Causality and Boundary of wave solutions

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- Joint work with Miguel Sánchez: *Class. Quant. Grav.* **20** (2003); Preprint (2007)

Wave type spacetimes

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- ⋆ Conformally flat iff $\mu_{ij} = -\mu\delta_{ij}$.

Causal hierarchy

Globally hyperbolic

$\Downarrow$

Causally simple

$\Downarrow$

Causally continuous

$\Downarrow$

Stably causal

$\Downarrow$

Strongly causal

$\Downarrow$

Distinguishing

$\Downarrow$

Causal

$\Downarrow$

Chronological

$\Downarrow$

Non–totally vicious

Causal ladder

- Causality: no closed causal curves.

$$\Downarrow \quad \not\Uparrow$$

Causal ladder

- Distinguishing: no points with same past and future.
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Globally hyperbolic	$\leftarrow$sometimes
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Causally continuous	$\leftarrow$sometimes
Strongly causal	$\leftarrow$sometimes
Distinguishing	$\leftarrow$sometimes
Causal	$\leftarrow$always

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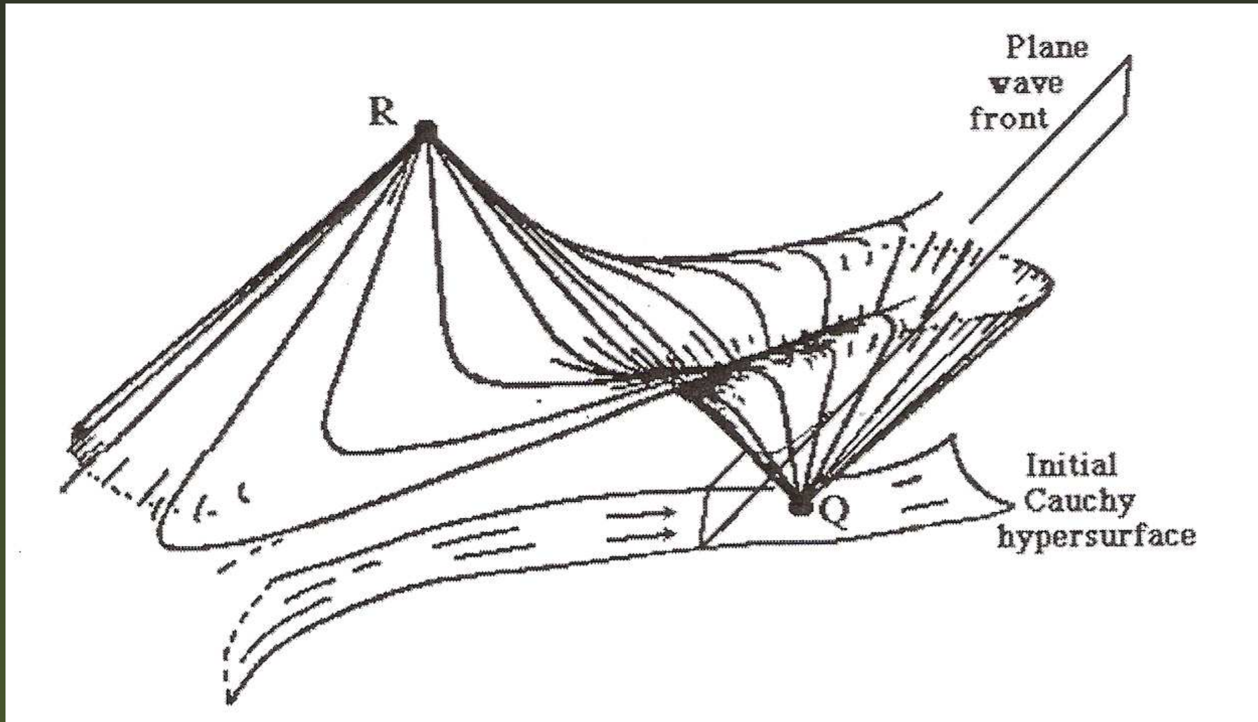
- But they are exact solutions to Einstein equations
- SCC: generic solutions to Einstein equations with reasonable matter and behavior at ∞ must be globally hyperbolic.
- The following question by Penrose becomes relevant:

  Is the bad causality of plane waves generic in some sense?
- The proof by Penrose lies on the exact symmetries of plane waves.



Focalization of null geodesics in plane waves

Reasonably generic waves

M p-waves

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$\star$ Involves very different mathematical tools.

Our functional

Fixed $x_0, x_1 \in M$, $u_0 \in \mathbb{R}$, $\Delta u > 0$ and $E(s) < 0$, put

$$\mathcal{J}(y) := \int_0^{\Delta u} (\langle \dot{y}, \dot{y} \rangle + H(y, u_0 + s) - E(s))ds$$

with domain $\mathcal{C}$ the curves $y : [0, \Delta u] \to M$ joining x_0, x_1.

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- Relates Δu, Δv along a causal curve $\gamma = (y, u, v)$ of energy E with $\mathrm{leng}(y)$.
- $\mathcal{J}$ contains deep information about the behavior of causal curves.
- The qualitative behavior of $\mathcal{J}$ can be analyzed in terms of $-H$.

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- superquadratic if $\exists \{x_n\}_n$ s.t. $d(x_n, \overline{x}) \to \infty$ and

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- at most quadratic if $\exists \; R_1(u), R_0(u) > 0$ s.t.

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- subquadratic if $\exists\ R_1(u), R_0(u) > 0$, s.t.

$$F(x, u) \leq R_1(u) d^{p(u)}(x, \overline{x}) + R_0(u) \quad p(u) < 2.$$

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(ii) M p-waves are globally hyperbolic if $-H$ is subquadratic (and M complete).

(iii) M p-waves are non-distinguishing if $-H$ is superquadratic.

$\star$ Very accurate result: examples of non-distinguishing M p-waves arbitrarily close to quadratic.

Conclusions

Causality becomes 'unstable' for $-H$ quadratic:

Global hyperbolic

Causally Symple

Causally continuous

Strongly causal $\leftarrow$ at most quadratic

Distinguishing

Causal

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- Deviations in the superquadratic direction may destroy strong causality.

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Causality becomes 'unstable' for $-H$ quadratic:

- Deviations in the superquadratic direction may destroy strong causality.
- Deviations in the subquadratic direction yield global hyperbolicity.

Global hyperbolic	$\leftarrow$subquadratic
Causally Symple	
Causally continuous	
Strongly causal	$\leftarrow$ <u>at most quadratic</u>
Distinguishing	
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Boundary of wave solutions

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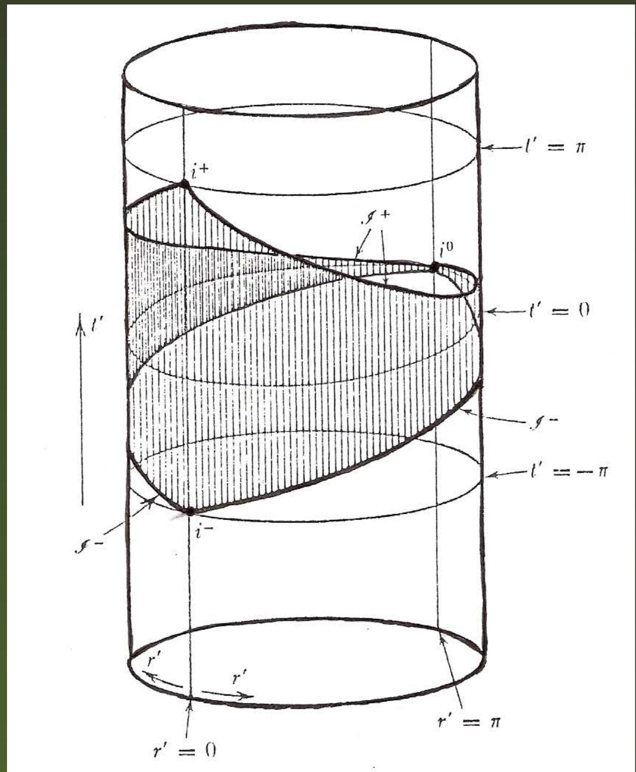

Minkowski into Einstein Static Universe

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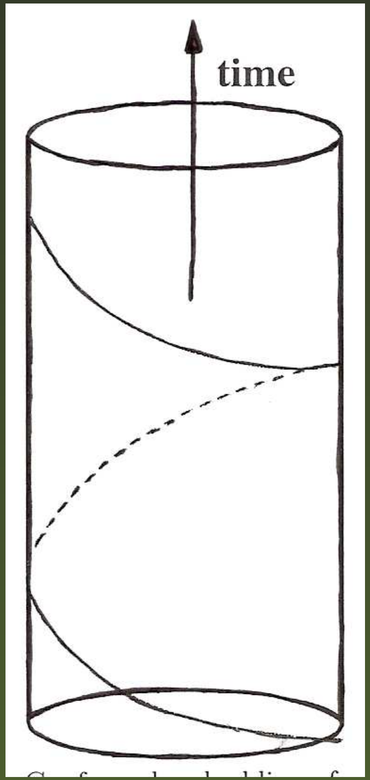
There are only two possibilities:

1. Homogeneous plane waves with $\mu_{ij} \equiv \mu\delta_{ij}$, $\mu > 0$
2. Homogeneous plane waves with $\mu_{ij} \equiv \mu\delta_{ij}$, $\mu < 0$

Case $\mu > 0$

In [BN'02] they found this explicit conformal embedding into ESU:



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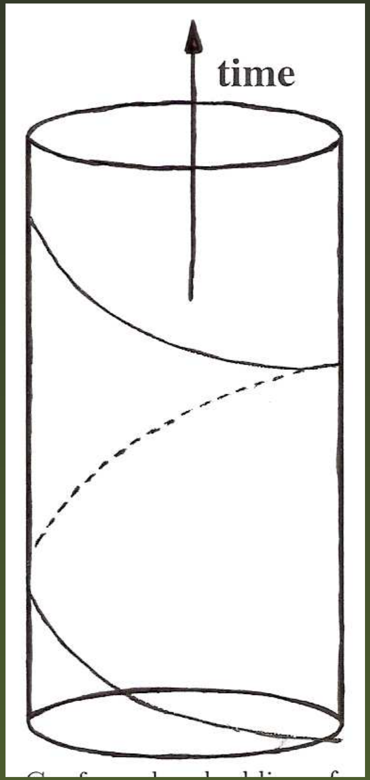
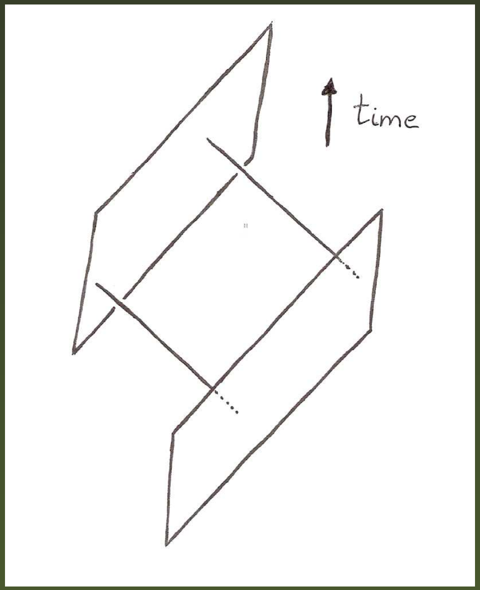


So, the conformal boundary is a 1-dimensional line which twists around the compact dimensions of ESU.

Case $\mu < 0$

In this case the conformal boundary is formed by two parallel null planes connected by two parallel null lines [MR'03].



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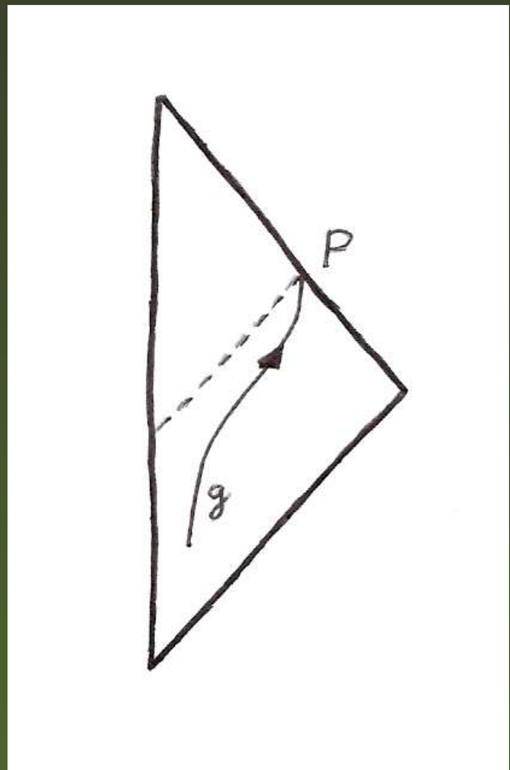

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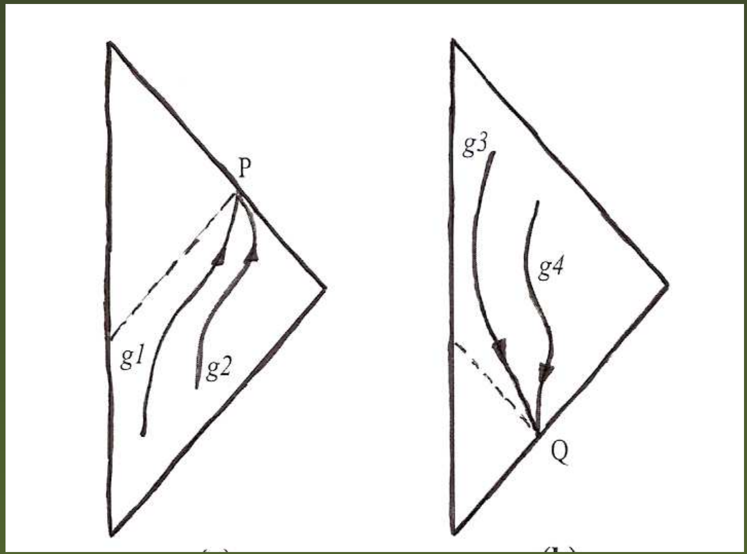

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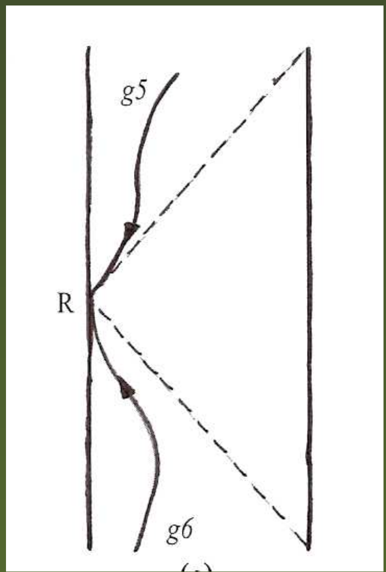

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- Identify some ideal points conveniently.
- The causal structure and topology of the spacetime can be extended to the boundary.

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$$f_{ij}(u) \to 0 \ \text{fast enough} \quad \Longrightarrow \text{dimension} \ > 1$$

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[HRa'03] also includes some discussion about the dimensionality of the boundary for pp-waves.

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- Needs solving, or analyzing in detail, solutions to differential equations involving H.
- Strategy only realizable for very special functions H.
- Conclusions on plane waves based on many particular examples.
- Few is said about the boundary of pp-waves.

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- The causal boundary construction does not require deep knowledge about geodesics.
- On the contrary, only the analysis of the rough behavior of causal curves is required.
- We only need to study the qualitative behavior of functional $\mathcal{J}$.
- Very general approach: applicable to *any* M p-wave.

More relevant growths

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- asymptotically quadratic if $\exists\, \lambda > \pi$, $R_0^- \in \mathbb{R}$ and $R_1(u), R_0(u) > 0$ s.t. for all $(x,u) \in M \times \mathbb{R}$

$$\frac{\lambda^2 d^2(x,\overline{x}) + R_0^-}{u^2+1} \le F(x,u) \le R_1(u) d^2(x,\overline{x}) + R_0(u)$$

Further relevant growths

- asymptotically quadratic if $\exists\ \lambda > \pi$, $R_0^- \in \mathbb{R}$ and $R_1(u), R_0(u) > 0$ s.t. for all $(x,u) \in M \times \mathbb{R}$

$$\frac{\lambda^2 d^2(x,\overline{x}) + R_0^-}{u^2+1} \leq F(x,u) \leq R_1(u)d^2(x,\overline{x}) + R_0(u)$$

- strongly subquadratic if $\exists\ \epsilon, R_1, R_0 > 0$ and $\delta \geq 0$ s.t.

$$F(x,u) \leq \frac{R_1 d^{2-\epsilon}(x,\overline{x}) + R_0}{|u|^\delta + 1} \quad \forall (x,u) \in M \times \mathbb{R}.$$

Our results

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- Reproduces previous results in [MR], [HRa].
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Teorema (–Sánchez): If M is complete and $|H|$ strongly subquadratic then the causal boundary coincides with that of the standard static spacetime

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Corollary: The causal boundary of any pp-wave with $|H|$ strongly subquadratic coincides with that of Minkowski, i.e. it is formed by a double cone with apexes i^+, i^- and base $\mathbb{S}^2$.

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- This fairly complicates the structure of the boundary, even though this situation is compatible with global hyperbolicity.
- Generic results are difficult here, but our technique can be exploited to describe the boundary for particular examples in these cases.

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- Is this generalizable to other spacetimes of physical interest?