Globally hyperbolic manifolds with special holonomy

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Aim of the talk: Construct globally hyperbolic manifolds with special holonomy

1. Holonomy groups of Lorentzian manifolds
2. Spinors on curved spaces
   - Introduction
   - Parallel spinors on Riemannian manifolds, curvature and holonomy
   - Parallel spinors on Lorentzian manifolds, curvature and holonomy
   - Codazzi spinors and parallel spinors on Lorentzian cylinders
3. Classification of complete Riemannian manifolds with Codazzi spinors
4. Globally hyperbolic manifolds with special holonomy
   $G \ltimes \mathbb{R}^n \subset SO(1, n+1)_{\mathbb{R}v_0} \simeq Sim(\mathbb{R}^n)$.

1. Holonomy groups of Lorentzian manifolds

(M,g) Lorentzian manifold, complete, simply-connected.
$Hol(M,g) = \{\mathcal{P}^g_\gamma \mid \mathcal{P}^g_\gamma : T_xM \to T_xM$ parallel transport along closed loops$\}$

Spitting Theorem: (H.Wu 1967)

$$(M,g) \;\simeq\; (N,h) \times (M_1,g_1) \times \cdots \times (M_k,g_k),$$

where (M_i,g_i) are flat or irreducible Riemannian manifolds and $(N^{1,n+1},h)$ is a Lorentzian manifold that is eigther

- flat: trivial holonomy
- irreducible: holonomy $= SO_0(1,n+1)$ or
- weakly irreducible & non-irreducible: holonomy $\subset SO(1,n+1)_{\mathbb{R}v_0}$, v_0 lightlike.

$$
\begin{aligned}
SO(1,n+1) &\simeq Isom(H^{n+1}) \\
&\simeq Conf(\partial H^{n+1}) \simeq Conf(S^n) \simeq Conf(\mathbb{R}^n \cup \{\infty\})
\end{aligned}
$$

$$
SO(1,n+1)_{\mathbb{R}v_0} \simeq Sim(\mathbb{R}^n) \simeq (\underset{\text{dilatation}}{\underset{\uparrow}{\mathbb{R}^*}} \times \underset{\text{lin.isometry}}{\underset{\uparrow}{SO(n)}}) \ltimes \underset{\text{translation}}{\underset{\uparrow}{\mathbb{R}^n}}
$$

$$
\begin{pmatrix} a^{-1} & x^t & -\frac{1}{2}a\|x\|^2 \\ 0 & A & -aAx \\ 0 & 0 & a \end{pmatrix} \longleftarrow (a, A, x)
$$

Theorem: (Th. Leistner 2003)

$(M^{1,n+1}, g)$ weakly irreducible, non-irreducible Lorentzian manifold $\Longrightarrow$

The connected holonomy group $H = Hol_0(M,g) \subset (\mathbb{R}^+ \times SO(n)) \ltimes \mathbb{R}^n$ is isomorphic to

1. $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$
2. $G \ltimes \mathbb{R}^n$
3. $(A^\phi \times B) \ltimes \mathbb{R}^n$ — $\phi : \mathbb{R}^+ \to SO(n)$ Homom.
   $A^\phi := \{a \cdot \phi(a) \mid a \in \mathbb{R}^+\} \subset \mathbb{R}^+ \times SO(n)$
4. $(B \times U^\psi) \ltimes V$ — $\mathbb{R}^n = U \oplus V, \ B \subset SO(V)$
   $\psi : U \to SO(V)$ Homom.
   $U^\psi := \{\psi(u) \cdot u \mid u \in U\} \subset SO(V) \times U$

where $G = proj_{SO(n)} H \subset SO(n)$ is the holonomy group of a Riemannian manifold, hence a product of 1, $U(k_1)$, $SU(k_2)$, $Sp(k_3)$, $Sp(k_4)Sp(1)$, G_2, $Spin(7)$ or $K =$ stabilizer of a Riemannian symmetric space.

$G = Z(G) \cdot B$ in case 3. and 4.

Theorem: (A. Galaev 2005)

Any of the groups in Leistner's list can be realized as holonomy group of a polynomial Lorentzian metric on $\mathbb{R}^{n+2}$

Task: Describe global geometric models for Lorentzian manifolds with special holonomy.

Question: Which of the special Lorentzian holonomy groups can be realized by globally hyperbolic metrics?

Globally hyperbolic Lorentzian manifolds with special holonomy

Definition:

A Lorentzian manifold (M, g) is called globally hyperbolic iff

- (M, g) is stongly causal (for example if there exists a continous function f on M which is strictly increasing along any future directed causal curve)
- $J^+(p) \cap J^-(q) \subset M$ is compact for all $p, q \in M$

$$J^{\pm}(p) := \{x \in M \mid \exists \gamma : p \longrightarrow x \text{ causal}, \uparrow_+ (\downarrow_-)\}$$

Some special properties of globally hyperbolic manifolds

- Normally hyperbolic operators have a global and unique forward and backward fundamental solution
- Existence of Cauchy surfaces
- Maximal causal geodesics: $p, q \in M$, $p \leq q$. Then there exists a causal geodesic from p to q of maximal length.

A partial answer:

Theorem (Baum/Müller 2005)

Any Lorentzian holonomy group of the form $G \ltimes \mathbb{R}^n \subset SO(1, n+1)$, where $G \subset SO(n)$ is the product of groups 1, $SU(k_1)$, $Sp(k_2)$, G_2 or $Spin(7)$ can be realized by a globally hyperbolic metric.

Theorem (Leistner 2003)

An indecomposable Lorentzian manifold has parallel spinors if and only if its holonomy group is $G \ltimes \mathbb{R}^n$, where G is a product of 1, $SU(k_1)$, $Sp(k_2)$, G_2 or $Spin(7)$.

$\Longrightarrow$ We can use spinors to construct such metrics.

The idea for the construction of such metrics was inspired by a paper of Ch. Bär, P. Gauduchon, A. Moroianu (2004)

2. Spinors on curved spaces

Let $(M^{p,q}, g)$ be a semi-Riemannian spin manifold ($w_2(M) = 0$). Then on (M, g) there is a special complex vector bundle $S := Q \times_{Spin(p,q)} \Delta$ (spinor bundle) with a covariant derivative $\nabla^S : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S)$ (spin connection) and a hermitian inner product $< \cdot, \cdot >$. $(n = p + q \geq 3)$

One can multiply vectors and spinors

$$X \in TM,\ \varphi \in S \longmapsto X \cdot \varphi \in S$$ Clifford product

such that the following rules hold

- $(X \cdot Y + Y \cdot X) \cdot \varphi = -2g(X, Y)\,\varphi$
- $< X \cdot \varphi, \psi > = (-1)^{p-1} < \varphi, X \cdot \psi >$
- $\nabla^S_X(Y \cdot \varphi) = (\nabla^g_X Y) \cdot \varphi + Y \cdot \nabla^S_X \varphi$
- $X(< \varphi, \psi >) = < \nabla^S_X \varphi, \psi > + < \varphi, \nabla^S_X \psi >$

φ spinor field $\Longrightarrow$ V_φ vector field : $g(V_\varphi, X) = i^{p+1}\langle X \cdot \varphi, \varphi\rangle$.

- (M,g) Riemannian $\Longrightarrow$ $Zero(\varphi) \subset Zero(V_\varphi)$
- (M,g) Lorentzian $\Longrightarrow$ $Zero(\varphi) = Zero(V_\varphi)$ and $g(V_\varphi, V_\varphi) \leq 0$.

A spinor field φ is called parallel if $\nabla^S \varphi = 0$.

$\Longrightarrow$ 1. V_φ is parallel and $g(V_\varphi, V_\varphi) =$ const

2. $Ric(X) \cdot \varphi = 0 \;\; \forall\, X$ $\quad (*)$
   - (M,g) Riemannian $\Longrightarrow$ $Ric = 0$
   - (M,g) Lorentzian $\Longrightarrow$ $Ric^2 = 0$ ($Ric(TM)$ totally null)

3. (M,g) has special holonomy

$$(*) \quad \sum_{k=1}^{n} \sigma^k \cdot R^S(X, s_k)\,\varphi = -\tfrac{1}{2} Ric(X) \cdot \varphi$$

Holonomy groups and parallel sections

E vector bundle over M with covariant derivative ∇, $x \in M$

$$Hol_x(E,\nabla) := \{\mathcal{P}_\gamma^\nabla : E_x \to E_x \text{ parallel transport along } \gamma \mid \gamma \text{ loop in } x\}$$

P G-principal bundle over M with principal bundle connection ω, $p \in P_x$

$$Hol_p(P,\omega) := \{g \in G \mid \exists \text{ loop } \gamma \text{ in } x \text{ such that } \gamma_p^*(1) = p \cdot g\}$$

Let $\rho : G \to GL(V)$ be a representation, $E := P \times_G V$ and $\nabla = \nabla^\omega$.
Fixing a $p \in P_x$ gives an isomorphism $E_x \simeq V$ such that

$$Hol_x(E,\nabla^\omega) = \rho(Hol_p(P,\omega))$$

Holonomy principle: There is a 1-1 correspondence between

$$\{\varphi \in \Gamma(E) \mid \nabla^\omega\varphi = 0\} \quad \text{and} \quad \{v \in V \mid \rho(Hol_p(P,\omega))v = v\}$$

$$= \{v \in V \mid \rho_*(\mathfrak{hol}_p(P,\omega))v = 0\} \quad \text{if } \pi_1(M) = 0$$

Parallel spinors and holonomy groups of metrics

Let $(M^{p,q}, g)$ be a spin manifold with the frame bundle P and spin structure (Q, f).

$\lambda : Spin(p,q) \longrightarrow SO(p,q)$ 2-fold covering

$\rho : Spin(p,q) \longrightarrow GL(\Delta)$ spin representation

$TM := P \times_{SO(p,q)} \mathbb{R}^{p,q}$ $\qquad p \in P_x$ frame in x

$S := Q \times_{Spin(p,q)} \Delta$ $\qquad q \in Q_x$ spin frame in x, $f(q) = p$

$$Hol_x(TM, \nabla^g) = Hol_p(P, \omega^{LC}) \subset SO(p,q)$$

$$Hol_x(S, \nabla^S) = \rho(Hol_q(Q, \omega^{LC})) \subset \rho(Spin(p,q))$$

Then

$$\lambda(Hol_q(Q, \omega^{LC})) = Hol_x(TM, \nabla^g) = Hol(M,g)$$

$$\mathfrak{hol}_q(Q, \omega^{LC}) = (\lambda_*)^{-1}\mathfrak{hol}(M,g)$$

$$\boxed{\{\varphi \in \Gamma(S) \mid \nabla^S \varphi = 0\} \equiv \{v \in \Delta \mid \rho_*(\lambda_*^{-1}(\mathfrak{hol}(M,g)))v = 0\}}$$

Special holonomy of manifolds with parallel spinors

Riemannian case

- M is not symmetric $(Ric = 0)$
- $Hol(M, g)$ is a product of 1, $SU(k_1)$, $Sp(k_2)$, G_2, $Spin(7)$.

Lorentzian case

- V_φ timelike and parallel $\Rightarrow$ $(M, g) \simeq (\mathbb{R}, -dt^2) \times$ (Riem. with parallel spinor)
- V_φ lightlike and parallel $\Rightarrow$
  - $Hol(M, g) \subset SO(1, n+1)_{\mathbb{R}v_0}$
  - $Hol(M, g)$ has no dilatation part
  - $G = proj_{SO(n)} Hol(M, g)$ has no center

$\Longrightarrow$ $Hol(M, g) = G \ltimes \mathbb{R}^n$

where G is a product of 1, $SU(k_1)$, $Sp(k_2)$, G_2, $Spin(7)$.

A special construction

Let (M, g_0) be a Riemannian spin manifold with a Codazzi tensor A

A is a symmetric (1,1)-tensor field with $(\nabla^{g_0}_X A)(Y) = (\nabla^{g_0}_Y A)(X)$.

A spinor field $\varphi \in \Gamma(S_M)$ is called A- Codazzi spinor if

$$\nabla^S_X \varphi = iA(X) \cdot \varphi \qquad \text{for all vector fields } X$$

Theorem: (Bär/Gauduchon/Moroianu'04)
Let (M, g_0) be a Riemannian spin manifold with an A-Codazzi spinor, then the Lorentzian cylinder

$$C := I \times M \ , \ g_C := -dt^2 + (1 - 2tA)^* g_0$$

has a parallel spinor field.

Questions:
- Find (all) Riemannian manifolds with A-Codazzi spinors.
- When such Lorentzian cylinder is globally hyperbolic?
- Which holonomy groups can be realized in this way?

Proof of Bär's Theorem:

Compare the spin geometry of the slice $M = M_0$ and the Lorentzian cylinder C:

Let φ be a spinor field on M_0. Consider the parallel displacement along the t-lines
$\widetilde{\varphi}(t,x) := \mathcal{P}_{\gamma_x(t)}(\varphi(x))$ (spinor field on C)

$$\nabla^C_X \widetilde{\varphi}\big|_{M_0} = \nabla^M_X \widetilde{\varphi} + \partial_t \cdot A(X) \cdot \widetilde{\varphi}\big|_{M_0} = \nabla^M_X \varphi - iA(X) \cdot \varphi \qquad X \in TM$$

- φ A-Codazzi spinor $\Rightarrow$ $\nabla^C_X \widetilde{\varphi} = 0$ on M_0.
- $\nabla^C_{\partial_t} \widetilde{\varphi} = 0$ by definition
- $R^C(\partial_t, X) = 0 \quad \forall\, X(x,t) = X(x)$ (A Codazzi tensor)
- $\nabla^C_{\partial_t}(\nabla^C_X \widetilde{\varphi}) = R^{S_C}(\partial_t, X)\widetilde{\varphi} = \frac{1}{2} R^C(\partial_t, X) \cdot \widetilde{\varphi} = 0$

$\Rightarrow$ $\nabla^C_X \widetilde{\varphi}$ is parallel along the t-lines and vanishes for $t = 0$

$\Longrightarrow$ $\widetilde{\varphi}$ is parallel on the Lorentzian cylinder C

3. Classification of Riemannian manifolds with Codazzi spinors (invertible case)

Theorem: (Baum/Müller'05)
Let (M,g) be a complete Riemannian manifold with an A-Codazzi spinor for an invertable Codazzi tensor A, and let $\|A^{-1}\| \leq c$. Then

$$(M,g) \simeq (\mathbb{R} \times F \,,\ (A^{-1})^*(ds^2 + e^{-4s} g_F)),$$

where (F,h) is a complete Riemannian manifold with a parallel spinor and A^{-1} is a Codazzi-tensor on the warped product $(\mathbb{R} \times F, ds^2 + e^{-4s} g_F)$.

And vice versa:

Let (F, g_F) be a complete Riemannian manifold with a parallel spinor and a Codazzi tensor T whose eigenvalues are uniformally bounded from below.
Then there are Codazzi tensors B on the warped product $(\mathbb{R} \times F, ds^2 + e^{-4s} g_F)$ such that $\|B^{-1}\| \leq c$. The Riemannian manifold

$$M = \mathbb{R} \times F \,,\ g = B^*(ds^2 + e^{-4s} g_F)$$

is complete and has an A-Codazzi spinor, where $A = B^{-1}$.

Proof:

- $\overline{g} = A^*g$ is a complete Riemannian metric. Let $\overline{\nabla} = \nabla^{\overline{g}}$
- $\overline{\nabla}_X = A^{-1} \circ \nabla^g_X \circ A$
- There is an isomorphism $\varphi \in S_{(M,g)} \mapsto \overline{\varphi} \in S_{(M,\overline{g})}$ such that

$$\varphi \ A\text{-Codazzi spinor for } g \quad \Longleftrightarrow \quad \overline{\varphi} \text{ imaginary Killing spinor for } A^*g$$

$$\nabla_X \varphi = iA(X) \cdot \varphi \qquad\qquad \overline{\nabla}_X \overline{\varphi} = iX \cdot \overline{\varphi}$$

- Splitting theorem for manifolds with imaginary Killing spinors (Baum 1989):

  Let $(M, \overline{g})$ be a complete Riemannian manifold with an imaginary Killing spinor, then

$$(M, \overline{g}) \simeq (\mathbb{R} \times F\, ,\, ds^2 + e^{-4s} g_F)$$

  where (F, g_F) is a complete Riemannian manifold with parallel spinor.
  And vice versa.

Codazzi tensors on warped products:

- T Codazzi tenor on (F, g_F).
- $b(s)$ smooth function on $\mathbb{R}$
- $E(s)$ s-parameter family of (1,1)-tensor fields on F:

$$E(s) := \frac{1}{f(s)}\Big(T + \int_0^s b(\tau)\dot{f}(\tau)d\tau \cdot Id_{|TF}\Big)$$

$\Longrightarrow$ $B = \begin{pmatrix} b & 0 \\ 0 & E \end{pmatrix}$ is a Codazzi tensor on the warped product

$$(\mathbb{R} \times F \,,\, ds^2 + f(s)^2\, g_F)$$

4. Globally hyperbolic manifolds with special holonomy

Theorem: (Baum/Müller'05)

Let (F, g_F) be a complete Riemannian manifold with a parallel spinor, T a Codazzi tensor on (F, g_F) with eigenvalues bounded from below. T defines Codazzi tensors B on the warped product $(\mathbb{R} \times F, ds^2 + e^{-4s} g_F)$ with $\|B^{-1}\| < c$. Let

$$C := I \times \mathbb{R} \times F \; , \; g_C := -dt^2 + (B - 2t)^*(ds^2 + e^{-4s} g_F).$$

Then

- (C, g_C) is a globally hyperbolic.
- C is decomposable if and only if (F, g_F) has a flat factor.
- If (F, g_F) is (locally) a product of irreducible factors, then C is weakly irreducible and
$$Hol^0_{(0,0,x)}(C, g_C) = (B^{-1} \circ Hol^0_x(F, g_F) \circ B) \ltimes \mathbb{R}^{dimF}$$

Proof:

1. (C, g_C) is globally hyperbolic:

Consider the cylinder $C = I \times M$ with the metric

$$g_C = -dt^2 + g_t$$

where g_t is a smooth family of Riemannian metrics on M.

Let $g_t = g_0(A_t \cdot, A_t \cdot) = A_t^* g_0$. If

- g_0 is complete
- $\|A_t^{-1}\| < c_t$ for all $t \in I$,

then (C, g_C) is globally hyperbolic.

2. Holonomy groups of the Lorentzian cylinder (C, g_C):

Calculate the parallel displacement $\mathcal{P}_\delta^C$ for $g_C = -dt^2 + (B-2t)^*(ds^2 + e^{-4s} g_F)$:

$TC \simeq \mathbb{R}P \oplus ATF \oplus \mathbb{R}Q$, where $A = B^{-1}$ and P is parallel and lightlike.

Let $\delta(r) = (t(r), s(r), \gamma(r))$ be a loop in $(0,0,x) \in C$. Then

$$\mathcal{P}_\delta^C = \begin{pmatrix} 1 & (a_{1\gamma}, \ldots, a_{n\gamma}) & * \\ 0 & B^{-1} \circ \mathcal{P}_\gamma^F \circ B & * \\ 0 & 0 & 1 \end{pmatrix} \subset SO(n) \ltimes \mathbb{R}^n \subset SO(1, n+1)_{\mathbb{R}P}$$

where

$$\begin{aligned} \dot{a}_{j\gamma}(r) &= 2 g_F(v_j(r), \dot{\gamma}(r)) & \qquad v_j(r) &= \mathcal{P}_\gamma^F(v_j) \\ a_{j\gamma}(0) &= 0 & \qquad (v_1, \ldots, v_n) &\text{ ON basis in } T_x F \end{aligned}$$

Solve the differential equation for a_γ.
Find the conditions under which the vectors a_γ generate the whole translation part.