On the topology of black holes in higher dimensions

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Introduction

- Recent developments in physics inspired by string theory have heightened interest in higher dimensional gravity.
- In particular, there has been a lot of recent research devoted to the study of black holes in higher dimensions.
- One of the first questions to arise is:

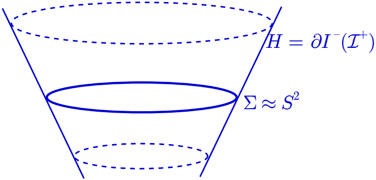
  Does black hole uniqueness (the "no hair theorems") hold in higher dimensions?

- And the answer is:

  No. In fact, one does not even have topological uniqueness.

Introduction

Hawking's black hole topology theorem: Suppose (M, g) is a $3+1$-dimensional AF stationary black hole spacetime obeying the dominant energy condition (DEC). Then cross sections of the event horizon are topologically 2-spheres.



But in 2002, Emparan and Reall published a remarkable example of a $4+1$ dimensional AF vacuum stationary black hole spacetime with horizon topology $S^2 \times S^1$ (a black ring; cf., hep-th/0608012).

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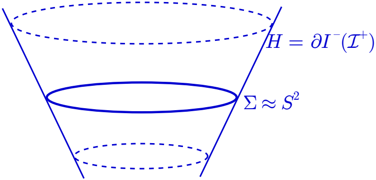

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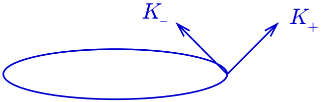
Introduction

- Question naturally arises: What restrictions are there on the topology of black holes in higher dimensions?
- In this talk I'm going to present recent joint work with Rick Schoen (CMP, 2006) in which we obtain a natural generalization of Hawking's black hole topology theorem to higher dimensions.
- Our conclusion in higher dimensions is that cross sections of the event horizon (in the stationary case) and outer apparent horizons (in the general case) must be of positive Yamabe type.
- This implies many well-known restrictions on the topology.
- Will also discuss some recent refinements of work with Schoen, and comment on other approaches.

Marginally trapped surfaces

$$M^{n+1} = \text{(n+1)-dimensional spacetime } (n \geq 3)$$
$$\Sigma^{n-1} = \text{co-dimension 2 spacelike submanifold of } M$$

Under suitable orientation assumptions, Σ admits two smooth nonvanishing future directed null normal vector fields K_+ and K_-:



By convention,

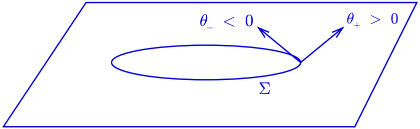
$$K_+ = \text{outward pointing}$$
$$K_- = \text{inward pointing}$$

Let θ_+, θ_- be the associated null expansion scalars,

$$\theta_+ = \mathrm{tr}\chi_+ = \mathrm{div}_\Sigma K_+$$
$$\theta_- = \mathrm{tr}\chi_- = \mathrm{div}_\Sigma K_-$$

Marginally trapped surfaces

For round spheres in Euclidean slices in Minkowski space (and, more generally, large "radial" spheres in AF spacelike hypersurfaces),



However, in a strong gravitational field one can have both,

$$\theta_+ < 0 \qquad \text{and} \qquad \theta_- < 0\,,$$

in which case Σ is called a trapped surface.

Under appropriate energy and causality conditions, the occurrence of a trapped surface signals the onset of gravitational collapse and the formation of a black hole (Penrose, Hawking).

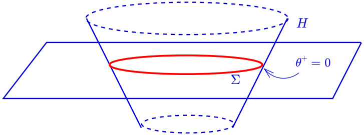
Marginally trapped surfaces

Focusing attention on the outward null normal only,

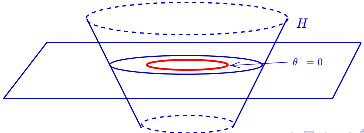
- If $\theta_+ < 0$ - we say Σ is an outer trapped surface
- If $\theta_+ = 0$ - we say Σ is a marginally outer trapped surface (MOTS).

MOTS arise in several natural situations.

- In stationary black hole spacetimes - cross sections of the event horizon are MOTS.

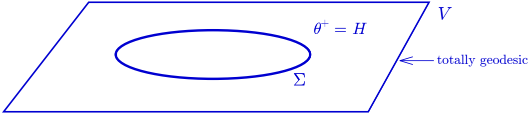


- In dynamical black hole spacetimes - MOTS typically occur inside the event horizon:



Marginally trapped surfaces

- There has been a lot of recent work concerned with properties of MOTS - e.g., Andersson, Mars, Simon, ...
- Current interest in MOTS is due to, e.g.: Renewed interest in quasi-local notions of black holes (*dynamical horizon* program, etc.) and connections between MOTS in spacetime and *minimal surfaces* in Riemannian manifolds.



  A MOTS Σ in a *time-symmetric* (i.e., totally geodesic) spacelike hypersurface V is simply a minimal surface ($H = 0$) in V.

- MOTS have been shown to satisfy a number of properties analogous to those of minimal surfaces.

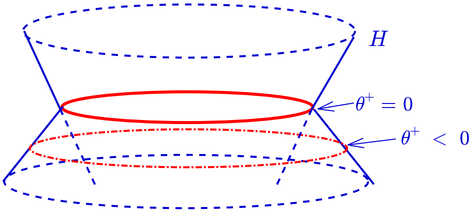
Hawking's Theorem

Theorem

Suppose (M, g) is a $(3+1)$-dimensional AF stationary black hole spacetime obeying the DEC. Then cross sections of the event horizon are topologically 2-spheres.

Idea of proof:

- By stationarity, Σ is a MOTS, $\theta^+ = 0$.
- If $\Sigma \not\approx S^2$, i.e., if $g \geq 1$ then using Gauss-Bonnet and DEC, Hawking shows that Σ can be deformed to an outer trapped surface, $\theta^+ < 0$, outside the black hole region.



Hawking's Theorem

Comments

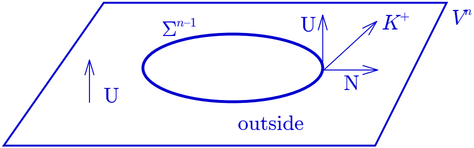
- Actually, the torus ($g = 1$) is borderline for Hawking's argument. But can occur only under special circumstances.
- Hawking showed by a variation of his original argument, that the conclusion of his theorem also holds for 'outer apparent horizons' in black hole spacetimes that are not necessarily stationary.
- In higher dimensions, one cannot appeal to the Gauss-Bonnet theorem. This is one of the complicating the issues.

Generalization of Hawking's Theorem

Let

$$\begin{aligned} M^{n+1} &= \text{ }(n+1)\text{-dimensional spacetime, } n \geq 3 \\ V^n &= \text{ spacelike hypersurface in } M^{n+1} \\ \Sigma^{n-1} &= \text{ closed } (n-1)\text{-surface in } V^n \end{aligned}$$

Suppose Σ separates V^n into an "inside" and an "outside":



Then we say that Σ^{n-1} is an outer apparent horizon provided:

- Σ is a MOTS, $\theta^+ = 0$
- There are no (strictly) outer trapped surfaces outside of Σ

Generalization of Hawking's Theorem

Theorem (G. and Schoen)

Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the DEC, and let Σ^{n-1} be an outer apparent horizon in V^n. Then

- Σ^{n-1} *is of positive Yamabe type, i.e., admits a metric of positive scalar curvature*

unless

- Σ *is Ricci flat (flat if $n = 3, 4$) in the induced metric, $\chi_+ \equiv 0$, and $\mathcal{T}(U, K) = T_{ab}U^aK^b \equiv 0$ on Σ*

Thus, apart from certain exceptional circumstances, Σ is positive Yamabe.

Topological restrictions

Σ being positive Yamabe implies many well-known restrictions on the topology.

E.g., if $\dim \Sigma = 2$ ($\dim M = 3 + 1$), then $\Sigma \approx S^2$ by G-B, and one recovers Hawking's theorem.

Focusing on the case $\dim \Sigma = 3$ ($\dim M = 4 + 1$), we have:

Theorem (Schoen-Yau, Gromov-Lawson)

If Σ is a closed orientable 3-manifold of positive Yamabe type then Σ must be diffeomorphic to:

- *a spherical space, or*
- $S^2 \times S^1$, *or*
- *a connected sum of the above two types.*

Thus, the basic horizon topologies in $\dim \Sigma = 3$ case are S^3 and $S^2 \times S^1$.

Topological restrictions

Proof: By the prime decomposition theorem, Σ must be a connected sum of (1) spherical spaces, (2) $S^2 \times S^1$'s, and (3) $K(\pi, 1)$ manifolds. But since Σ is positive Yamabe, it cannot contain any $K(\pi, 1)$'s in its prime decomposition.

Here is a simple obstruction that holds in arbitrary dimensions:

Theorem (Gromov-Lawson)

A compact manifold that admits a metric of nonpositve sectional curvatures, $K \leq 0$, cannot carry a metric of positive scalar curvature.

This rules out many obvious topologies.

Proof of the theorem

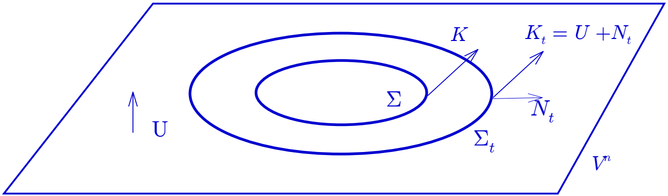
We consider normal variations of Σ in V, i.e., variations $t \to \Sigma_t$ of $\Sigma = \Sigma_0$ with variation vector field

$$V = \frac{\partial}{\partial t}\Big|_{t=0} = \phi N, \quad \phi \in C^\infty(\Sigma)\,.$$

Let

$$\theta(t) = \text{ the null expansion of } \Sigma_t,$$

where $K_t = U + N_t$ and N_t is the unit normal field to Σ_t in V.



Proof, cont.

A computation shows

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = L(\phi)\,,$$

where,

$$L(\phi) = -\triangle\phi + 2\langle X, \nabla\phi\rangle + \left(Q + \operatorname{div} X - |X|^2\right)\phi\,,$$

$$Q = \frac{1}{2}S - \mathcal{T}(U,K) - \frac{1}{2}|\chi|^2 \quad \text{and} \quad X = \tan\left(\nabla_N U\right).$$

<u>*Remark:*</u> In analogy with minimal surface theory, L is the 'stability operator' associated with variations in θ. Note, however, that L is *not* in general self-adjoint.

Proof, cont.

- $\lambda_1(L)$, the principal eigenvalue of L, is real (Krein-Rutman).
- No outer trapped surfaces outside $\Sigma \implies \lambda_1(L) \geq 0$ (i.e., Σ is *stable*).
- Consider the "symmetrized operator",

$$L_0(\phi) = -\triangle\phi + Q\,\phi\,.$$

Key fact: $\lambda_1(L_0) \geq \lambda_1(L)$. Thus, $\lambda_1(L_0) \geq 0$.

- Let $\phi > 0$ be an eigenfunction corresponding to $\lambda_1(L_0)$. The scalar curvature $\tilde{S}$ of Σ in the conformally rescaled metric $\tilde{h} = \phi^{\frac{2}{n-2}} h$ is given by,

$$\begin{aligned}\tilde{S} &= \phi^{-\frac{n}{n-2}}(-2\triangle\phi + S\phi + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi}) \\ &= \phi^{-\frac{2}{n-2}}(2\lambda_1(L_0) + 2\mathcal{T}(U,K) + \frac{1}{2}|\chi|^2 + \frac{n-1}{n-2}\frac{|\nabla\phi|^2}{\phi^2}) \\ &\geq 0\end{aligned}$$

The borderline case

Theorem (G. and Schoen)

Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the DEC, and let Σ^{n-1} be an outer apparent horizon in V^n. Then

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Note, for example, the theorem does not rule out the possibility of a vacuum black hole spacetime with toroidal horizon topology.

The theorem can be read this way: if Σ is *not* positive Yamabe then get *infinitesimal* rigidity. But I expect more rigidity to hold, which could be used to rule out the exceptional case.

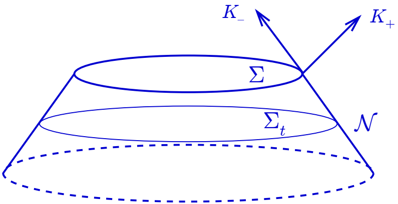
The borderline case

Theorem (Rigidity result, G.)

Let (M^{n+1}, g), $n \geq 3$, be a spacetime satisfying the DEC, and let Σ^{n-1} be a compact co-dimension two spacelike submanifold of M with null expansions $\theta_\pm$, such that the following conditions hold.

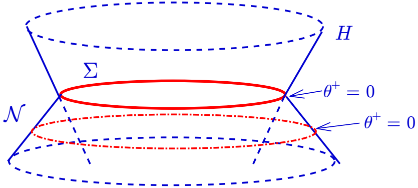
- *Σ is a MOTS, i.e, $\theta_+ = 0$.*
- *There are no (strictly) outer trapped surfaces along $\mathcal{N}$*
- *$\theta_- < 0$ on Σ.*

Then, if Σ does not admit a metric of positive scalar curvature, $\mathcal{N}$ is foliated by MOTSs Σ_t, $t \in [0, \epsilon)$, near Σ.



The borderline case

- It follows from this rigidity result that *without exception* (but subject to the sign condition), cross sections of the event horizon in *stationary* black hole spacetimes obeying the the DEC are positive Yamabe.



- The proof of the rigidity result involves two main steps:
  - Show $\mathcal{N}$ is foliated near Σ by surfaces Σ_t of constant null expansion, $\theta_t = const.$
  - Show for each leaf Σ_t, $const. = 0$.

Spacelike versions of the rigidity result

- One can prove in a completely analogous way a spacelike version of the rigidity result, i.e., where $\mathcal{N}$ is spacelike and Σ is an outer apparent horizon. But the natural sign condition in this case is that $\mathcal{N}$ be *maximal*, i.e., have mean curvature $= 0$.
- (In progress) It now appears that one can derive a spacelike version of rigidity as a *consequence* of the null version. The sign condition will be as in the latter: $\theta_- < 0$ along Σ.
- (In progress) There is an approach to the eliminating the 'exceptional case' in our result with Schoen that does not require maximality and does not require that Σ be inner trapped, but instead assumes a very mild asymptotic condition. This involves an application of Schoen's recent existence result for MOTS.

Other approaches to black hole topology

- An entirely different approach to the study of black hole topology was developed in the 90's based on topological censorship, see especially, G., Schleich, Witt and Woolgar, Phys. Rev. D (1999).
- In the AF case, topological censorship is simply the statement that if the null energy condition holds, and the domain of outer communications (DOC) is globally hyperbolic then the DOC must be *simply connected*. This can be used to show in the dim $M = 3 + 1$ case that cross sections of the event horizon are spherical.
- In the case dim $M = 5 + 1$, dim $\Sigma = 4$, Helfgott et al. have recently argued (JHEP, 2006), using topological censorship, that if Σ is *simply connected* then either it is homeomorphic to S^4 or is a connected sum of $S^2 \times S^2$'s.