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On the geometry of three-dimensional homogeneous Lorentz manifolds

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HOMOGENEOUS SPACES

A pseudo-Riemannian (p.R.) manifold (M, g) is *homogeneous* if for any points $p, q \in M$ there is an isometry ϕ such that $\phi(p) = q$. Pseudo-Riemannian homogeneous spaces are known as one of the most interesting research fields in p.R. geometry.

Recently, many authors investigated the problem of extending several results concerning homogeneous Riemannian manifolds, to p.R. geometry (in particular, to Lorentzian geometry).

HOMOGENEOUS STRUCTURES

Gadea and Oubiña introduced the notion of *homogeneous pseudo-Riemannian structure*, in order to obtain a characterization of reductive homogeneous p.R. manifolds, similar to the one given in the Riemannian case by Ambrose and Singer.

Definition: *A homogeneous p.R. structure on (M, g) is a tensor field T of type $(1, 2)$ on M, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies*

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} T = 0.$$

Theorem [Gadea-Oubiña] *Let (M, g) be a connected, simply connected and complete p.R. manifold. (Mg) admits a p.R. structure if and only if it is a reductive homogeneous space.*

Proof. Let $(M = G/H, g)$ be a homogeneous reductive p.R. manifold, G and H being a group of isometries acting on (M, g) and the isotropy group at an arbitrary point $p \in M$, respectively.
Let α belong to the Lie algebra $\mathfrak{g}$ of G and α^* be the vector field on M generated by the one-parameter group of isometries $\{\exp(t\alpha) : t \in \mathbb{R}\}$.
The Lie algebra of H is $\mathfrak{h} = \{\alpha \in \mathfrak{g} : \alpha^*_p = 0\}$.

The *canonical connection* $\tilde{\nabla}$ associated to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, is determined by

$$(\tilde{\nabla}_{\alpha^*}\beta^*)_p = [\alpha^*, \beta^*]_p = -[\alpha, \beta]^*_p \quad \forall\, \alpha, \beta \in \mathfrak{g}.$$

Then, $T = \nabla - \tilde{\nabla}$ is a homogeneous p.R. structure.

Conversely: $\exists$ a p.R. homogeneous structure T on (M, g)
$\Rightarrow \exists$ a connection $\tilde{\nabla} = \nabla - T$ on M, which is complete and ensures the existence, given two points $p, q \in M$, of a global isometry mapping p to q.
Then, a group G of isometries acts transitively on M, $M = G/H$ is reductive, and $\tilde{\nabla}$ is the canonical connection associated to this reductive decomposition.

REMARKS: a) a Riemannian homogeneous space is necessarily reductive, a p.R. one needs not to be reductive.
b) two homogeneous structures T_1 and T_2 on a p.R. homogeneous manifold (M, g) can give rise either to the same Lie algebra $\mathfrak{g}$ with different decompositions, or to non-isomorphic Lie algebras.

Theorem [Sekigawa] *A connected, simply connected and complete homogeneous Riemannian 3-manifold is either symmetric or it is a Lie group equipped with a left-invariant Riemannian metric.*

KEY POINT: to show that (unless $M = G/H$ is symmetric), in all cases determined by the different possibilities for the Ricci eigenvalues, there exists a homogeneous structure T such that $T_{\alpha^*}\beta^* = \nabla_{\alpha^*}\beta^*$ for all $\alpha, \beta \in \mathfrak{g}$.
$\Rightarrow \mathfrak{h} = 0$
$\Rightarrow M = G$ is a Lie group.

Together with the classification of three-dimensional Riemannian Lie groups [Milnor], this result permits to determine all three-dimensional homogeneous Riemannian manifolds.

LORENTZIAN VERSION

Theorem [C] *A connected, simply connected, complete homogeneous Lorentzian 3-manifold (M, g) is either symmetric, or isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric.*

KEY POINT: to prove the existence (unless $M = G/H$ is symmetric) of a p.R. homogeneous structure T such that $T_{\alpha^*}\beta^* = \nabla_{\alpha^*}\beta^*$ for all $\alpha, \beta \in \mathfrak{g}$.
$\Rightarrow \mathfrak{h} = 0$
$\Rightarrow M = G$ is a Lie group.

ESSENTIAL DIFFERENCE:
(M, g) Riemannian $\Rightarrow$ the *Ricci operator* Q is diagonal.
(M, g) Lorentz $\Rightarrow$ Q can take four different standard forms, called *Segre types*.

$$\text{Segre type } \{11,1\} : Q = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$\text{Segre type } \{1z\bar{z}\} : Q = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}$$

$$\text{Segre type } \{21\} : Q = \begin{pmatrix} a & 0 & 0 \\ 0 & b & \varepsilon \\ 0 & -\varepsilon & b - 2\varepsilon \end{pmatrix}$$

$$\text{Segre type } \{3\} : Q = \begin{pmatrix} b & a & -a \\ a & b & 0 \\ a & 0 & b \end{pmatrix},$$

with respect to a suitable pseudo-orthonormal frame $\{e_1, e_2, e_3\}$, with e_3 timelike.

CLASSIFICATION RESULT:

Theorem [C] *If (M, g) is a nonsymmetric connected, simply connected, complete homogeneous Lorentzian 3-manifold, then $M = G$ is a Lie group and g is left-invariant. Precisely:*

a) If G is unimodular, then its Lie algebra is one of the following:

$$(\mathfrak{g}_1): \quad \begin{aligned} [e_1, e_2] &= \alpha e_1 - \beta e_3, \\ [e_1, e_3] &= -\alpha e_1 - \beta e_2, \\ [e_2, e_3] &= \beta e_1 + \alpha e_2 + \alpha e_3, \ \alpha \neq 0. \end{aligned}$$

$G = O(1,2)$ *or* $SL(2, \mathbb{R})$ *if* $\beta \neq 0$,
$G = E(1,1)$ *if* $\beta = 0$.

$$(\mathfrak{g}_2): \quad \begin{aligned} [e_1, e_2] &= \gamma e_2 - \beta e_3, \\ [e_1, e_3] &= -\beta e_2 + \gamma e_3, \ \gamma \neq 0, \\ [e_2, e_3] &= \alpha e_1. \end{aligned}$$

$G = O(1,2)$ *or* $SL(2, \mathbb{R})$ *if* $\alpha \neq 0$,
$G = E(1,1)$ *if* $\alpha = 0$.

$$(\mathfrak{g}_3): \quad \begin{aligned} [e_1, e_2] &= -\gamma e_3, \\ [e_1, e_3] &= -\beta e_2, \\ [e_2, e_3] &= \alpha e_1. \end{aligned}$$

Lie groups admitting a Lie algebra g_3*:*

G	α	β	γ
$O(1,2)$ or $SL(2,\mathbb{R})$	$+$	$+$	$+$
$O(1,2)$ or $SL(2,\mathbb{R})$	$+$	$-$	$-$
$SO(3)$ or $SU(2)$	$+$	$+$	$-$
$E(2)$	$+$	$+$	0
$E(2)$	$+$	0	$-$
$E(1,1)$	$+$	$-$	0
$E(1,1)$	$+$	0	$+$
H_3	$+$	0	0
H_3	0	0	$-$
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	0	0	0

$$(\mathfrak{g}_4): \quad \begin{aligned} [e_1, e_2] &= -e_2 + (2\varepsilon - \beta) e_3, \ \eta = \pm 1, \\ [e_1, e_3] &= -\beta e_2 + e_3, \\ [e_2, e_3] &= \alpha e_1. \end{aligned}$$

Lie groups admitting a Lie algebra g_4*:*

G	α	β
$O(1,2)$ or $SL(2,\mathbb{R})$	$\neq 0$	$\neq \eta$
$E(1,1)$	0	$\neq \eta$
$E(1,1)$	< 0	η
$E(2)$	> 0	η
H_3	0	η

b) If G is non-unimodular, then its Lie algebra is one of the following:

$$(\mathfrak{g}_5): \quad \begin{aligned} [e_1, e_2] &= 0, \\ [e_1, e_3] &= \alpha e_1 + \beta e_2, \\ [e_2, e_3] &= \gamma e_1 + \delta e_2, \end{aligned}$$

with $\alpha + \delta \neq 0, \;\; \alpha\gamma + \beta\delta = 0.$

$$(\mathfrak{g}_6): \quad \begin{aligned} [e_1, e_2] &= \alpha e_2 + \beta e_3, \\ [e_1, e_3] &= \gamma e_2 + \delta e_3, \\ [e_2, e_3] &= 0. \end{aligned}$$

with $\alpha + \delta \neq 0, \;\; \alpha\gamma - \beta\delta = 0.$

$$(\mathfrak{g}_7): \quad \begin{aligned} [e_1, e_2] &= -\alpha e_1 - \beta e_2 - \beta e_3, \\ [e_1, e_3] &= \alpha e_1 + \beta e_2 + \beta e_3, \\ [e_2, e_3] &= \gamma e_1 + \delta e_2 + \delta e_3, \end{aligned}$$

with $\alpha + \delta \neq 0$, $\alpha\gamma = 0.$

REMARK: Classification above uses the previous works by Cordero-Parker and Rahmani on three-dimensional Lorentz Lie groups.

HOMOGENEOUS GEODESICS:

$(M = K/H, g)$ p.R. *reductive* homogeneous space, $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$ a reductive split.

A geodesic γ through $o \in M = K/H$ is called *homogeneous* (h.g.) if it is the orbit of a one-parameter subgroup.
In the Riemannian case, this is equivalent to writing γ in the form

$$\gamma(t) = exp(tZ)(o), \qquad t \in \mathbb{R},$$

where Z is a nonzero vector of $\mathfrak{k}$. In the pseudo-Riemannian case, if $\dot{\gamma}(t)$ is a null vector, one needs to change its parametrization in order to write γ in the form $exp(sZ)(o)$.

Z is called a *geodesic vector* (g.v.).

Geometric problem: $\longleftrightarrow$ **algebraic problem:**

to find ALL h.g. to find ALL g.v.

Proposition ([Philip],[Dusek-Kowalski]): $X \in \mathfrak{k}$ is a g.v. if and only if

$$< [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} >= k < X_{\mathfrak{m}}, Y >,$$

for all $Y \in \mathfrak{m}$ and some $k \in \mathbb{R}$.

$X_{\mathfrak{m}}$ either spacelike or timelike $\Rightarrow$ $k = 0$

$X_{\mathfrak{m}}$ null vector $\Rightarrow$ k may be any real constant.

PHYSICAL RELEVANCE: homogeneous Lorentzian spaces for which all null geodesics are homogeneous, are candidates for constructing solutions to the 11-dimensional supergravity, which preserve more than 24 of the available 32 supersymmetries.
In fact, all Penrose limits, preserving the amount of supersymmetry of such a solution, must preserve homogeneity, which is the case for the Penrose limit of a reductive homogeneous spacetime along a null homogeneous geodesic [Meessen].

H.g. of ALL three-dimensional Lie groups (G, g), equipped with a left-invariant Lorentzian metric. [C-Marinosci]

$(\{e_1, e_2, e_3\}$ pseudo-orthonormal frame field with e_3 timelike.)

$$(\mathfrak{g}_1): \qquad x_2(e_2 \pm e_3)$$

$(\mathfrak{g}_2)$: $x_1 e_1$,

$$x_1\left(e_1 \pm \left(\sqrt{\frac{(\alpha-\beta)^2+\gamma^2}{2|\gamma|}} - \frac{1}{2}\right) e_2 \pm \left(\sqrt{\frac{(\alpha-\beta)^2+\gamma^2}{2|\gamma|}} + \frac{1}{2}\right) e_3\right)$$

$(\mathfrak{g}_3)$:

$\alpha \neq \beta \neq \gamma \neq \alpha$: $x_1 e_1$, $x_2 e_2$, $x_3 e_3$, $x_1\left(e_1 \pm \sqrt{\frac{\gamma-\alpha}{\beta-\gamma}} e_2 \pm \sqrt{\frac{\beta-\alpha}{\beta-\gamma}} e_3\right)$ (if $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$)
$\alpha = \beta \neq \gamma$: $x_3 e_3 + a_3 A_3$, $x_1 e_1 + x_2 e_2 + x_3 e_3 + (\gamma - \alpha) A_3$
$\alpha = \gamma \neq \beta$: $x_2 e_2 + a_2 A_2$, $x_1 e_1 + x_2 e_2 + x_3 e_3 + (\beta - \alpha) A_2$
$\beta = \gamma \neq \alpha$: $x_1 e_1 + a_1 A_1$, $x_1 e_1 + x_2 e_2 + x_3 e_3 + (\alpha - \beta) A_1$
$\alpha = \beta = \gamma$: all vectors

$(\mathfrak{g}_4)$:	$\alpha \neq \beta - \varepsilon$: $x_1 e_1$, $x_3(-\varepsilon e_2 + e_3)$, $x_3 e_3$, $x_3\left(\pm\sqrt{\frac{(\varepsilon+1)(2\beta-2\alpha-\varepsilon-1)}{(\beta-\alpha)^2}}e_1 + \frac{(\beta-\alpha-\varepsilon-1)}{\beta-\alpha}e_2 + e_3\right)$ (if $2(\beta-\alpha) \geq \varepsilon+1$), $x_3\left(\pm\sqrt{\frac{(\varepsilon-1)(2\beta-2\alpha-\varepsilon+1)}{(\beta-\alpha)^2}}e_1 + \frac{(\alpha-\beta+\varepsilon-1)}{\beta-\alpha}e_2 + e_3\right)$ (if $2(\beta-\alpha) \geq \varepsilon+1$)
	$\alpha = \beta - \varepsilon$: $x_3(-\varepsilon e_2 + e_3)$, $x_1 e_1 + x_2 e_2 + x_3 e_3 - (x_2 + \varepsilon x_3)A$

$(\mathfrak{g}_5)$:

$x_1 e_1 + x_2 e_2$ with $\alpha x_1^2 + (\beta+\gamma)x_1 x_2 + \delta x_2^2 = 0$,

$x_3 e_3$,

$x_1(\delta e_1 - \gamma e_2) + x_3 e_3$ but $\alpha = \beta = 0$,

$x_2(-\beta e_1 + \alpha e_2) + x_3 e_3$ but $\gamma = \delta = 0$,

$x_1 e_1 + x_2 e_2 \pm \sqrt{x_1^2 + x_2^2}\, e_3$
with $\gamma x_1^2 + (\delta - \alpha)x_1 x_2 - \beta x_2^2 = 0$.

Some interesting behaviours:

For some particular cases of unimodular Lie algebra $(\mathfrak{g}_3)$ and non-unimodular Lie algebras $(\mathfrak{g}_5)$ and $(\mathfrak{g}_6)$, there are no null h.g.

In many cases, there are not three linearly independent h.g. through a point.

NATURALLY REDUCTIVE AND G.O. SPACES:

A reductive p.R. hom. space $(M = K/H, g)$ is a *g.o. space* if all its geodesics are homogeneous, it is *naturally reductive* if there exists at least one reductive split $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$ such that

$$< [X, Y]_{\mathfrak{m}}, Z > + < [X, Z]_{\mathfrak{m}}, Y >= 0,$$

for all $X, Y, Z \in \mathfrak{m}$.

To decide whether (M, g) is or is not nat. reductive, condition above must be checked for all groups of isometries acting transitively on M.

(M, g) nat. reductive
$\Leftrightarrow$ the Levi-Civita connection of (M, g) and the canonical connection (of the reductive split $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{h}$) have the same geodesics.

(M, g) nat. reductive $\Rightarrow$ (M, g) is g.o.

$\not\Leftarrow$, but a 3-dim. Riemannian homogeneous g.o. space is nat. reductive.

RESULTS FOR LORENTZ 3-SPACES

Theorem [C-Marinosci] *Given a connected, simply connected Lorentz 3-space (M, g):*

(a) (M, g) is a g.o. space.
$\Leftrightarrow$ (b) (M, g) is naturally reductive.
$\Leftrightarrow$ (c) Either (M, g) is symmetric, or it is a unimodular Lie group G, equipped with a left-invariant Lorentz metric, having one of the following Lie algebras:

- $\mathfrak{g} = \mathfrak{g}_3$, *with either* $\alpha = \beta \neq \gamma$, $\alpha = \gamma \neq \beta$ *or* $\beta = \gamma \neq \alpha$.
- $\mathfrak{g} = \mathfrak{g}_4$, *with* $\alpha = \beta - \varepsilon$.

Theorem [C-Marinosci] *A connected, simply connected Lorentz 3-space (M, g) is a non-symmetric nat. reductive space if and only if it is isometric to either $SL(2, \mathbb{R})$, $SU(2)$ or H_3, equipped with a suitable left-invariant Lorentz metric.*

LORENTZ SYMMETRIC 3-SPACES:

Symmetric Lorentz 3-spaces only can occur for some of possible Segre types of the Ricci operator:

i): The Ricci operator of (M, g) is of Segre type $\{11, 1\}$ with eigenvalues $q_1 = q_2 = q_3$.

$\Rightarrow$ (M, g) has constant sectional curvature. If M is connected and simply connected, then (M, g) is one of the *Lorentzian space forms* S^3_1, $\mathbb{R}^3_1$ or $\mathbb{H}^3_1$.

ii): The Ricci operator of (M, g) is of Segre type $\{11, 1\}$ with eigenvalues $q_1 = q_2 \neq q_3$, and e_3 is a timelike parallel vector field

$\Rightarrow$ M is reducible as a direct product $M^2 \times \mathbb{R}$, where M^2 is a Riemannian surface of constant curvature. If M is connected and simply connected, (M, g) is then isometric to either $S^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$.

iii) The Ricci operator of (M, g) is of Segre type $\{11, 1\}$ with eigenvalues $q_1 \neq q_2 = q_3$, and e_1 is a spacelike parallel vector field

$\Rightarrow$ M is reducible as a direct product $\mathbb{R} \times M^2_1$, where M^2_1 is a Lorentzian surface of constant curvature. When M is connected and simply connected, (M, g) is isometric to either $\mathbb{R} \times S^2_1$ or $\mathbb{R} \times \mathbb{H}^2_1$.

iv): The Ricci operator of (M,g) is of Segre type $\{21\}$ with $a-b=\varepsilon$, and $u=e_2-e_3$ is a parallel null vector field. Three-dimensional Lorentz spaces admitting a parallel null vector field were studied by Chaichi,Garcia-Rio and Vazquez-Abal:

a locally symmetric Lorentz 3-space (M,g), having a parallel null vector field u, admits local coordinates (t,x,y) such that, with respect to $\{(\frac{\partial}{\partial t}),(\frac{\partial}{\partial x}),(\frac{\partial}{\partial y})\}$, the Lorentz metric g and the Ricci operator are given by

$$g=\begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \qquad Q=\begin{pmatrix} 0 & 0 & -\frac{1}{\varepsilon}\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (1)$$

where $\varepsilon=\pm 1$, $u=\frac{\partial}{\partial t}$ and

$$f(x,y)=x^2\alpha+x\beta(y)+\xi(y), \quad (2)$$

for any constant $\alpha\in\mathbb{R}$ and any functions β,ξ. It is easy to build a (local) pseudo-orthonormal frame field from $\{(\frac{\partial}{\partial t}),(\frac{\partial}{\partial x}),(\frac{\partial}{\partial y})\}$, and to check that, whenever $\alpha f\neq 0$ (that is, g is not flat), the Ricci operator described by (2) is of Segre type $\{21\}$.

Theorem 1 [C] *A connected, simply connected Lorentz symmetric 3-space (M,g) is either*

i) a Lorentzian space form S^3_1, $\mathbb{R}^3_1$ or $\mathbb{H}^3_1$, or

ii) a direct product $\mathbb{R}\times S^2_1$, $\mathbb{R}\times\mathbb{H}^2_1$, $S^2\times\mathbb{R}$ or $\mathbb{H}^2\times\mathbb{R}$, or

iii) a space with a Lorentzian metric g described by (1)-(2).

EINSTEIN-LIKE LORENTZ METRICS:

Class $\mathcal{A}$: a p.R. manifold (M, g) belongs to $\mathcal{A}$ if and only if its Ricci tensor ϱ is *cyclic-parallel*, that is,

$$(\nabla_X\varrho)(Y,Z)+(\nabla_Y\varrho)(Z,X)+(\nabla_Z\varrho)(X,Y)=0,$$

equivalently, if ϱ is a *Killing tensor*, that is,

$$(\nabla_X\varrho)(X,X)=0.$$

Class $\mathcal{B}$: (M, g) belongs to $\mathcal{B}$ if and only if ϱ is a *Codazzi tensor*, that is,

$$(\nabla_X\varrho)(Y,Z)=(\nabla_Y\varrho)(X,Z).$$

ϱ is parallel $\Leftrightarrow$ (M, g) belongs to $\mathcal{A}\cap\mathcal{B}$.

Theorem [Abbena-Garbiero-Vanhecke]: a connected, simply connnected homogeneous Riemannian 3-space belongs to class $\mathcal{A}$ if and only if it is *naturally reductive.*

Theorem [C]: *Let (M, g) be a connected, simply connected Lorentz 3-space. (M, g) belongs to class $\mathcal{A}$ if and only if one of the following cases occurs:*

a) (M, g) is naturally reductive;

b) $M = G$, $\mathfrak{g} = \mathfrak{g}_5$ and either $\alpha = \beta = 0 \neq \delta$, with $\gamma \neq 0$, or $\gamma = \delta = 0 \neq \alpha$, with $\beta \neq 0$;

c) $M = G$, $\mathfrak{g} = \mathfrak{g}_6$ and either $\alpha = \beta = 0 \neq \delta$, with $\gamma \neq 0$ and $\gamma \neq \varepsilon\delta$, or $\gamma = \delta = 0 \neq \alpha$, with $\beta \neq 0$ and $\beta \neq \varepsilon\alpha$;

d) $M = G$, $\mathfrak{g} = \mathfrak{g}_7$ and $\alpha = \beta = 0 \neq \delta$, with $\gamma \neq 0$.

Theorem [Abbena-Garbiero-Vanhecke]: a connected, simply connnected homogeneous Riemannian 3-space belongs to class $\mathcal{B}$ if and only if it is *symmetric*.

Theorem [C] *Let (M, g) be a connected, simply connected* Lorentz *3-space. (M, g) belongs to class $\mathcal{B}$ if and only if one of the following cases occurs:*

a) (G, g) is symmetric;

b) $M = G$, $\mathfrak{g} = \mathfrak{g}_1$ with $\beta = 0$. In this case, $G = E(1, 1)$;

c) $M = G$, $\mathfrak{g} = \mathfrak{g}_2$ with $\alpha = -2\beta$, $\gamma = \pm\sqrt{3}\beta$ and $\beta \neq 0$. In this case, $G = O(1, 2)$ or $SL(2, \mathbb{R})$;

d) $M = G$, $\mathfrak{g} = \mathfrak{g}_7$ and $\gamma = 0 \neq \alpha\delta(\alpha \pm \delta)$.

CONFORMALLY FLAT LORENTZ METRICS:

A homogeneous p.R. manifold (M,g) has constant scalar curvature.
If $\dim M = 3$, then (M,g) is conformally flat if and only if it belongs to class $\mathcal{B}$. In particular:

Theorem [C] *A homogeneous Lorentz 3-space (M,g) is conformally flat if and only if one of the following cases occurs:*

a) (M,g) is symmetric;

b) $M = G$ and either
$\mathfrak{g} = \mathfrak{g}_1$ *with* $\beta = 0$*, or*
$\mathfrak{g} = \mathfrak{g}_2$ *with* $\alpha = -2\beta$, $\gamma = \pm\sqrt{3}\beta$, $\beta \neq 0$*, or*
$\mathfrak{g} = \mathfrak{g}_7$ *and* $\gamma = 0 \neq \alpha\delta(\alpha \pm \delta)$.

(M,g) Riemannian conf. flat hom. space
$\Rightarrow$ (M,g) is symmetric.
So, conformal flatness is a weaker assumption in Lorentzian than in Riemannian geometry.

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