A new Laplacian acting on tensor fields: potentials and Hodge decompositions.

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The classical Helmholtz theorem

Any vector field $\vec{V}$ on Euclidean space $\mathbb{R}^3$ can be written in terms of two potentials, scalar and vector respectively,

$$\vec{V} = \nabla\Phi - \nabla \times \vec{\Psi}$$

(of course, this is local or requires suitable boundary conditions).

(Physicist say: "any $\vec{V}$ is the sum of an irrotational part and a solenoidal part".)

The proof follows from the (local!, or with decaying assumptions at ∞) existence of a solution to the vector version of Poisson's equation

$$\nabla^2 \overset{o}{\vec{V}} = \vec{V}$$

for a "superpotential" vector field $\overset{o}{\vec{V}}$. Then, the potentials follow immediately from the vector operator identity

$$\vec{V} = \nabla^2 \overset{o}{\vec{V}} \equiv \nabla(\nabla\cdot \overset{o}{\vec{V}}) - \nabla \times (\nabla\times \overset{o}{\vec{V}}).$$

$\Longrightarrow$ Given suitable boundary conditions, any vector field can be fully reconstructed from its divergence and curl

Potentials for curvature tensors?

There is a well-known and largely exploited correspondence between electromagnetism and gravity:

$$F_{\mu\nu} \longleftrightarrow R^{\alpha}{}_{\beta\mu\nu}$$

$$A_{\mu} \longleftrightarrow \;??$$

A_μ has a fundamental GAUGE freedom: $\boldsymbol{A} \to \boldsymbol{A} + d\phi$.

Standard wisdom says that the connection $\Gamma^{\alpha}{}_{\beta\mu}$ plays the role of $\boldsymbol{A}$, and the gauge freedom is related to the choice of coordinates and/or bases. **BUT** the connection is not exactly the same, in the sense that $\boldsymbol{A}$ is a tensorial object.

Is it possible to define something analogous to $\boldsymbol{A}$ for the Riemann tensor?

The Lanczos potential

A (partial) answer was given by Lanczos in 1962. He proved (*) that, in Ricci-flat manifolds ($R_{ab} = 0$), the Riemann(=Weyl) tensor can always be written as:

$$C^{ab}{}_{cd} = 2L^{ab}{}_{[c;d]} + 2L_{cd}{}^{[a;b]} - 2\delta^{[a}_{[c}\left(L^{b]e}{}_{d];e} + L_{d]e}{}^{b];e}\right)$$

for a <u>tensor</u> potential L_{abc} with the properties

$$L_{abc} = L_{[ab]c}, \qquad L_{[abc]} = 0, \qquad L_{ab}{}^{b} = 0$$

Furthermore, the potential is affected by a gauge freedom so that

$$L_{abc}{}^{;c}$$

can be chosen at will (if this is set to zero, the gauge is called Lanczos' gauge).

However, and unfortunately, it is known that the Lanczos potential

- exists exclusively in 4 dimensions
- there is no such potential for the Riemann tensor when $R_{ab} \neq 0$.

Interest?

- Inspiration from electromagnetism.
- Definition of energy and momentum.
- Possibility of "massive gravitons", then the potential becomes dynamic.
- Quantization.
- Dealing with (probably) a simpler object.
- Gravitational entropy (in analogy with the entropy measure for quantum fields in flat space).
- Wave equation for the potential
- Finding a symmetric hyperbolic system for gravity in terms of the potential.
- Potentials for arbitrary tensors?
- Generalization of Helmholtz/Hodge decompositions.
- ...

De Rham-Hodge standard results for p-forms.

Λ = exterior algebra; Λ^p = set of exterior p-forms.

$\boldsymbol{\eta} \in \Lambda^n$ = the canonical volume element n-form ($\eta_{a_1 \dots a_n} = \eta_{[a_1 \dots a_n]}$.)

<u>Hodge dual operator</u> $* : \Lambda^p \longrightarrow \Lambda^{n-p}$ defined by

$$\overset{*}{\Sigma}_{a_{p+1} \dots a_n} \equiv \frac{1}{p!} \eta_{a_1 \dots a_n} \Sigma^{a_1 \dots a_p} \qquad \forall\, \Sigma \in \Lambda^p.$$

$$** \;= \epsilon (-1)^{p(n-p)}, \quad \epsilon = \pm 1 = \mathrm{sign}(\det(g_{ab}))$$

Scalar product $<, >$ on each Λ^p:

$$\forall \boldsymbol{\Sigma}, \boldsymbol{\Phi} \in \Lambda^p : \quad < \Sigma, \Phi > \equiv \int_{V_n} (\Sigma, \Phi)\, \boldsymbol{\eta} = p! \int_{V_n} \boldsymbol{\Sigma} \wedge \overset{*}{\boldsymbol{\Phi}} \ .$$

where $(\Sigma, \Phi) \equiv \Sigma_{a_1 \dots a_p} \Phi^{a_1 \dots a_p}$. (Compact/compact support).
This scalar product is bi-linear, symmetric and non-degenerate.

<u>Exterior differential</u> $d : \Lambda^p \longrightarrow \Lambda^{p+1}$

$$(d\Sigma)_{a_1 \dots a_{p+1}} \equiv (p+1) \nabla_{[a_1} \Sigma_{a_2 \dots a_{p+1}]} = (-1)^p (p+1) \Sigma_{[a_1 \dots a_p ; a_{p+1}]}$$

<u>Co-differential or divergence</u> $\delta : \Lambda^p \longrightarrow \Lambda^{p-1}$

$$\delta \equiv (-1)^p *^{-1} d * = \epsilon\, (-1)^{(n-p)(p-1)+1} * d *$$

$$(\delta \Sigma)_{a_2 \dots a_p} \equiv -\nabla^{a_1} \Sigma_{a_1 a_2 \dots a_p} = (-1)^p \Sigma_{a_2 \dots a_p a_1}{}^{;a_1}$$

- $d^2 \equiv 0$
- $\delta^2 \equiv 0$
- d and δ are mutually adjoint with respect to $<\,,\,>$:

$$< d\Sigma, \Gamma > = < \Sigma, \delta\Gamma > \quad \forall\ \Sigma \in \Lambda^p,\ \ \Gamma \in \Lambda^{p+1}$$

- Σ is *closed* if $d\Sigma = 0$ and *exact* if $\Sigma = d\Psi$.
- Σ is *co-closed* if $\delta\Sigma = 0$ and *co-exact* if $\Sigma = \delta\Gamma$.
- every statement on p-forms has a dual statement replacing d for δ and the form by its Hodge dual.
- The de Rham cohomology class of order p is defined as the quotient of the set of closed p-forms by the set of exact p-forms.
- The de Rham Laplacian operator $\Delta : \Lambda^p \longrightarrow \Lambda^p$:

$$\boxed{\Delta \equiv d\delta + \delta d}$$

With index notation:

$$(\Delta\Sigma)_{a_1\ldots a_p} = -\nabla^c\nabla_c\Sigma_{a_1\ldots a_p} + pR_{c[a_1}\Sigma^c{}_{a_2\ldots a_p]} - \frac{p(p-1)}{2}R_{cd[a_1a_2}\Sigma^{cd}{}_{a_3\ldots a_p]}\,.$$

- Δ is linear,
- Δ is self-adjoint with respect to $<,>$:

$$< \Delta\Sigma, \Phi > = < \Sigma, \Delta\Phi > \quad \forall\, \Sigma, \Phi \in \Lambda^p$$

- Δ commutes with $*$, d and δ:

$$*\Delta = \Delta *, \qquad d\Delta = \Delta d, \qquad \delta\Delta = \Delta\delta\,.$$

- $< \Sigma, \Delta\Sigma > = < d\Sigma, d\Sigma > + < \delta\Sigma, \delta\Sigma >$
- In *proper* Riemannian manifolds Δ is a positive operator: $< \Sigma, \Delta\Sigma > \geq 0$ for all $\Sigma \in \Lambda^p$

Result 1 (Helmholtz-Hodge local decomposition) *Given any p-form* $\Sigma \in \Lambda^p$ *there always exists a pair of local potentials* (Ψ, Γ) *with* $\Psi \in \Lambda^{p-1}$ *and* $\Gamma \in \Lambda^{p+1}$ *such that*

$$\Sigma = d\Psi + \delta\Gamma$$

Proof. Assuming analiticity, according to the Cauchy-Kovalewski theorem there always exists $\overset{o}{\Sigma} \in \Lambda^p$ such that $\Delta \overset{o}{\Sigma} = \Sigma$. The result then follows immediately from $\Delta \equiv d\delta + \delta d$ by setting $\Psi = \delta \overset{o}{\Sigma}$ and $\Gamma = d \overset{o}{\Sigma}$. □

Observe that this result is independent of any field equations for Σ, and is *local*. Obviously, the potentials have a very large gauge freedom.

Actually, the previous Result can be strengthened in some important cases:

Result 2 (Global Hodge decomposition theorem) *In the case of compact without boundary proper Riemannian manifolds, any* $\Sigma \in \Lambda^p$ *admits a* **unique** *global decomposition as*

$$\Sigma = d\Psi + \delta\Gamma + \Upsilon$$

where $\Psi \in \Lambda^{p-1}$, $\Gamma \in \Lambda^{p+1}$ *and* $\Upsilon \in \Lambda^p$ *is a* harmonic *p-form (that is,* Υ *is closed and co-closed, i.e.,* $\Delta\Upsilon = 0$*) in the same co-homology class as* Σ.

Generalization to arbitrary tensors?

The two crucial properties have been

i. the ability to identify a superpotential via a Laplace-like equation for the de Rham operator,

ii. and then being able to link potentials to derivatives of the superpotential, via the particular structure of the de Rham operator.

Unfortunately, the de Rham operator is defined only for differential forms.

The Lichnerowicz operator

In 1961 Lichnerowicz proposed a generalised Laplacian for arbitrary tensor fields:

$$(\Delta_L T)_{a_1\ldots a_m} \equiv -\nabla^c \nabla_c T_{a_1\ldots a_m} + \sum_{s=1}^{m} R^c{}_{a_s} T_{a_1\ldots a_{s-1} c a_{s+1}\ldots a_m}$$
$$- \sum_{s\neq t}^{m} R^c{}_{a_s}{}^d{}_{a_t} T_{a_1\ldots a_{s-1} c a_{s+1}\ldots a_{t-1} d a_{t+1}\ldots a_m}.$$

This operator has the following important properties:

- Δ_L respects the symmetry properties of $T_{a_1\ldots a_m}$.
- Δ_L commutes with traces.
- Δ_L is self-adjoint with respect to the scalar product $\{\,,\,\}$ defined by
$$\{T, S\} \equiv \int_{V_n} T_{a_1\ldots a_m} S^{a_1\ldots a_m} \boldsymbol{\eta}$$
for arbitrary $T, S \in T_m(V_n)$. Therefore $\{\Delta_L T, S\} = \{T, \Delta_L S\}$.
- If $\nabla_a R_{bc} = 0$ the following two properties hold:
  - when acting on rank-1 tensors, Δ_L commutes with the covariant derivative;
  - when acting on rank-2 tensors, Δ_L commutes with the divergence operator.
- *When acting on p-forms*, Δ_L coincides with the de Rham Laplacian Δ:
$$\Delta_L \Sigma = \Delta \Sigma, \qquad \forall\ \Sigma \in \Lambda^p\,.$$

$\boxed{\Delta_L \text{ does NOT have any links to first-derivative operators.}}$

The solution: tensors as r-fold forms!

Given any rank-m covariant tensor $T_{a_1 \dots a_m}$,

- there exists a *minimum* $r \in \mathbb{N}$, $r \leq m$
- and a unique set of r natural numbers $n_1, \dots, n_r \in \mathbb{N}$, with $\sum_{i=1}^{r} n_i = m$,
- such that $T_{a_1 \dots a_m}$ is a linear map on $\Lambda_{n_1} \times \cdots \times \Lambda_{n_r}$.

In other words: there always exists a minimum r such that

$$\tilde{T} \in \Lambda^{n_1} \otimes \cdots \otimes \Lambda^{n_r}$$

where $\tilde{T}_{a_1 \dots a_m}$ is the appropriate permuted version of $T_{a_1 \dots a_m}$ which selects the natural order for the $n_1, \dots, n_r$ entries.

Tensors seen in this way are called $\boxed{r\text{-fold } (n_1, \dots, n_r)\text{-forms.}}$

In semi-Riemannian manifolds (V_n, g) this extends to all tensors by means of the metric isomorphism between TV_n and T^*V_n (raising and lowering indices)

In short:

All tensors can be considered, in a precise way, as r-fold forms.

Definition 1 (Form-structure number and block ranks) *For any tensor T, the uniquely defined number r will be called its* form-structure number*, and each of the n_i, the* i-th block rank.

Examples

- Any p-form Σ is trivially a single (that is, 1-fold) p-form
- $\nabla\Sigma$ is a double $(1,p)$-form, with $r = 2$ and $n_1 = 1, n_2 = p$
- R_{abcd} has $r = 2$, $n_1 = n_2 = 2$: it is a double (2,2)-form which is *symmetric* (the pairs can be interchanged!)
- R_{ab} is a double symmetric (1,1)-form ($r = 2$, $n_1 = n_2 = 1$)
- $T_{a_1 \ldots a_r} = T_{(a_1 \ldots a_r)}$ is a symmetric r-fold (1,1,...,1)-form.
- A 3-tensor A_{abc} with the property $A_{abc} = -A_{cba}$ is a double (2,1)-form and the corresponding $\tilde{A}$ is given by $\tilde{A}_{abc} = \tilde{A}_{[ab]c} \equiv A_{acb}$.

(The standard index version of familiar tensors such as Riemann tensors $R_{abcd} = R_{[ab][cd]}$, Weyl tensors $C_{abcd} = C_{[ab][cd]}$, torsion tensors $T_{abc} = T_{[ab]c}$, or Lanczos tensors $L_{abc} = L_{[ab]c}$, already have the indices in the appropriate permuted version, so that they coincide with their tilded versions. In these cases we shall dispense with the ~ label.)

The new scalar product

For arbitrary tensors of the same type (i.e., with the same form-structure number r and block ranks):

$$< T, S > \equiv \int_{V_n} (T, S)\, \boldsymbol{\eta}$$

where now

$$(T, S) \equiv \tilde{T}_{a_1 \ldots a_m} \tilde{S}^{a_1 \ldots a_m}$$

Observe:

This scalar product is adapted to the structure as r-fold forms of the tensor fields T and S, and therefore it is different from the product defined before:

$$< T, S > = \{\tilde{T}, \tilde{S}\} \neq \{T, S\}$$

This scalar product is bi-linear, symmetric and non-degenerate.

(And positive-definite in proper Riemannian manifolds.)

The new operators

Let r be the form-structure number of T. Let us focus on the i-th block, with block rank n_i:

$$\tilde{T}^{a_1\ldots a_h}{}_{b_1\ldots b_{n_i}}{}^{a_{h+1}\ldots a_k} = \tilde{T}^{a_1\ldots a_h}{}_{[b_1\ldots b_{n_i}]}{}^{a_{h+1}\ldots a_k}$$

<u>The i-Hodge dual</u>: $*_{(i)}T$

$$(*_{(i)}T)^{a_1\ldots a_h}{}_{b_{n_i+1}\ldots b_n}{}^{a_{h+1}\ldots a_k} \equiv \frac{1}{n_i!}\eta_{b_1\ldots b_n}\tilde{T}^{a_1\ldots a_h b_1\ldots b_{n_i} a_{h+1}\ldots a_k}\,.$$

(As before, $*_{(i)}*_{(i)} = \epsilon(-1)^{n_i(n-n_i)}$ when acting on $\tilde{T}$.)

<u>The i-differential</u>: $d_{(i)}T$

$$(d_{(i)}T)^{a_1\ldots a_h}{}_{b_1\ldots b_{n_i+1}}{}^{a_{h+1}\ldots a_k} \equiv (-1)^{n_i}(n_i{+}1)\tilde{T}^{a_1\ldots a_h}{}_{[b_1\ldots b_{n_i}}{}^{a_{h+1}\ldots a_k}{}_{;b_{n_i+1}]}$$

<u>The i-codifferential</u>: $\delta_{(i)}T$

$$\delta_{(i)} = (-1)^{n_i} *_{(i)}^{-1} d_{(i)}*_{(i)} \equiv \epsilon\,(-1)^{(n-n_i)(n_i-1)+1} *_{(i)} d_{(i)}*_{(i)}$$

<u>The i-Laplacian</u>: $\Delta_{(i)}T$

$$\Delta_{(i)} \equiv d_{(i)}\delta_{(i)} + \delta_{(i)}d_{(i)}\,.$$

$$\boxed{\text{The covariant derivatives act on } \textit{all} \text{ indices.}}$$

- $(\delta_{(i)}T)^{a_1\ldots a_h}{}_{b_1\ldots b_{n_i-1}}{}^{a_{h+1}\ldots a_k} = -\tilde{T}^{a_1\ldots a_h}{}_{cb_1\ldots b_{n_i-1}}{}^{a_{h+1}\ldots a_k;c}$

  $$(\Delta_{(i)}T)^{a_1\ldots a_h}{}_{b_1\ldots b_{n_i}}{}^{a_{h+1}\ldots a_k} = -\nabla^c\nabla_c\tilde{T}^{a_1\ldots a_h}{}_{b_1\ldots b_{n_i}}{}^{a_{h+1}\ldots a_k}$$
  $$+n_i R_{c[b_1}\tilde{T}^{a_1\ldots a_h c}{}_{b_2\ldots b_{n_i}]}{}^{a_{h+1}\ldots a_k} - \frac{n_i(n_i-1)}{2}R_{cd[b_1 b_2}\tilde{T}^{a_1\ldots a_h cd}{}_{b_3\ldots b_{n_i}]}{}^{a_{h+1}\ldots a_k}$$
  $$-n_i\sum_{s=1}^{k} R_c{}^{a_i}{}_{d[b_1}\,\tilde{T}^{a_s\ldots a_{s-1}\,c\,a_{s+1}\ldots a_h d}{}_{b_2\ldots b_{n_i}]}{}^{a_{h+1}\ldots a_k}\,.$$

- The operators $d_{(i)}$ and $\delta_{(i)}$ are adjoint to each other with respect to $<,>$:

  $$< d_{(i)}T, U > = < T, \delta_{(i)}U >$$

  $[n_i(T) + 1 = n_i(U)]$.

- For each $i = 1, \ldots, r$, $\Delta_{(i)}T$ respects the skew-symmetry on the i-th antisymmetric block of T, and the symmetries and trace properties on the extra indices not in that block; this implies, in particular, that $\Delta_{(i)}T$ has the same form-structure number and block ranks as T.

- However, any *mixed trace*, or *mixed index symmetry*, involving indices from *both* the explicit i-th antisymmetric block and the rest of the indices is *not* preserved in general.

- For each $i = 1, \ldots, r$, $\Delta_{(i)}$ is self-adjoint : $< \Delta_{(i)}T, S > = < T, \Delta_{(i)}S >$.

  Moreover, one can prove the identities

  $$< T, \Delta_{(i)}T > = < d_{(i)}T, d_{(i)}T > + < \delta_{(i)}T, \delta_{(i)}T > \ \forall i \in \{1, \ldots, r\}$$

- When $r = 1$: $\Delta_{(1)}\Sigma = \Delta\Sigma = \Delta_L\Sigma$ for all $\Sigma \in \Lambda^p$.

i-Co-Homology?

- $d^2_{(i)} \neq 0$ in curved spaces

$$
\begin{aligned}
&(d^2_{(i)}T)^{a_1\ldots a_h}{}_{b_1\ldots b_{n_i+2}}{}^{a_{h+1}\ldots a_k} \\
&\qquad = \frac{1}{2}(n_i+1)(n_i+2)\sum_{s=1}^{k} R^{a_s}{}_{c[b_{n_i+1}b_{n_i+2}}\,\tilde{T}^{a_1\ldots a_{s-1}ca_{s+1}\ldots a_h}{}_{b_1\ldots b_{n_i}]}{}^{a_{h+1}\ldots a_k}\,,
\end{aligned}
$$

- $\delta^2_{(i)} \neq 0$ in curved spaces

$$
(\delta^2_{(i)}T)^{a_1\ldots a_h}{}_{b_1\ldots b_{n_i-2}}{}^{a_{h+1}\ldots a_k} = -\frac{1}{2}\sum_{s=1}^{k} R^{a_s}{}_{c}{}^{de}\,\tilde{T}^{a_1\ldots a_{s-1}ca_{s+1}\ldots a_h}{}_{deb_1\ldots b_{n_i-2}}{}^{a_{h+1}\ldots a_k}\,.
$$

- $d_{(i)}\Delta_{(i)} - \Delta_{(i)}d_{(i)} = d^2_{(i)}\delta_{(i)} - \delta_{(i)}d^2_{(i)}$
- $\delta_{(i)}\Delta_{(i)} - \Delta_{(i)}\delta_{(i)} = \delta^2_{(i)}d_{(i)} - d_{(i)}\delta^2_{(i)}$
- $*_{(j)}\ \Delta_{(i)} = \Delta_{(i)}\ *_{(j)} \qquad \forall i,j \in \{1,\ldots,r\}$

Definition 2 (i-harmonic and fully harmonic tensors) *A tensor field T with form-structure number r is said to be* i-harmonic, *for $i \in \{1,\ldots,r\}$, if and only if $\Delta_{(i)}T = 0$. Such a tensor will be called* fully harmonic *if it is i-harmonic* for all $i = 1,\ldots,r$.

Note that the harmonic property in the sense of Lichnerowicz (i.e., $\Delta_L T = 0$) is different from these new harmonic properties.

Nevertheless, in <u>FLAT</u> semi-Riemannian manifolds these operators are nilpotent with $d^2_{(i)} = 0$ and $\delta^2_{(i)} = 0$.

Furthermore, a generalised version of Poincaré's Lemma is valid in such flat manifolds. Namely:

Result 3 (Poincaré for r-fold forms in absence of curvature) *Let (V_n, g) be an n-dimensional semi-Riemannian manifold of any signature and zero curvature. Then, for any $x \in V_n$ there is a neighbourhood $U(x)$ of x such that, for any tensor field T and for any i,*

$$d_{(i)}T = 0 \quad \Rightarrow \quad T = d_{(i)}A$$

on $U(x)$, where A has the same form structure number than T and $n_i(T) = n_i(A) + 1$.

Corollary 1 *Under the same hypothesis,*

$$\delta_{(i)}T = 0 \quad \Rightarrow \quad T = \delta_{(i)}B$$

on $U(x)$, where B has the same form structure number than T and $n_i(T) = n_i(B) - 1$.

- $d_{(i)}$ ($\delta_{(i)}$) produces another tensor with one more (one less) index in general, but
- there are some special situations:
  - if $n_i = n$ then $d_{(i)}T = 0$
  - for any tensor T with form structure number r, ∇T has form-structure number $r+1$ and can be considered as a definition of '$d_{(r+1)}T$'
  - if $n_i = 1$ then $\delta_{(i)}T$ has $r-1$ as form-structure number. In this case, in order to compute $\Delta_{(i)}$ one has to allow the operator $d_{(i)}$ in the combination $d_{(i)}\delta_{(i)}$ to act on the missing block

    [as if $\delta_{(i)}T$ were an r-fold $(n_1, \dots, n_{i-1}, 0, n_{i+1} \dots n_r)$-form, in the same way as in (ii).]
- thus, for $\ell \neq 1, \dots, r$
  - $d_{(\ell)}T = \widehat{\nabla \tilde{T}}\ , \qquad \delta_{(\ell)}T = 0, \qquad \ell \notin \{1, \dots, r\}$
  - $(\Delta_{(\ell)}T)_{a_1 \dots a_m} = -\nabla^c \nabla_c \tilde{T}_{a_1 \dots a_m}$

where in the first case the $\widehat{\ }$ means that the extra index provided by the covariant derivative must be placed in the appropriate place within $\{1, \dots, r+1\}$.

A weighted de Rham operator and associated potentials.

Theorem 1 *The operator $\bar{\Delta}$ given by*

$$\boxed{\bar{\Delta} \equiv \tfrac{1}{r}(\Delta_{(1)} + \Delta_{(2)} + \cdots + \Delta_{(r)}) = \tfrac{1}{r}\textstyle\sum_{i=1}^{r} \Delta_{(i)}}$$

i. is linear

ii. self-adjoint with respect to $< , >$

iii. respects all index symmetry properties

iv. commutes with all trace operations

v. It is related to the Lichnerowicz operator by

$$\bar{\Delta} = \frac{1}{r}\Delta_L - \frac{r-1}{r}\nabla^c\nabla_c \ .$$

Of course, for single p-forms we have

$$\bar{\Delta}\Sigma = \Delta\Sigma = \Delta_L\Sigma, \ \ \forall\ \Sigma \in \Lambda^p \, .$$

Important:

An extremely important consequence of the new operator $\bar{\Delta}$ is that for any given tensor field T, there exists an *associated* superpotential $\overset{o}{T}$, by which we mean a superpotential, not just with the same form-structure number and block ranks, but also *with the same index symmetries and trace properties* as T.

Crucial:

$\bar{\Delta}$ has the useful properties of Δ_L and, in addition, has direct links with $d_{(i)}$ and $\delta_{(i)}$.

Thus, using

$$\bar{\Delta}\,\overset{o}{T} \equiv \frac{1}{r}\sum_{i=1}^{r}[\delta_{(i)}(d_{(i)}\,\overset{o}{T}) + d_{(i)}(\delta_{(i)}\,\overset{o}{T})]\,.$$

we get

Theorem 2 *Given any tensor field T with form-structure number r, there always exists a set of $2r$* local *potentials $(Y_{(i)}, Z_{(i)})$, $i = 1, 2, \ldots r$, such that*

$$T = \frac{1}{r}\sum_{i=1}^{r}(\delta_{(i)}Y_{(i)} + d_{(i)}Z_{(i)})$$

where $Y_{(i)} = d_{(i)}\,\overset{o}{T}$ and $Z_{(i)} = \delta_{(i)}\,\overset{o}{T}$ are the potentials.

Harmonic tensors

Definition 3 *A tensor field T will be called* harmonic *if and only if*

$$\boxed{\boxed{\bar{\Delta} T = 0}}\,.$$

Obviously, any fully harmonic tensor is trivially harmonic. The converse, however, does not hold in general. Nevertheless,

$$< T, \bar{\Delta} T > = \frac{1}{r} \sum_{i=1}^{r} < T, \Delta_{(i)} T > =$$

$$\frac{1}{r} \sum_{i=1}^{r} \left(< d_{(i)} T, d_{(i)} T > + < \delta_{(i)} T, \delta_{(i)} T > \right)$$

for arbitrary tensor fields.

Therefore, it is straightforward to obtain the following converse in proper Riemannian manifolds.

Theorem 3 *Let (V_n, g) be a compact without boundary proper Riemannian manifold. Then, a tensor T is harmonic if and only if it is fully harmonic, and if and only if*

$$d_{(i)} T = 0, \qquad \delta_{(i)} T = 0, \qquad \forall\ i \in \{1, \ldots, r\}\,.$$

(As usual, the compactness can be replaced by appropriate decaying properties at infinity.)

Hodge decompositions (in flat manifolds)

Combining the previous results, it seems very plausible that one can easily prove the following generalization of the global Hodge decomposition theorem:

Conjecture 1 *Let (V_n, g) be a compact without boundary* <u>*flat*</u> *proper Riemannian manifold (of any topology). Any tensor field T, whose form structure number is r, admits r orthogonal (unique) global decompositions*

$$T = d_{(i)}A + \delta_{(i)}B + H_i$$

where A is i-closed, B is i-co-closed, and H_i is i-harmonic.

(One can also speak of i-cohomology classes.)

The complete proof of these conjectures requires some technical details, concerning the continuity of $(\Delta_{(i)} + Id)^{-1}$, and the finite dimensionality of the set of i-harmonic tensors.

More ambitious: is it possible to combine all these decompositions into a unique one?

In other words, is it possible to prove a unique decomposition of type

$$T = H + \frac{1}{r}\sum_{i=1}^{r}(\delta_{(i)}Y_{(i)} + d_{(i)}Z_{(i)})$$

where H is harmonic $\bar{\Delta}H = 0$ (ergo fully harmonic)?

Back to the general case: local potentials

The case of double (q,p)-forms

$$\tilde{T}^{a_1\dots a_q}{}_{b_1\dots b_p} = \tilde{T}^{[a_1\dots a_q]}{}_{[b_1\dots b_p]}.$$

$$(\Delta_{(1)}T)^{a_1\dots a_q}{}_{b_1\dots b_p} = -\nabla^c\nabla_c\tilde{T}^{a_1\dots a_q}{}_{b_1\dots b_p} + qR^{c[a_1}\tilde{T}_c{}^{a_2\dots a_q]}{}_{b_1\dots b_p}$$
$$-\frac{q(q-1)}{2}R^{cd[a_1a_2}\tilde{T}_{cd}{}^{a_3\dots a_q]}{}_{b_1\dots b_p} + (-1)^{q-1}qp\,R^{[a_1}{}_{cd[b_1}\tilde{T}^{a_2\dots a_q]cd}{}_{b_2\dots b_p]}$$

$$(\Delta_{(2)}T)^{a_1\dots a_q}{}_{b_1\dots b_p} = -\nabla^c\nabla_c\tilde{T}^{a_1\dots a_q}{}_{b_1\dots b_p} + pR_{c[b_1}\tilde{T}^{a_1\dots a_q c}{}_{b_2\dots b_p]}$$
$$-\frac{p(p-1)}{2}R_{cd[b_1b_2}\tilde{T}^{a_1\dots a_q cd}{}_{b_3\dots b_p]} + (-1)^{q-1}pqR^{[a_1}{}_{cd[b_1}\tilde{T}^{a_2\dots a_q]cd}{}_{b_2\dots b_p]}$$

$$(\bar{\Delta}T)^{a_1\dots a_q}{}_{b_1\dots b_p} = -\nabla^c\nabla_c\tilde{T}^{a_1\dots a_q}{}_{b_1\dots b_p} + \frac{q}{2}R^{c[a_1}\tilde{T}_c{}^{a_2\dots a_q]}{}_{b_1\dots b_p}$$
$$+\frac{p}{2}R_{c[b_1}\tilde{T}^{a_1\dots a_q c}{}_{b_2\dots b_p]} - \frac{q(q-1)}{4}R^{cd[a_1a_2}\tilde{T}_{cd}{}^{a_3\dots a_q]}{}_{b_1\dots b_p}$$
$$-\frac{p(p-1)}{4}R_{cd[b_1b_2}\tilde{T}^{a_1\dots a_q cd}{}_{b_3\dots b_p]} + (-1)^{q-1}qpR^{[a_1}{}_{cd[b_1}\tilde{T}^{a_2\dots a_q]cd}{}_{b_2\dots b_p]}$$

$$\bar{\Delta} = \frac{1}{2}(\Delta_L - \nabla_c\nabla^c).$$

Specialising the previous Theorem we obtain,

Corollary 2 *Given any tensor field T with the structure of a double (q, p)-form there always exist local potentials $Y_{(1)}, Y_{(2)}, Z_{(1)}, Z_{(2)}$ such that*

$$T = \frac{1}{2}\left(\delta_{(1)}Y_{(1)} + \delta_{(2)}Y_{(2)} + d_{(1)}Z_{(1)} + d_{(2)}Z_{(2)}\right).$$

Traces and transposes for double forms

- The *trace* of a double (q,p)-form T is the double $(q-1,p-1)$-form $tr(T)$ given by

$$(\widetilde{tr(T)})^{a_2\ldots a_q}{}_{b_2\ldots b_p} \equiv \tilde{T}^{ca_2\ldots a_q}{}_{cb_2\ldots b_p}$$

- We must remark that if $q=1$ (or $p=1$), then the first (second) block disappears after taking the trace, so that the resulting tensor has a form-structure number less than 2. In these situations sometimes it is necessary to consider the resulting tensor $\mathrm{tr}(T)$ as an equivalent double $(0,p-1)$-form (or double $(q-1,0)$-form).
- $\mathrm{tr}(d_{(1)}T) = -d_{(1)}\mathrm{tr}(T) - \delta_{(2)}T$
- $\mathrm{tr}(d_{(2)}T) = -d_{(2)}\mathrm{tr}(T) - \delta_{(1)}T$
- $\mathrm{tr}(\delta_{(1)}T) = -\delta_{(1)}\mathrm{tr}(T) \;\; (q \geq 2)$
- $\mathrm{tr}(\delta_{(2)}T) = -\delta_{(2)}\mathrm{tr}(T) \;\; (p \geq 2)$
- The *(generalised) transpose* ${}^tT$ of a double (q,p)-form T is the double (p,q)-form given by interchange of the blocks:

$$(\widetilde{{}^tT})^{a_1\ldots a_p}{}_{b_1\ldots b_q} \equiv \tilde{T}_{b_1\ldots b_q}{}^{a_1\ldots a_p}.$$

- ${}^{tt}T = T$
- $d_{(2)}({}^tT) = {}^t(d_{(1)}T)$
- $\delta_{(2)}({}^tT) = {}^t(\delta_{(1)}T)$
- ${}^t(\mathrm{tr}(T)) = \mathrm{tr}({}^tT)$

For the case of double (q,p)-forms, a straightforward computation provides the commutation properties of the operators $d_{(i)}$ and $\delta_{(j)}$ for $i,j \in \{1,2\}$

$$\left([d_{(1)}, d_{(2)}]T\right)^{a_1\ldots a_{q+1}}{}_{b_1\ldots b_{p+1}} = \frac{(-1)^{p+q}}{2}(p+1)(q+1) \times$$
$$\left(qR_{c[b_{p+1}}{}^{[a_q a_{q+1}}\tilde{T}^{a_1\ldots a_{q-1}]c}{}_{b_1\ldots b_p]} - pR^{c[a_{q+1}}{}_{[b_p b_{p+1}}\tilde{T}^{a_1\ldots a_q]}{}_{b_1\ldots b_{p-1}]c}\right),$$

$$\left([d_{(1)}, \delta_{(2)}]T\right)^{a_1\ldots a_{q+1}}{}_{b_1\ldots b_{p-1}} = (-1)^q(q+1)\left(\frac{q}{2}R^c{}_d{}^{[a_q a_{q+1}}\tilde{T}^{a_1\ldots a_{q-1}]d}{}_{cb_1\ldots b_{p-1}}\right.$$
$$\left.+\frac{p-1}{2}R^{cd}{}_{[b_1}{}^{[a_{q+1}}\tilde{T}^{a_1\ldots a_q]}{}_{b_2\ldots b_{p-1}]cd} + R^{d[a_{q+1}}\tilde{T}^{a_1\ldots a_q]}{}_{db_1\ldots b_{p-1}}\right),$$

$$\left([\delta_{(1)}, \delta_{(2)}]T\right)^{a_1\ldots a_{q-1}}{}_{b_1\ldots b_{p-1}} = \frac{p-1}{2}R^{ce}{}_{d[b_1}\tilde{T}^{da_1\ldots a_{q-1}}{}_{b_2\ldots b_{p-1}]ce}$$
$$-\frac{q-1}{2}R_{ce}{}^{d[a_1}\tilde{T}^{a_2\ldots a_{q-1}]ce}{}_{db_1\ldots b_{p-1}}$$

(Observe again, that in flat space these operators commute.)

Double (p,p)-forms: Curvature tensors

The transpose ${}^tT$ of T is of special relevance for the special case of the double (p,p)-forms

Definition 4 (Symmetric and antisymmetric double forms) *A double (p,p)-form is* symmetric *if $T = {}^tT$, and* antisymmetric *if $T = -{}^tT$ (in this case only for $p > 1$).*

Of course, for $p > 1$ any double (p,p)-form can be decomposed uniquely into a symmetric and an antisymmetric one. Hence, without loss of generality, in what follows we will only consider these two cases,

$$T = \pm{}^tT \qquad \text{or} \qquad \tilde{T}^{a_1\ldots a_p}{}_{b_1\ldots b_p} = \pm\tilde{T}_{b_1\ldots b_p}{}^{a_1\ldots a_p}\,.$$

Then it follows trivially

$$d_{(2)}T = \pm\,{}^t(d_{(1)}T), \qquad \delta_{(2)}T = \pm\,{}^t(\delta_{(1)}T)$$

and therefore $\Delta_{(2)}T = \pm\,{}^t(\Delta_{(1)}T)$ so that

$$\bar{\Delta}T = \frac{1}{2}\left(\Delta_{(2)}T \pm {}^t(\Delta_{(2)}T)\right).$$

Since the associated superpotential $\overset{o}{T}$ will have the same symmetry properties as T, it follows that the four potentials $Y_{(1)}$, $Y_{(2)}$, $Z_{(1)}$ and $Z_{(2)}$ defined before satisfy

$$\begin{aligned} Y_{(2)} &= \pm{}^t Y_{(1)} \equiv (-1)^{p+1} Y, \\ Z_{(2)} &= \pm{}^t Z_{(1)} \equiv (-1)^{p-1} Z\,, \end{aligned}$$

Furthermore the completely antisymmetric part $\mathcal{A}[Y_{(2)}]$ of $Y_{(2)}$ (or Y) vanishes identically in the next cases

$$Y_{[a_1\dots a_p b_1\dots b_{p+1}]} = 0 \text{ for } \left\{ \begin{array}{ll} T = {}^t T & p \text{ odd,} \\ T = -{}^t T & p \text{ even.} \end{array} \right. \tag{1}$$

Theorem 4 *Given any tensor T with the structure of a double (anti)symmetric (p,p)-form there always exist a* PAIR *of local potentials $Y_{(2)}, Z_{(2)}$ satisfying (1) such that*

$$T = \frac{1}{2}\left[\delta_{(2)} Y_{(2)} \pm {}^t(\delta_{(2)} Y_{(2)}) + d_{(2)} Z_{(2)} \pm {}^t(d_{(2)} Z_{(2)})\right].$$

In index notation

$$\begin{aligned} \tilde{T}^{a_1\dots a_p}{}_{b_1\dots b_p} = \frac{1}{2}\Big(& Y^{a_1\dots a_p}{}_{b_1\dots b_p c}{}^{;c} \pm Y_{b_1\dots b_p}{}^{a_1\dots a_p c}{}_{;c} \\ & + pZ^{a_1\dots a_p}{}_{[b_1\dots b_{p-1};b_p]} \pm pZ_{b_1\dots b_p}{}^{[a_1\dots a_{p-1};a_p]}\Big) \end{aligned}$$

Traceless double (p,p)-forms

Suppose now that, in addition to the (anti)symmetry between blocks, the double (p,p)-form T is *traceless*, i.e.,

$$\mathrm{tr}(T) = 0\,.$$

Then the potentials $Y_{(2)}$ and $Z_{(2)}$ *are not independent*

$$Z_{(2)} = \mp\ \mathrm{tr}({}^tY_{(2)})$$

or with indices

$$Z^{a_1\ldots a_p}{}_{b_1\ldots b_{p-1}} = \mp\, Y_{cb_1\ldots b_{p-1}}{}^{ca_1\ldots a_p}\,.$$

Furthermore

$$\mathrm{tr}(\mathrm{tr}(Y_{(2)})) = 0$$

which becomes in index notation

$$Y_{cdb_1\ldots b_{p-2}}{}^{cda_1\ldots a_{p-1}} = 0\,. \tag{2}$$

Theorem 5 *Given any tensor T with the structure of a double (anti)symmetric traceless (p,p)-form there always exists a double $(p,p{+}1)$-form local potential $Y_{(2)}$ satisfying (1) and $\mathrm{tr}(\mathrm{tr}(Y_{(2)})) =$ 0 such that*

$$T = \frac{1}{2}\left(\delta_{(2)}Y_{(2)} \pm {}^t(\delta_{(2)}Y_{(2)}) - d_{(1)}\mathrm{tr}(Y_{(2)}) \mp {}^t(d_{(1)}\mathrm{tr}(Y_{(2)}))\right),$$

The index version is

$$\begin{aligned}\tilde{T}^{a_1\ldots a_p}{}_{b_1\ldots b_p} \;&=\; \frac{1}{2}\Big(Y^{a_1\ldots a_p}{}_{b_1\ldots b_p c}{}^{;c} \pm Y_{b_1\ldots b_p}{}^{a_1\ldots a_p c}{}_{;c}\\ &\qquad -pY^{c[a_1\ldots a_{p-1}}{}_{cb_1\ldots b_p}{}^{;a_p]} \mp pY_{c[b_1\ldots b_{p-1}}{}^{ca_1\ldots a_p}{}_{;b_p]}\Big)\end{aligned}$$

where the potential Y satisfies (1) and (2).

Symmetric rank-2 tensors

$$T_{ab} \;=\; \frac{1}{2}\left(Y_{abc}{}^{;c} + Y_{bac}{}^{;c} + Z_{a;b} + Z_{b;a}\right).$$

For a *traceless* symmetric 2-tensor $(T^a{}_a = 0)$

$$T_{ab} \;=\; \frac{1}{2}\left(Y_{abc}{}^{;c} + Y_{bac}{}^{;c} - Y^c{}_{ca;b} - Y^c{}_{cb;a}\right).$$

In both cases the double $(1,2)$-form $Y^a{}_{bc} = Y^a{}_{[bc]}$ satisfies $Y_{[abc]} = 0$.

[There is a well known decomposition for symmetric rank-2 tensors in *three* dimensional spaces, but it is restricted to *proper Riemannian space* with the further condition of being either compact or asymptotically Euclidean. The decomposition coincides with the previous one with the added desirable properties that (i) it is unique and (ii) the part $Y_{(ab)c}{}^{;c}$ is divergence-free. It seems plausible that, using the same kind of techniques, some of our more general decompositions, which are valid in arbitrary dimension and signature, can be enforced to be unique in the case of positive-definite metric for either of the mentioned cases: compact manifold or asymptotic flatness.]

Application to general curvature tensors

Let $\mathcal{R}_{abcd}$ be a *Riemann candidate*, that is to say

$$\mathcal{R}_{abcd} = \mathcal{R}_{[ab][cd]}, \qquad \mathcal{R}_{a[bcd]} = 0 \ (\Longrightarrow \ \mathcal{R}_{abcd} = \mathcal{R}_{cdab}),$$

so that $\mathcal{R}_{abcd}$ is in particular a *symmetric* double (2,2)-form.

Given the additional cyclic symmetry property, its potentials satisfy the additional symmetries

$$Y_{a[bcde]} = 0, \qquad Z_{[abc]} = 0, \tag{3}$$

the first of which implies the following useful properties

$Y^{e}{}_{[bcd]e} = 0$, $Y_{[abcd]e} = 0$, $Y_{abcde} = 3Y_{[cde]ab} = 3Y_{a[cde]b}$, $Y_{a[bc]de} = -Y_{a[de]bc}$.

Theorem 6 *Any Riemann candidate tensor $\mathcal{R}^{ab}{}_{cd}$ has a pair of local potentials given by a double $(2,3)$-form $Y^{ab}{}_{cde}$ and a double (2,1)-form $Z^{ab}{}_{c}$ with the properties (3) such that*

$$\mathcal{R}_{abcd} = \frac{1}{2}\left(Y_{abcde}{}^{;e} + Y_{cdabe}{}^{;e} + 2Z_{ab[c;d]} + 2Z_{cd[a;b]}\right) .$$

Let us consider now a *Weyl candidate*, that is, a double (2,2)-form $\mathcal{C}_{abcd} = \mathcal{C}_{[ab]cd} = \mathcal{C}_{ab[cd]}$ with the algebraic properties of the Weyl conformal curvature tensor:

$$\mathcal{C}^{a}{}_{bca} = 0, \quad \mathcal{C}_{a[bcd]} = 0, \ (\Longrightarrow \ \mathcal{C}_{abcd} = \mathcal{C}_{cdab}),$$

so that $\mathcal{C}_{abcd}$ is in particular a *traceless and symmetric* double (2,2)-form.

Theorem 7 *Any Weyl candidate tensor field* $\mathcal{C}_{abcd}$ *has a double* $(2,3)$*-form local potential* P_{abcde} *with the properties*

$$P_{a[bcde]} = 0, \qquad P^{ab}{}_{abc} = 0$$

such that

$$\mathcal{C}^{ab}{}_{cd} = \frac{1}{2}\left(P^{ab}{}_{cde}{}^{;e} + P_{cd}{}^{abe}{}_{;e} - 2P_{e[c}{}^{abe}{}_{;d]} - 2P^{e[a}{}_{cde}{}^{;b]}\right) .$$

Immediate consequences are the following useful properties,

$$P^{e}{}_{[bcd]e} = 0,\ P_{[abcd]e} = 0,\ P^{ab}{}_{cde} = 3P_{[cde]}{}^{ab} = 3P^{[a}{}_{[cde]}{}^{b]},\ P_{a[bc]de} = -P_{a[de]bc} .$$

Number of independent components of the potential:

$$(n+2)n(n-3)(n^2-n+4)/24$$

(16 if $n=4$, 70 if $n=5$).

This is LARGER than the number of independent components of a Weyl candidate:

$$(n+2)(n+1)n(n-3)/12$$

(that is, 10 if $n=4$, 35 if $n=5$).

It is also larger (equal, in the case $n=4$) than the number of independent Ricci rotation coefficients, or of independent components of the connection in a given basis.

$$\boxed{\Longrightarrow \quad \text{GAUGE}}$$

$\boxed{\text{What about the Lanczos potential in } n = 4?}$

- Observe that a double (2,3)-form is equivalent, via dualization with the Hodge * operator, to a double (2,1)-form in $n = 4$.
- Also, for any *traceless* double (2,2)-form:

$$(*W*)_{abcd} \equiv \frac{1}{4}\eta_{abef}\eta_{cdgh}W^{efgh} \quad \Longrightarrow \quad (*W*)_{abcd} = \epsilon\, W_{abcd}\,.$$

- Similarly, for any double(2,3)-form P_{abcde}:

$$(*P*)_{abc} \equiv \frac{1}{12}\eta_{abef}\eta_{dghc}P^{efdgh} \Longrightarrow$$

$$P^{ab}{}_{cde} = 6\epsilon\,(*P*)^{[a}{}_{[cd}\delta^{b]}_{e]}, \;\; P^{i}{}_{cabi} = \epsilon\,(*P*)_{abc}.$$

- Observe that the symmetry and trace properties of P translate for the double dual into

$$(*P*)_{[abc]} = 0, \qquad (*P*)_{ab}{}^{b} = 0$$

  which are the Lanczos potential properties exactly!

Hence, by taking the double dual of in the basic formula <u>in four dimensions</u>

$$\epsilon\, \mathcal{C}^{ab}{}_{cd} = 2(*P*)^{ab}{}_{[c;d]} + 2(*P*)_{cd}{}^{[a;b]} - 2\delta^{[a}_{[c}\left((*P*)^{b]e}{}_{d];e} + (*P*)_{d]e}{}^{b];e}\right)$$

so that

$$\boxed{L_{abc} = \epsilon(*P*)_{abc} \iff P_{abcde} = \epsilon\,(*L*)_{abcde}.}$$

$$\boxed{L = {}^{t}(\mathrm{tr}P),\text{ that is, } L_{abc} = P^{e}{}_{cabe}}$$

Case of Lorentzian manifolds

* Symmetric hyperbolic systems.

Take the following set of equations

$$d_{(1)}A = (s+1)J, \qquad \delta_{(1)}A = -j$$

for a tensor field A (here the first block $_{(1)}$ is used for the sake of simplicity, without loss of generality; and we write $s = n_1$). With index notation this reads

$$\nabla_{[\mu_0}A_{\mu_1\dots\mu_s]\mu_{s+1}\dots\mu_m} = J_{\mu_0\dots\mu_m},\ \nabla^\rho A_{[\rho\mu_2\dots\mu_s]\mu_{s+1}\dots\mu_m} = j_{\mu_2\dots\mu_m}$$

These can be "hyperbolized" as

$$Q^{\alpha\ \sigma_1\dots\sigma_s\,\rho_1\dots\rho_{n_2}\dots\tau_1\dots\tau_{n_r}}_{\quad\gamma_1\dots\gamma_s\,\epsilon_1\dots\epsilon_{n_2}\dots\zeta_1\dots\zeta_{n_r}}\nabla_\alpha A_{\sigma_1\dots\sigma_s\rho_1\dots\rho_{n_2}\dots\tau_1\dots\tau_{n_r}} = \mathcal{J}_{\gamma_1\dots\gamma_s\epsilon_1\dots\epsilon_{n_2}\dots\zeta_1\dots\zeta_{n_r}}$$

where the vector-valued matrices $\boldsymbol{Q}$ (endomorphisms acting on the set of r-fold $(s,\dots,n_r)$-forms) are defined by

$$Q^{\alpha\ \sigma_1\dots\sigma_s\dots\tau_1\dots\tau_{n_r}}_{\quad\rho_1\dots\rho_s\dots\nu_1\dots\nu_{n_r}} \equiv E^{\alpha}{}_{\mu_1\lambda_2\mu_2\dots\lambda_r\mu_r}{}^{[\sigma_1\dots\sigma_s]\dots[\tau_1\dots\tau_{n_r}]}_{[\rho_1\dots\rho_s]\dots[\nu_1\dots\nu_{n_r}]}v^{\mu_1}u_2^{\lambda_2}v_2^{\mu_2}\dots u_r^{\lambda_r}v_r^{\mu_r}$$

for arbitrary timelike future-directed vectors $\{v^{\mu_1}, u_2^{\lambda_2}, v_2^{\mu_2}, \dots, u_r^{\lambda_r}, v_r^{\mu_r}\}$, and with

$$E_{\lambda_1\mu_1\dots\lambda_r\mu_r}{}^{\sigma_1\dots\sigma_{n_1}\dots\tau_1\dots\tau_{n_r}}_{\rho_1\dots\rho_{n_1}\dots\nu_1\dots\nu_{n_r}} \equiv \frac{1}{(n_1-1)!}\delta^{\sigma_2\dots\sigma_{n_1}}_{\rho_2\dots\rho_{n_1}}\left(2\delta^{\sigma_1}_{(\lambda_1}g_{\mu_1)\rho_1} - \frac{1}{n_1}\delta^{\sigma_1}_{\rho_1}g_{\lambda_1\mu_1}\right)\times\cdots$$
$$\times\frac{1}{(n_r-1)!}\delta^{\tau_2\dots\tau_{n_r}}_{\nu_2\dots\nu_{n_r}}\left(2\delta^{\tau_1}_{(\lambda_r}g_{\mu_r)\nu_1} - \frac{1}{n_r}\delta^{\tau_1}_{\nu_1}g_{\lambda_r\mu_r}\right)$$

Indeed, for arbitrary r-fold $(s,\dots,n_r)$-forms A and B one has that

$$Q^\alpha(A,B) = Q^\alpha(B,A)\,,$$

and furthermore, for any timelike future-directed 1-form u_α, $u_\alpha Q^\alpha(\cdot,\cdot)$ is positive definite.

This positive-definite property follows from the identity

$$u_\alpha Q^\alpha(A, A) = u_\alpha T^\alpha{}_{\mu_1 \ldots \lambda_r \mu_r}\{A\} v_1^{\mu_1} \ldots u_r^{\lambda_r} v_r^{\mu_r} > 0$$

where $T\{A\}$ is the so-called "superenergy tensor" of A, which always satisfy the dominant property, that is, the outcome when they are saturated with timelike future-pointing vectors is always positive.

The hyperbolicity of the above general system is related, of course, to the existence of a "wave equation" for A. This is immediate from the definition above of the operator $\Delta_{(1)} = d_{(1)}\delta_{(1)} + \delta_{(1)}d_{(1)}$. Thus, from the original system one deduces

$$\Delta_{(1)} A = -d_{(1)} j + (s+1)\delta_{(1)} J$$

which is manifestly hyperbolic in Lorentzian signature.

Observe also that

- The constraints (which do exist) are always complete, and the system is causal.
- All characteristics of the original system are physical and they, together with the extra characteristics of the hyperbolization, are directly related to special principal null directions of the corresponding solutions.
- General energy estimates and inequalities can always be constructed by using the properties of the superenergy tensors. Conservation laws also arise if there are Killing vectors.
- Even if only the first relation $d_{(1)}A = (s+1)J$ is given, one can deal with the system by adding the second relation as GAUGE equations for arbitrary sources.

Conclusions/Perspectives

- $\bar{\Delta}$!!
- For any tensor field, $2r$ potentials
- If further symmetry/trace properties, then less than $2r$
- Harmonic tensors (in the new sense).
- Global Hodge decomposition for tensors ?? (Flat!).
- Two potentials for Riemann-type tensors, or Ricci-type tensors.
- A potential for the Weyl (and traceless Ricci) tensors.
- Implications of the Bianchi on the potentials? ($\Longrightarrow \boldsymbol{A}$).
- Energy
- Lowering differentiability
- First order symmetric hyperbolic system for P
- Wave/Laplace equation for P
- GAUGE !!!!