.
It induces a (nondegenerate) symplectic form ω on

$$\begin{aligned} V(M,E,P) &:= C_c^\infty(M,E)/P(C_c^\infty(M,E)) \\ &= C_c^\infty(M,E)/ker(G_+ - G_-) \end{aligned}$$

Quantization Functor

$\mathfrak{A}_M := CCR(M, E, P) := CCR(V(M, E, P), \omega)$ defines a functor

$$\boxed{\begin{array}{c}\text{globally hyperbolic}\\ \text{manifolds equipped}\\ \text{with a formally}\\ \text{self-adjoint wave operator}\end{array}} \Longrightarrow \boxed{\begin{array}{c}C^*\text{-algebras}\\ \text{with unit}\end{array}}$$

Haag-Kastler Axioms, II

Theorem

- *If* $O_1 \subset O_2$, *then* $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$ *for all* $O_1, O_2 \in I$.
- $\mathfrak{A}_M = \overline{\bigcup_{\substack{O \in I \\ O \neq \emptyset, O \neq M}} \mathfrak{A}_O}$.
- $\mathfrak{A}_M$ *is simple.*
- *The* $\mathfrak{A}_O$*'s have a common unit* 1.
- *For all* $O_1, O_2 \in I$ *with* $J(\overline{O_1}) \cap \overline{O_2} = \emptyset$ *the subalgebras* $\mathfrak{A}_{O_1}$ *and* $\mathfrak{A}_{O_2}$ *of* $\mathfrak{A}_M$ *commute:* $[\mathfrak{A}_{O_1}, \mathfrak{A}_{O_2}] = \{0\}$.
- *Time-slice axiom. Let* $O_1 \subset O_2$ *be nonempty elements of* I *admitting a common Cauchy hypersurface. Then* $\mathfrak{A}_{O_1} = \mathfrak{A}_{O_2}$.
- *Let* $O_1, O_2 \in I$ *and let the Cauchy development* $D(O_2)$ *be relatively compact in* M. *If* $O_1 \subset D(O_2)$, *then* $\mathfrak{A}_{O_1} \subset \mathfrak{A}_{O_2}$.

Comparison of the Two Approaches

Given a Cauchy hypersurface $S \subset M$, a twist structure, and the corresponding quantum field Φ_S

$$W_S(f) := \exp(i\Phi_S(f))$$

defines a CCR-representation for $V(M, E, P)$.

Problems

- Construct physically satisfactory representations (Hadamard states)
- Construct n-point functions (Singularities, renormalization)
- Construct nonlinear fields (Energy-momentum tensor)

Applications in Physics

- Hawking radiation of black holes
- Unruh effect

Brunetti, Dimock, Fewster, Fredenhagen, Hollands, Radzikowski, Verch, Wald, ...