Spheres with prescribed m-curvature in warped product manifolds

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UFC

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HIPOTHESIS 1: $\kappa(t) > 0$ for all $t \in (0, a)$ and decreasing.

Graphics over Σ in $\bar{M}$

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The second fundamental form of M is given by $\bar{A} = (\bar{a}_{ij})$ where

$$-\bar{\nabla}_{E_i} N = \sum \bar{a}_{ij} E_j \ .$$

The m-curvatures

The basic invariants associated to $\bar{A}$ are the m-curvatures S_m of M, $0 < m \leq n$, given by:

$$S_m = \sum_{1 \leq i_1 < \ldots < i_m \leq n} \det \bar{A}(i_1, \ldots, i_m)$$

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When $\bar{A}$ is diagonal, that is, $\bar{a}_{ij} = k_i\delta_{ij}$ then

$$S_m = \sum_{1\leq i_1<\ldots<i_m\leq n} k_{i_1} \ldots k_{i_m}$$

The Problem

Given a function $\psi : \bar{M} \to \mathbb{R}$, does exist a function $z : \Sigma \to (0, a)$ such that it satisfies the equation

$$(*) \quad \bar{S}_m(z(u), u) = \psi(z(u), u) \quad \text{for all} \quad u \in \Sigma$$

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When $\psi = \mathrm{const.}$, the problem is trivial. The solution will be one of the $\Sigma(t)$.

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c) $\partial\psi/\partial t + m\kappa\psi < 0$

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If we further assume that the curvature of $\bar{M}$ *is nonnegative than we can also exhibit an a priori* C^2 *bound for the solution.*

Hence, if the curvature of $\bar{M}$ *is nonnegative, and if we have the validity of the above hypothesis then there exists a solution of* $(*)$.

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Consider the set $T = \{s \in [0, 1];\ S_m(z(u), u) = \Psi(s, z(u), u)$ *has solution*$\}$ and show it is nonempty, open and closed.

Applying continuity method

For $s \in [0,1]$ define

$$\Psi(s,t,u) = s\psi(t,u) + (1-s)\phi(t)\binom{n}{m}\kappa(t)^m$$

where $\phi : (0,a) \to \mathbb{R}$ satisfies

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&\phi > 0 \\
&\phi(t) > 1 \quad \text{when} \quad t \le r_1 \\
&\phi(t) < 1 \quad \text{when} \quad t \ge r_2 \\
&\phi'(t) < 0
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Lemma: *the function* Ψ *satisfies the properties listed as hypothesis for* ψ.

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c) $\partial\Psi/\partial t + m\kappa(t)\Psi < 0$

Proof of the lemma

Since $\Psi(s,t,u) = s\psi(t,u) + (1-s)\phi(t)\binom{n}{m}\kappa(t)^m$ it is clear that:

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On the R.H.S., the first two terms are clearly negative. If the last one is not negative, it possible to choose the function ϕ in such way that the sum of the last two terms is also negative.

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Take $\phi(t) = e^{-b^2 t + c}$ choosing:
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Hence, the result is proved, that is, for each s the function Ψ satisfies the same set of hypothesis as ψ.

The continuity method

Since $\Psi(s,t,u) = s\psi(t,u) + (1-s)\phi(t)\binom{n}{m}\kappa(t)^m$, for $s=0$ and $t=t_0$, since $\phi(t_0)=1$, we have $\Psi = \binom{n}{m}k(t_0)^m$. Hence $z=t_0$ is solution of the equation $S_m = \Psi$.

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The closeness of T depends on the existence of C^0, C^1 and C^2 a priori estimates.

C^0 estimate

Assume there exists a solution $z(u)$ of the equation $(*)$. I claim that $r_1 \leq z(u) \leq r_2$ for each $u \in \Sigma$.

C^0 estimate

PROOF: Assume $\max(z(u)) = t_0 > r_2$.
Let $z(u_0) = t_0$. Then M is above of $\Sigma(t_0)$ and has a point of contact (t_0, u_0) with M.

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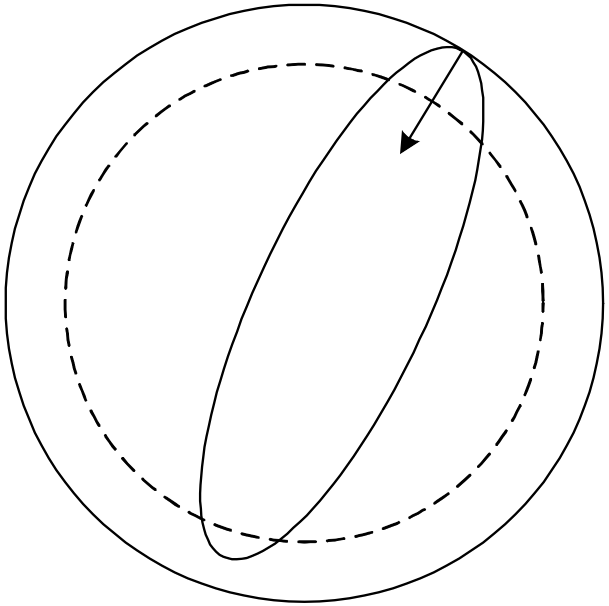

Since $\bar{S}_m = \psi < \binom{n}{m} k(t_0)^m$ by maximum principle we get a contradiction.

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