Completeness of non-reversible Finsler metrics and geodesics in standard stationary spacetimes

Erasmo Caponio
Polytechnic of Bari, Italy

a joint work with
Miguel Angel Javaloyes (University of São Paolo, Brazil)
Antonio Masiello (Polytechnic of Bari)

Overview

- Quick overview of Finsler metrics

Overview

- Quick overview of Finsler metrics
- The Fermat metric associated to a standard stationary space-time (which is a Finsler metric of Randers type)

Overview

- Quick overview of Finsler metrics
- The Fermat metric associated to a standard stationary space-time (which is a Finsler metric of Randers type)
- Relations between causal properties of standard stationary spacetimes and the Fermat metric

Overview

- Quick overview of Finsler metrics
- The Fermat metric associated to a standard stationary spacetime (which is a Finsler metric of Randers type)
- Relations between causal properties of standard stationary spacetimes and the Fermat metric
- Existence and multiplicities results for geodesics on standard stationary spacetimes

Finsler metrics

Definition

Let M be a manifold. A Finsler metric on M is a function $F\colon TM \to \mathbb{R}$ such that:

- F is C^0 on TM and C^k, $k \geq 2$, on $TM \setminus 0$
- $F(x, y) \geq 0$, and F vanishes only on the zero section
- $F(x, \lambda y) = \lambda F(x, y)$, for any $\lambda \geq 0$.
- with fiberwise strictly convex square, i. e. the tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y)$$

is positively defined for any $(x, y) \in TM \setminus 0$

Definition

Let M be a manifold. A Finsler metric on M is a function $F\colon TM \to \mathbb{R}$ such that:

- F is C^0 on TM and C^k, $k \geq 2$, on $TM \setminus 0$
- $F(x, y) \geq 0$, and F vanishes only on the zero section
- $F(x, \lambda y) = \lambda F(x, y)$, for any $\lambda \geq 0$.
- with fiberwise strictly convex square, i. e. the tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y)$$

is positively defined for any $(x, y) \in TM \setminus 0$

Definition

Let M be a manifold. A Finsler metric on M is a function $F\colon TM \to \mathbb{R}$ such that:

- F is C^0 on TM and C^k, $k \geq 2$, on $TM \setminus 0$
- $F(x, y) \geq 0$, and F vanishes only on the zero section
- $F(x, \lambda y) = \lambda F(x, y)$, for any $\lambda \geq 0$.
- with fiberwise strictly convex square, i. e. the tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y)$$

is positively defined for any $(x, y) \in TM \setminus 0$

Definition

Let M be a manifold. A Finsler metric on M is a function $F\colon TM \to \mathbb{R}$ such that:

- F is C^0 on TM and C^k, $k \geq 2$, on $TM \setminus 0$
- $F(x, y) \geq 0$, and F vanishes only on the zero section
- $F(x, \lambda y) = \lambda F(x, y)$, for any $\lambda \geq 0$.
- with fiberwise strictly convex square, i. e. the tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y)$$

is positively defined for any $(x, y) \in TM \setminus 0$

Definition

Let M be a manifold. A Finsler metric on M is a function $F\colon TM \to \mathbb{R}$ such that:

- F is C^0 on TM and C^k, $k \geq 2$, on $TM \setminus 0$
- $F(x, y) \geq 0$, and F vanishes only on the zero section
- $F(x, \lambda y) = \lambda F(x, y)$, for any $\lambda \geq 0$.
- with fiberwise strictly convex square, i. e. the tensor

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2(F^2)}{\partial y^i \partial y^j}(x, y)$$

is positively defined for any $(x, y) \in TM \setminus 0$

By homogeneity, F^2 is C^1 on TM and it reduces to the square of the norm of a Riemannian metric if and only if its second order fibrative derivative is continuous up to the zero section, Warner 1965.

By homogeneity, F^2 is C^1 on TM and it reduces to the square of the norm of a Riemannian metric if and only if its second order fibrative derivative is continuous up to the zero section, Warner 1965.

Since F is only positive homogeneous of degree 1, we have that, in general, $F(x,v) \neq F(x,-v)$. If for all $(x,v) \in TM$ $F(x,v) = F(x,-v)$, the Finsler metric F is said *reversible* otherwise it will be called *non-reversible*.

The length of a piecewise smooth curve $\gamma\colon [a,b] \subset \mathbb{R} \to M$ with respect to the Finsler structure F is defined by

$$L(\gamma) = \int_a^b F(\gamma(s), \dot{\gamma}(s)) \mathrm{d}s.$$

The length of a piecewise smooth curve $\gamma\colon [a,b] \subset \mathbb{R} \to M$ with respect to the Finsler structure F is defined by

$$L(\gamma) = \int_a^b F(\gamma(s), \dot{\gamma}(s))\mathrm{d}s.$$

The distance between two arbitrary points $p,\ q \in M$ is given by

$$\mathrm{dist}(p,q) = \inf_{\gamma \in C(p,q)} L(\gamma),$$

where $C(p,q)$ is the set of all piecewise smooth curves $\gamma\colon [a,b] \to \mathbb{R}$ with $\gamma(a) = p$ and $\gamma(b) = q$.

Finsler geometry as Riemannian geometry

Finsler geometry as Riemannian geometry

exponential map

exponential map $\surd$

exponential map $\surd$
normal convex neighborhoods

Finsler geometry as Riemannian geometry

exponential map $\surd$

normal convex neighborhoods $\surd$

Finsler geometry as Riemannian geometry

exponential map	$\surd$
normal convex neighborhoods	$\surd$
Hopf-Rinow theorem	

Finsler geometry as Riemannian geometry

exponential map	$\surd$
normal convex neighborhoods	$\surd$
Hopf-Rinow theorem	$\surd$

exponential map $\surd$

normal convex neighborhoods $\surd$

Hopf-Rinow theorem $\surd$

Variational properties of geodesics:

Finsler geometry as Riemannian geometry

exponential map	$\surd$
normal convex neighborhoods	$\surd$
Hopf-Rinow theorem	$\surd$

Variational properties of geodesics:

- they are the critical points of the energy functional

$$J(x) = \frac{1}{2}\int F^2(x,\dot{x})\mathrm{d}s$$

Finsler geometry as Riemannian geometry

exponential map	$\surd$
normal convex neighborhoods	$\surd$
Hopf-Rinow theorem	$\surd$

Variational properties of geodesics:

- they are the critical points of the energy functional

$$J(x) = \frac{1}{2}\int F^2(x, \dot{x})\mathrm{d}s$$

- $J(x)$ satisfies the Palais-Smale condition, under analogous assumptions to those guaranteeing that the Riemannian energy functional does:

Finsler geometry as Riemannian geometry

exponential map	$\surd$
normal convex neighborhoods	$\surd$
Hopf-Rinow theorem	$\surd$

Variational properties of geodesics:

- they are the critical points of the energy functional

$$J(x) = \frac{1}{2}\int F^2(x,\dot{x})\mathrm{d}s$$

- $J(x)$ satisfies the Palais-Smale condition, under analogous assumptions to those guaranteeing that the Riemannian energy functional does: some technical difficulties related to the lack of regularity of F^2 on the zero section, but everything works fine

Finsler geometry as Riemannian geometry

exponential map	$\surd$
normal convex neighborhoods	$\surd$
Hopf-Rinow theorem	$\surd$

Variational properties of geodesics:

- they are the critical points of the energy functional

$$J(x) = \frac{1}{2}\int F^2(x, \dot{x})\mathrm{d}s$$

- $J(x)$ satisfies the Palais-Smale condition, under analogous assumptions to those guaranteeing that the Riemannian energy functional does: some technical difficulties related to the lack of regularity of F^2 on the zero section, but everything works fine $\surd$

Finsler geometry as Riemannian geometry

"Finsler geometry is just Riemannian geometry without the quadratic restriction $F^2(x,v) = g(x)[v,v]$" , Chern 1996

Finsler geometry as Riemannian geometry

> *"Finsler geometry is just Riemannian geometry without the quadratic restriction $F^2(x,v) = g(x)[v,v]$" , Chern 1996*

That is fully true whenever F is reversible

Finsler geometry as Riemannian geometry

"Finsler geometry is just Riemannian geometry without the quadratic restriction $F^2(x,v) = g(x)[v,v]$" , Chern 1996

That is fully true whenever F is reversible

On the other hand, if F is non-reversible ...

Finsler geometry vs Riemannian geometry

the distance is not symmetric

Finsler geometry vs Riemannian geometry

- there are Finsler metrics on S^2 which have only two prime distinct closed geodesics (actually they are two closed geodesics with different Finslerian length, the same images but different orientations, Katok 1973)

the distance is not symmetric

Finsler geometry vs Riemannian geometry

- there are Finsler metrics on S^2 which have only two prime distinct closed geodesics (actually they are two closed geodesics with different Finslerian length, the same images but different orientations, Katok 1973)

- every Riemannian metric on S^2 has infinitely many prime distinct closed geodesics, Bangert 1993 and Franks 1992

$\boxed{\text{the distance is not symmetric}}$

Finsler geometry vs Riemannian geometry

- there are Finsler metrics on S^2 which have only two prime distinct closed geodesics (actually they are two closed geodesics with different Finslerian length, the same images but different orientations, Katok 1973)
- every Riemannian metric on S^2 has infinitely many prime distinct closed geodesics, Bangert 1993 and Franks 1992

For non-reversible Finsler manifold one has to distinguish between two notions of completeness since

$$\boxed{\text{the distance is not symmetric}}$$

forward

backward

forward

$$B^+(x_0, r) = \{x \in M \mid \mathrm{dist}(x_0, x) < r\}$$

backward

$$B^-(x_0, r) = \{x \in M \mid \mathrm{dist}(x, x_0) < r\}$$

Finsler geometry vs Riemannian geometry

forward

$$B^+(x_0, r) = \{x \in M \mid \mathrm{dist}(x_0, x) < r\}$$

backward

$$B^-(x_0, r) = \{x \in M \mid \mathrm{dist}(x, x_0) < r\}$$

The topologies induced by the forward and the backward balls coincide with the underlaying manifold topology

Finsler geometry vs Riemannian geometry

forward

$\{x_n\}$ is a forward Cauchy sequence iff for every $\varepsilon > 0$ there exist $i \in \mathbb{N}$ such that for any $i \le m \le n$: $\operatorname{dist}(x_m, x_n) < \varepsilon$

backward

$\{x_n\}$ is a backward Cauchy sequence iff for every $\varepsilon > 0$ there exist $i \in \mathbb{N}$ such that for any $i \le m \le n$: $\operatorname{dist}(x_n, x_m) < \varepsilon$

Finsler geometry vs Riemannian geometry

forward

A geodesic $\gamma\colon [a,b) \to M$ is forward complete if it can be extended, as a geodesic, to the interval $[a,+\infty)$

backward

A geodesic $\gamma\colon (b,a] \to M$ is backward complete if it can be extended, as a geodesic, to the interval $(-\infty,a]$

Finsler geometry vs Riemannian geometry

The two notions of completeness are not equivalent

Finsler geometry vs Riemannian geometry

The two notions of completeness are not equivalent

An example is provided by a Randers metric.

Randers metric

A Randers metric is a Finsler metric of the type

$$F(x,v) = \sqrt{h(x)[v,v]} + \omega(x)[v]$$

where h is a Riemannian metric on M and ω is a 1-form on M such that

$$\|\omega\|_x = \sup_{v \in T_xM \setminus 0} \frac{|\omega(x)[v]|}{\sqrt{h(x)[v,v]}} < 1.$$

The Fermat metric

Standard stationary Lorentzian manifold

- $L = M \times \mathbb{R}$, M is endowed with a Riemannian metric g_0
- δ is a vector field on M
- β is a positive function on M
- the Lorentzian metric l on L is given by

$$l(x,t)[(y,\tau),(y,\tau)] = g_0(x)[y,y] + 2g_0(x)[\delta(x),y]\tau - \beta(x)\tau^2,$$

  for any $(x,t) \in M \times \mathbb{R}$ and $(y,\tau) \in T_xM \times \mathbb{R}$
- it is an oriented space-time with the timelike Killing vector field ∂_t

Standard stationary Lorentzian manifold

- $L = M \times \mathbb{R}$, M is endowed with a Riemannian metric g_0
- δ is a vector field on M
- β is a positive function on M
- the Lorentzian metric l on L is given by

$$l(x,t)[(y,\tau),(y,\tau)] = g_0(x)[y,y] + 2g_0(x)[\delta(x),y]\tau - \beta(x)\tau^2,$$

  for any $(x,t) \in M \times \mathbb{R}$ and $(y,\tau) \in T_xM \times \mathbb{R}$
- it is an oriented space-time with the timelike Killing vector field ∂_t

Standard stationary Lorentzian manifold

- $L = M \times \mathbb{R}$, M is endowed with a Riemannian metric g_0
- δ is a vector field on M
- β is a positive function on M
- the Lorentzian metric l on L is given by

$$l(x,t)[(y,\tau),(y,\tau)] = g_0(x)[y,y] + 2g_0(x)[\delta(x),y]\tau - \beta(x)\tau^2,$$

for any $(x,t) \in M \times \mathbb{R}$ and $(y,\tau) \in T_xM \times \mathbb{R}$

- it is an oriented space-time with the timelike Killing vector field ∂_t

Standard stationary Lorentzian manifold

- $L = M \times \mathbb{R}$, M is endowed with a Riemannian metric g_0
- δ is a vector field on M
- β is a positive function on M
- the Lorentzian metric l on L is given by

$$l(x,t)[(y,\tau),(y,\tau)] = g_0(x)[y,y] + 2g_0(x)[\delta(x),y]\tau - \beta(x)\tau^2,$$

  for any $(x,t) \in M \times \mathbb{R}$ and $(y,\tau) \in T_xM \times \mathbb{R}$
- it is an oriented space-time with the timelike Killing vector field ∂_t

Standard stationary Lorentzian manifold

- $L = M \times \mathbb{R}$, M is endowed with a Riemannian metric g_0
- δ is a vector field on M
- β is a positive function on M
- the Lorentzian metric l on L is given by

  $$l(x,t)[(y,\tau),(y,\tau)] = g_0(x)[y,y] + 2g_0(x)[\delta(x),y]\tau - \beta(x)\tau^2,$$

  for any $(x,t) \in M \times \mathbb{R}$ and $(y,\tau) \in T_xM \times \mathbb{R}$
- it is an oriented space-time with the timelike Killing vector field ∂_t

Standard stationary Lorentzian manifold

- $L = M \times \mathbb{R}$, M is endowed with a Riemannian metric g_0
- δ is a vector field on M
- β is a positive function on M
- the Lorentzian metric l on L is given by

  $$l(x,t)[(y,\tau),(y,\tau)] = g_0(x)[y,y] + 2g_0(x)[\delta(x),y]\tau - \beta(x)\tau^2,$$

  for any $(x,t) \in M \times \mathbb{R}$ and $(y,\tau) \in T_xM \times \mathbb{R}$
- it is an oriented space-time with the timelike Killing vector field ∂_t

The Fermat principle

Among all lightlike curves connecting some event p with some timelike curve γ, lightlike geodesics are, up to reparameterizations, critical points of the arrival time, that is, the parameter of the timelike curve in the point where the lightlike curve meets it, Kovner 1990

The Fermat principle

> Among all lightlike curves connecting some event p with some timelike curve γ, lightlike geodesics are, up to reparameterizations, critical points of the arrival time, that is, the parameter of the timelike curve in the point where the lightlike curve meets it, Kovner 1990

For a standard stationary lorentzian manifold, Perlick 1990:

The Fermat principle

Among all lightlike curves connecting some event p with some timelike curve γ, lightlike geodesics are, up to reparameterizations, critical points of the arrival time, that is, the parameter of the timelike curve in the point where the lightlike curve meets it, Kovner 1990

For a standard stationary lorentzian manifold, Perlick 1990: lightlike curves $(x(s), t(s))$ has to satisfy:

$$g_0(x)[\dot{x}, \dot{x}] + 2g_0(x)[\delta(x), \dot{x}]\dot{t} - \beta(x)\dot{t}^2 = 0,$$

The Fermat principle

Among all lightlike curves connecting some event p with some timelike curve γ, lightlike geodesics are, up to reparameterizations, critical points of the arrival time, that is, the parameter of the timelike curve in the point where the lightlike curve meets it, Kovner 1990

For a standard stationary lorentzian manifold, Perlick 1990: lightlike curves $(x(s), t(s))$ has to satisfy:

$$g_0(x)[\dot{x}, \dot{x}] + 2g_0(x)[\delta(x), \dot{x}]\dot{t} - \beta(x)\dot{t}^2 = 0,$$

solving with respect to $\dot{t}$ and integrating, we get:

$$t(s) = \varrho_0 + \int_0^s \left(\frac{1}{\beta(x)} g_0(x)[\delta(x), \dot{x}] + \frac{1}{\beta(x)} \sqrt{g_0(x)[\delta(x), \dot{x}]^2 + \beta(x) g_0(x)[\dot{x}, \dot{x}]} \right) \mathrm{d}v \tag{1}$$

The Fermat principle

Thus lightlike geodesics $(\tilde{x}(s), \tilde{t}(s))$ connecting the event $(x_0, \varrho_0) \in L$ with the timelike curve $\varrho \in \mathbb{R} \mapsto (x_1, \varrho) \in L$ are reparameterizations of the curves $(x(s), t(s))$ such that $x(s)$ is a critical point of the functional

$$I(x) = \varrho_0 + L(x), \tag{2}$$

where

The Fermat principle

Thus lightlike geodesics $(\tilde{x}(s), \tilde{t}(s))$ connecting the event $(x_0, \varrho_0) \in L$ with the timelike curve $\varrho \in \mathbb{R} \mapsto (x_1, \varrho) \in L$ are reparameterizations of the curves $(x(s), t(s))$ such that $x(s)$ is a critical point of the functional

$$I(x) = \varrho_0 + L(x), \tag{2}$$

where

$$L(x) = \int_0^1 \left(\frac{1}{\beta(x)} g_0(x)[\delta(x), \dot{x}] + \frac{1}{\beta(x)} \sqrt{g_0(x)[\delta(x), \dot{x}]^2 + \beta(x) g_0(x)[\dot{x}, \dot{x}]} \right) \mathrm{d}v,$$

and $t(s)$ is given by (1).

Global hyperbolicity and completeness of the Fermat metric

A characterization of global hyperbolicity

- Let $p_0 = (x_0, t_0) \in L$

A characterization of global hyperbolicity

- Let $p_0 = (x_0, t_0) \in L$
- $\mathrm{C}^+(p_0, \mu) = \bigcup_{s \in [0,\mu)} \bar{B}_s^+(x_0) \times \{t_0 + s\}$

A characterization of global hyperbolicity

- Let $p_0 = (x_0, t_0) \in L$
- $\mathrm{C}^+(p_0, \mu) = \bigcup_{s\in[0,\mu)} \bar{B}^+_s(x_0) \times \{t_0 + s\}$
- $\mathrm{C}^-(p_0, \mu) = \bigcup_{s\in[0,\mu)} \bar{B}^-_s(x_0) \times \{t_0 - s\}$,

- Let $p_0 = (x_0, t_0) \in L$
- $\mathrm{C}^+(p_0, \mu) = \bigcup_{s \in [0,\mu)} \bar{B}^+_s(x_0) \times \{t_0 + s\}$
- $\mathrm{C}^-(p_0, \mu) = \bigcup_{s \in [0,\mu)} \bar{B}^-_s(x_0) \times \{t_0 - s\}$,

Theorem 1 *Let (L, l) be a standard stationary Lorentzian manifold and let $\bar{t} \in \mathbb{R}$. Then the following properties are equivalent:*

A characterization of global hyperbolicity

- Let $p_0 = (x_0, t_0) \in L$
- $\mathrm{C}^+(p_0, \mu) = \bigcup_{s\in[0,\mu)} \bar{B}^+_s(x_0) \times \{t_0 + s\}$
- $\mathrm{C}^-(p_0, \mu) = \bigcup_{s\in[0,\mu)} \bar{B}^-_s(x_0) \times \{t_0 - s\}$,

Theorem 1 *Let (L, l) be a standard stationary Lorentzian manifold and let $\bar{t} \in \mathbb{R}$. Then the following properties are equivalent:*

(a) (L, l) *is globally hyperbolic with Cauchy surface* $S = M \times \{\bar{t}\}$,

A characterization of global hyperbolicity

- Let $p_0 = (x_0, t_0) \in L$
- $\mathrm{C}^+(p_0, \mu) = \bigcup_{s \in [0,\mu)} \bar{B}^+_s(x_0) \times \{t_0 + s\}$
- $\mathrm{C}^-(p_0, \mu) = \bigcup_{s \in [0,\mu)} \bar{B}^-_s(x_0) \times \{t_0 - s\}$,

Theorem 1 *Let (L, l) be a standard stationary Lorentzian manifold and let $\bar{t} \in \mathbb{R}$. Then the following properties are equivalent:*

(a) *(L, l) is globally hyperbolic with Cauchy surface $S = M \times \{\bar{t}\}$,*

(b) *the Fermat metric on M is forward complete,*

A characterization of global hyperbolicity

- Let $p_0 = (x_0, t_0) \in L$
- $\mathrm{C}^+(p_0, \mu) = \bigcup_{s \in [0,\mu)} \bar{B}^+_s(x_0) \times \{t_0 + s\}$
- $\mathrm{C}^-(p_0, \mu) = \bigcup_{s \in [0,\mu)} \bar{B}^-_s(x_0) \times \{t_0 - s\}$,

Theorem 1 *Let (L, l) be a standard stationary Lorentzian manifold and let $\bar{t} \in \mathbb{R}$. Then the following properties are equivalent:*

(a) (L, l) *is globally hyperbolic with Cauchy surface* $S = M \times \{\bar{t}\}$,

(b) the Fermat metric on M is forward complete,

(c) $J^+(p_0) = \mathrm{C}^+(p_0, +\infty)$ *and* $J^-(p_0) = \mathrm{C}^-(p_0, +\infty)$ *for every* $p_0 = (x_0, t_0) \in L$, *and the balls* $\bar{B}^+_s(x_0)$ *are compact.*

A characterization of global hyperbolicity

In the statement of Theorem 1 forward can be replaced by backward and the compactness of the forward balls by that of the backward ones.

A characterization of global hyperbolicity

In the statement of Theorem 1 forward can be replaced by backward and the compactness of the forward balls by that of the backward ones.

Therefore for the Fermat metric any of the two completeness conditions implies the other.

A characterization of global hyperbolicity

We observe that any Randers metric is forward and backward if the Riemannian metric (M, h) is complete and

$$\|\omega\| := \sup_{x \in M} \|\omega\|_x < 1,$$

A characterization of global hyperbolicity

We observe that any Randers metric is forward and backward if the Riemannian metric (M, h) is complete and

$$\|\omega\| := \sup_{x \in M} \|\omega\|_x < 1,$$

For the Fermat metric, using the Cauchy-Schwarz inequality, $g_0(y, y) \geq g_0(\delta, y)^2 / |\delta|_0^2$, we obtain that $\|\omega\| < 1$ if

$$\sup_{x \in M} \frac{|\delta(x)|_0}{\sqrt{|\delta(x)|_0^2 + \beta(x)}} < 1.$$

We observe that any Randers metric is forward and backward if the Riemannian metric (M, h) is complete and

$$\|\omega\| \colon = \sup_{x \in M} \|\omega\|_x < 1,$$

For the Fermat metric, using the Cauchy-Schwarz inequality, $g_0(y, y) \geq g_0(\delta, y)^2 / |\delta|_0^2$, we obtain that $\|\omega\| < 1$ if

$$\sup_{x \in M} \frac{|\delta(x)|_0}{\sqrt{|\delta(x)|_0^2 + \beta(x)}} < 1.$$

Therefore the Fermat metric is complete and by Theorem 1 the spacetime is globally hyperbolic if

$\beta(x)^{-1} g_0$ is complete and $\displaystyle\inf_{x \in M} \frac{\beta(x)}{|\delta(x)|_0^2} > 0$

Completeness of the Fermat metric and Lorentzian geodesics

Geodesics connectedness

A result by Candela, Flores, and Sánchez 2006 can be restated as follows:

Geodesics connectedness

A result by Candela, Flores, and Sánchez 2006 can be restated as follows:

a standard stationary Lorentzian manifold, such that the Riemannian metric (M, g_0) is complete and the Fermat metric (M, F) is forward or backward complete, is geodesically connected.

Geodesics connectedness

A result by Candela, Flores, and Sánchez 2006 can be restated as follows:

a standard stationary Lorentzian manifold, such that the Riemannian metric (M, g_0) is complete and the Fermat metric (M, F) is forward or backward complete, is geodesically connected.

In a recent paper Bartolo, Candela, and Flores 2006 show that the assumptions

$$|\delta(x)|_0^2 \leq c_1 \mathrm{dist}_0^2(x, x_0) + c_2 \qquad\qquad \beta(x) \leq c_3 \mathrm{dist}_0^2(x, x_0) + c_4$$

are optimal for the applications of variational methods on the problem of geodesic connectedness of a standard stationary Lorentzian manifold. They provide a fine counterexample where $|\delta|^2$ has superquadratic growth. The Fermat metric in their example is not forward complete.

Lightlike geodesics

Lightlike geodesics

In a conformal standard stationary Lorentzian manifold, such that (M, F) is forward or backward complete, and M is non-contractible, there exist infinitely many lightlike geodesics $\gamma_n = (x_n, t_n)$ joining the point $(\bar{x}, \varrho_0)$ with the curve $v(\varrho) = (\tilde{x}, \varrho)$ and having arrival time (see (2)) $I(x_n) \to +\infty$, as $n \to \infty$.

Lightlike geodesics

In a conformal standard stationary Lorentzian manifold, such that (M, F) is forward or backward complete, and M is non-contractible, there exist infinitely many lightlike geodesics $\gamma_n = (x_n, t_n)$ joining the point $(\bar{x}, \varrho_0)$ with the curve $v(\varrho) = (\tilde{x}, \varrho)$ and having arrival time (see (2)) $I(x_n) \to +\infty$, as $n \to \infty$.

Piccione 1997 introduced the following notion of compactness on the set of lightlike curves between a point and an integral line of the field ∂_t:

Lightlike geodesics

In a conformal standard stationary Lorentzian manifold, such that (M, F) is forward or backward complete, and M is non-contractible, there exist infinitely many lightlike geodesics $\gamma_n = (x_n, t_n)$ joining the point $(\bar{x}, \varrho_0)$ with the curve $v(\varrho) = (\tilde{x}, \varrho)$ and having arrival time (see (2)) $I(x_n) \to +\infty$, as $n \to \infty$.

Piccione 1997 introduced the following notion of compactness on the set of lightlike curves between a point and an integral line of the field ∂_t:

let $C \in \mathbb{R}$ be a positive constant and let $\mathcal{L}_{p,v} \subset H^1([0,1], M) \times H^1([0,1], \mathbb{R})$ be the manifold of curves such that $l[\dot{z}, \dot{z}] = 0$ a. e., z is future pointing a. e. on $[0,1]$, $z(0) = p$ and $z(1) \in v(\mathbb{R})$. $\mathcal{L}_{p,v}$ is said *C-precompact* if every sequence $\{z_k = (x_k, t_k)\}_{k \in \mathbb{N}} \subset \mathcal{L}_{p,v}$ such that $I(x_k) \leq C$ admits a subsequence converging uniformly, up to reparameterization, in L.

Lightlike geodesic

Assuming C-precompactness, for every C, of the space of lightlike curves connecting an event p to the world-line v and assuming that M is non-contractible, Piccione proved existence of infinitely many lightlike geodesics connecting p to v.

Lightlike geodesic

Assuming C-precompactness, for every C, of the space of lightlike curves connecting an event p to the world-line v and assuming that M is non-contractible, Piccione proved existence of infinitely many lightlike geodesics connecting p to v.

Theorem 2 *Let (L, l) be a standard stationary Lorentzian manifold, $p \in L$ and $v = v(\varrho)$ an integral line of the vector field ∂_t. The condition*

- *$\mathcal{L}_{p,v}$ is C-precompact for every $C > 0$ and for every p and v in M*

is equivalent to forward or backward completeness of the Fermat metric.

Timelike geodesic

The Fermat metric on a one dimensional higher Riemannian manifold can be used to prove existence, multiplicity and finitness results for timelike geodesics with fixed energy in a Lorentzian standard stationary manifold.

Timelike geodesic

The Fermat metric on a one dimensional higher Riemannian manifold can be used to prove existence, multiplicity and finitness results for timelike geodesics with fixed energy in a Lorentzian standard stationary manifold.

We seek for timelike geodesics γ parameterized on a given interval $[a, b]$, connecting a point $(x_0, \varrho_0) \in L$ with a timelike curve $\varrho \in \mathbb{R} \mapsto (x_1, \varrho) \subset L$ and having a priori fixed energy $l(\gamma(s))[\dot\gamma(s), \dot\gamma(s)] = -E < 0$, for all $s \in [a, b]$.

Timelike geodesic

The Fermat metric on a one dimensional higher Riemannian manifold can be used to prove existence, multiplicity and finitness results for timelike geodesics with fixed energy in a Lorentzian standard stationary manifold.

We seek for timelike geodesics γ parameterized on a given interval $[a, b]$, connecting a point $(x_0, \varrho_0) \in L$ with a timelike curve $\varrho \in \mathbb{R} \mapsto (x_1, \varrho) \subset L$ and having a priori fixed energy $l(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] = -E < 0$, for all $s \in [a, b]$.

We extend the Riemannian manifold M to the manifold $N = M \times \mathbb{R}$ endowed with the metric $n = g_0 + \mathrm{d}u^2$ where u is the natural coordinate on $\mathbb{R}$.

Timelike geodesic

The Fermat metric on a one dimensional higher Riemannian manifold can be used to prove existence, multiplicity and finitness results for timelike geodesics with fixed energy in a Lorentzian standard stationary manifold.

We seek for timelike geodesics γ parameterized on a given interval $[a, b]$, connecting a point $(x_0, \varrho_0) \in L$ with a timelike curve $\varrho \in \mathbb{R} \mapsto (x_1, \varrho) \subset L$ and having a priori fixed energy $l(\gamma(s))[\dot\gamma(s), \dot\gamma(s)] = -E < 0$, for all $s \in [a, b]$.

We extend the Riemannian manifold M to the manifold $N = M \times \mathbb{R}$ endowed with the metric $n = g_0 + \mathrm{d}u^2$ where u is the natural coordinate on $\mathbb{R}$.

We associate to the manifold N a one-dimensional higher Lorentzian manifold $(\bar N, \bar n)$, with the metric $\bar n$ defined as

$$\bar n(x, u, t)[(y, v, \tau), (y, v, \tau)] = g_0(x)[y, y] + v^2 + 2g_0(x)[\delta(x), y]\tau - \beta(x)\tau^2.$$

Timelike geodesic

Lightlike geodesics for the metric $\bar{n}$ satisfy the following equation

$$g_0[\dot{x}, \dot{x}] + 2g_0[\delta, \dot{x}]\dot{t} - \beta\dot{t}^2 = -\dot{u}^2 = \text{const.}$$

Timelike geodesic

Lightlike geodesics for the metric $\bar{n}$ satisfy the following equation

$$g_0[\dot{x},\dot{x}] + 2g_0[\delta,\dot{x}]\dot{t} - \beta\dot{t}^2 = -\dot{u}^2 = \text{const.}$$

Thus in order to find timelike geodesics $\gamma = (x,t)$ in (L,l) with fixed energy $-E < 0$ it is enough to find lightlike geodesics in $(\bar{N},\bar{n})$ whose u component has derivative equal to $\sqrt{E}$.

Lightlike geodesics for the metric $\bar{n}$ satisfy the following equation

$$g_0[\dot{x},\dot{x}] + 2g_0[\delta,\dot{x}]\dot{t} - \beta\dot{t}^2 = -\dot{u}^2 = \text{const.}$$

Thus in order to find timelike geodesics $\gamma = (x,t)$ in (L,l) with fixed energy $-E < 0$ it is enough to find lightlike geodesics in $(\bar{N},\bar{n})$ whose u component has derivative equal to $\sqrt{E}$.

The Fermat metric associated to the manifold $(\bar{N},\bar{n})$ is given by

$$F((x,u),(y,v)) = \sqrt{\frac{1}{\beta(x)}(g_0[y,y]+v^2) + \frac{1}{\beta(x)^2}g_0[\delta(x),y]^2} + \frac{1}{\beta(x)}g_0[\delta(x),y],$$

for all $((x,u),(y,v)) \in TN$.

Timelike geodesic

The following improves some results of Bartolo, Germinario, and Sánchez 2002 and Germinario 2006

Theorem 3 *Let (L, l) be a standard stationary Lorentzian manifold, such that (M, F) is forward or backward complete, and M is non-contractible, then there exist infinitely many timelike geodesics $\gamma_n = (x_n, t_n)$ connecting the point $(x_0, \varrho_0) \in L$ with the timelike curve $\upsilon(\varrho) = (x_1, \varrho)$, parameterized on the interval $[a, b]$, having fixed energy $-E$ and diverging arrival time.*

References

References

F. W. Warner. The conjugate locus of a Riemannian manifold. *Amer. J. Math.*, 87:575–604, 1965

A. B. Katok. Ergodic perturbations of degenerate integrable Hamiltonian systems. *Izs. Akad. Nauk SSSR Ser. MAT.*, **37**, *539–576. Engl. trans. Math. USSR Izv.*, 7:535–572, 1973

V. Bangert. On the existence of closed geodesics on two-spheres. *Int. J. Math.*, 4(1):1–10, 1993

J. Franks. Geodesics on S^2 and periodic points of annulus homeomorphisms. *Invent. Math.*, 108(2):403–418, 1992

G. Randers. On an asymmetrical metric in the fourspace of General Relativity. *Phys. Rev. (2)*, 59:195–199, 1941

References

G. S. Asanov. *Finsler Geometry, Relativity and Gauge Theories*. Fundamental Theories of Physics. D. Reidel Publishing Co., Dordrecht, 1985

I. Kovner. Fermat principles for arbitrary space-times. *Astrophysical Journal*, 351:114–120, 1990

V. Perlick. On Fermat's principle in General Relativity. I. The general case. *Classical Quantum Gravity*, 7(8):1319–1331, 1990

A. M. Candela, J. L. Flores, and M. Sánchez. Global hyperbolicity and Palais-Smale condition for action functionals in stationary spacetimes, 2006. URL `http://arxiv.org/math.DG/0610175` ki

References

R. Bartolo, A.M. Candela, and J.L. Flores. Geodesic connectedness of stationary spacetimes with optimal growth. *J. Geom. Phys.*, 56 (10):2025–2038, 2006

P. Piccione. On the existence of light-like geodesics on conformally stationary Lorentzian manifolds. *Nonlinear Anal.*, 28(4):611–623, 1997

R. Bartolo, A. Germinario, and M. Sánchez. A note on the boundary of a static Lorentzian manifold. *Differ. Geom. Appl.*, 16(2): 121–131, 2002

A. Germinario. Geodesics in stationary spacetime and classical Lagrangian systems, *J. Diff. Eq.* to apperar, 2006