Topological censorship for spacetimes with timelike boundary

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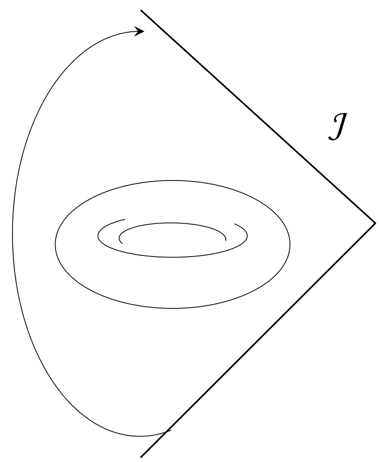
Theorem. *(Gannon′75, Lee′76) Let* (M, g) *be a globally hyperbolic and asymptotically flat spacetime that satisfies the null energy condition. If* M *admits a Cauchy surface* S *with* $\pi(S) \neq 0$ *then* M *is null geodesically incomplete.*



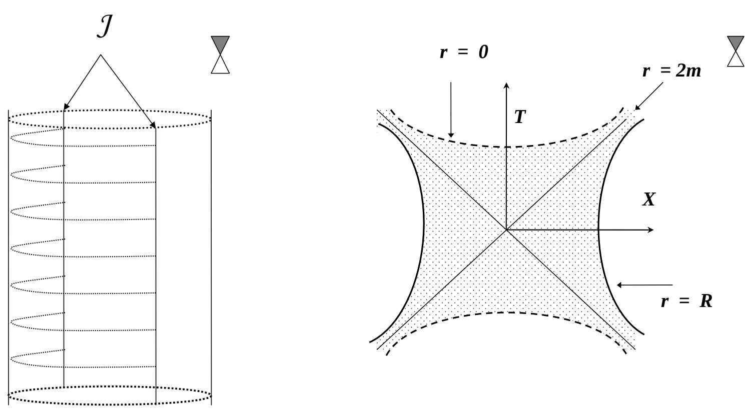
Principle of Topological Censorship (PTC) (Friedman, Schleich, Witt ′93):

Let $(\tilde{M}, \tilde{g})$ be a spacetime admitting a conformal completion (M, g) with null $\mathcal{J}$. Then every causal curve on M whose initial and final endpoints lie on $\mathcal{J}$ is fixed endpoint homotopic to a curve on $\mathcal{J}$.

In (Galloway, '96) a quasi-local version of PTC was established. In this formulation, we have a globally hyperbolic spacetime with timelike boundary $\mathcal{I} := \partial M$ such that $\mathcal{I}_\alpha \approx \mathbb{R} \times S_\alpha$ where

(a) $S_\alpha \approx S^{n-2}$ $\qquad\qquad$ (b) S_α is null convex

Examples:

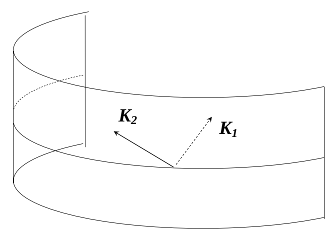


Our goal: To prove PTC in the timelike setting under weaker assumptions.

- S_α is compact.

- S_α is <u>weakly</u> null convex.

Let $S \subset \partial M$ be spacelike hypersurface of ∂M.



For $i = 1, 2$ define the *null Weingarten map of* S *relative to* K_i by

$$b_i \colon T_pS \to T_pS, \qquad b_i(X) := \tan \nabla_X K_i,$$

the corresponding *null second fundamental forms*

$$B_i \colon T_pS \times T_pS \to \mathbb{R}, \qquad B_i(X, Y) = g(b_i(X), Y)$$

and the *null expansions of* S

$$\theta_i = \mathrm{Tr}(b_i).$$

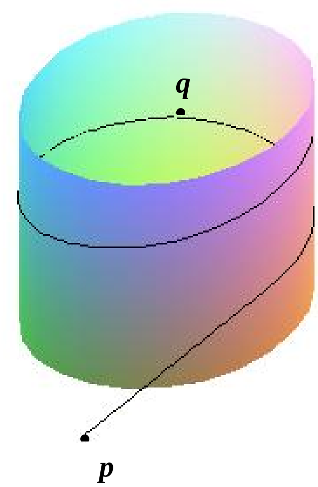
- S is **null convex** if B_1 is negative definite and B_2 is positive definite.

- S is **weakly null convex** if $\theta_1 < 0$ and $\theta_2 \geq 0$.

Let (M, g) be a spacetime with timelike boundary, then the following holds:

1.- The relation I^+ is open in M.

2.- If $p \ll q$ and $q \leq r$ then $p \ll r$.



Further, if (M, g) is globally hyperbolic then

1.- $J^+(A)$ is closed for all compact $A \subset M$.

2.- Limit curve theory carries on without significant changes.

Proposition. *(Fastest null geodesics).*

Let M be a globally hyperbolic spacetime with timelike boundary and assume

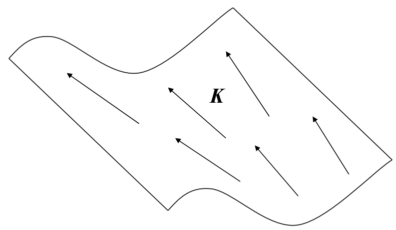
- $\partial M = \partial_1 \cup \partial_2$

- *The Cauchy surfaces of ∂_i are compact and acausal in M.*

- $J^+(S_1) \cap \partial_2 \neq \emptyset$.

Then there exists a geodesic $\eta \subset \partial I^+(S)$ from S to ∂_2.

A **Smooth null hypersurface** is an embedded submanifold $S \subset M$ such that the pullback metric $\imath^* g$ is degenerate.

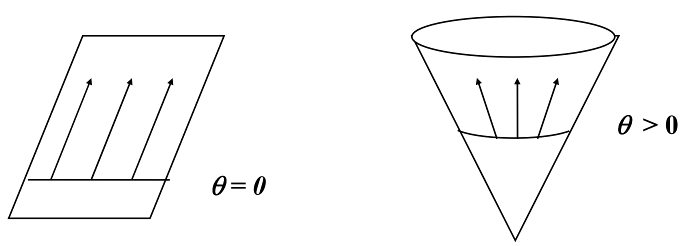


$$\overline{X} = \overline{Y} \quad \Leftrightarrow \quad X_p - Y_p = \lambda K_p$$

$$h(\overline{X}, \overline{Y}) = g(X, Y)$$

$$b_K \colon T_pS/K \to T_pS/K, \qquad b_K(\overline{X}) = \overline{\nabla_X K}.$$

$$\theta_K = \mathrm{Tr}(b_K)$$



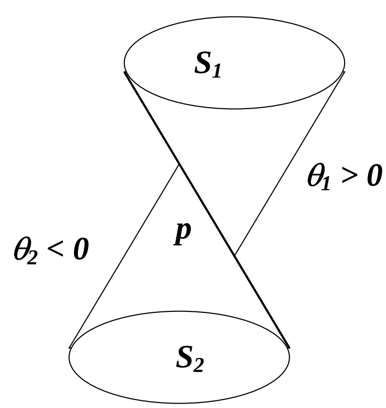
Theorem. *(Maximum Principle for Smooth Null Hypersurfaces).*

Let S_1 and S_2 be smooth null hypersurfaces in a spacetime (M, g). Suppose that S_1 and S_2 meet at p and

1. S_2 lies to the future of S_1 near p.

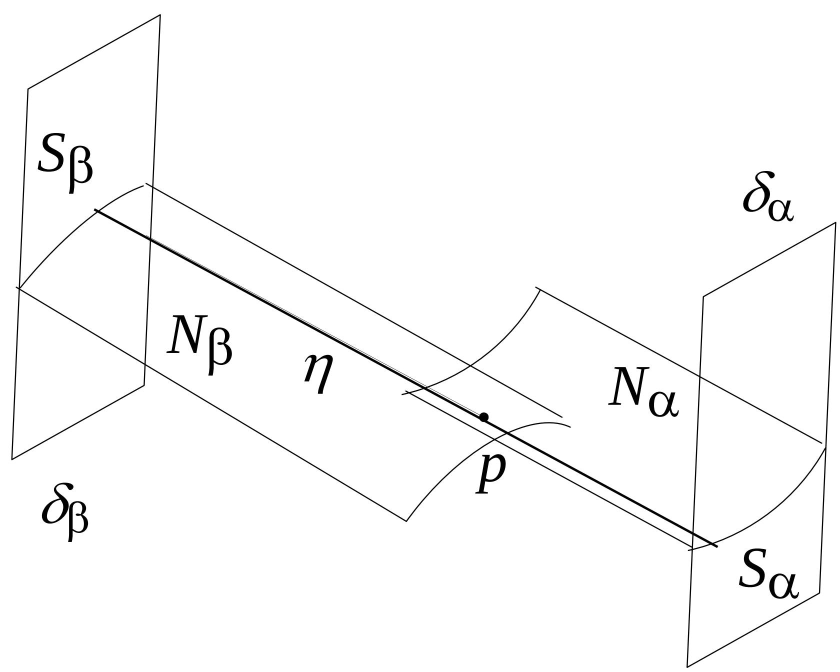
2. The null mean curvature scalars θ_i satisfy $\theta_2 \leq 0 \leq \theta_1$.

Then S_1 and S_2 coincide near p and this common null hypersurface has null mean curvature $\theta = 0$.



Lemma. *Let M be a globally hyperbolic spacetime with timelike boundary satisfying the null energy condition and ∂_α, ∂_β two different connected components of ∂M. Further, let $S_\alpha \subset \partial_\alpha$, $S_\beta \subset \partial_\beta$ be weakly null convex Cauchy surfaces with $J^+(S_\alpha) \cap S_\beta \neq \emptyset$. If $I^+(S_\alpha) \cap S_\beta = \emptyset$ then every future causal curve joining S_α and S_β must meet ∂M at a point other than its endpoints.*

Proof:



Corollary. *(Wormhole non-tranversability).*

Let M be a globally hyperbolic spacetime with timelike boundary and assume

- $\partial M = \partial_1 \cup \partial_2$

- *The Cauchy surfaces of ∂_i are compact, weakly null convex and acausal in M.*

Then $J^+(\partial_1) \cap J^-(\partial_2) = \emptyset$.

Theorem. *Let M be a spacetime with timelike boundary with $\mathcal{I} = \partial M$ connected and assume $\mathcal{D} := I^{+}(\mathcal{I}) \cap I^{-}(\mathcal{I})$ is globally hyperbolic. Further assume the Cauchy surfaces of $\mathcal{I}$ are compact, weakly null convex and acausal in $\mathcal{D}$. Then the PTC holds on $\mathcal{D}$.*

Proof: