The Cauchy problem

Lars Andersson
University of Miami and Albert Einstein Institute

February 5, 2007

Overview

I. Cauchy problem for Quasi-linear wave equations

- Wave equation in Minkowski space
- Example: Wave maps equation
- Yang-Mills and gauge fixing
- Spacetimes and causality
- Wave equation in spacetime
- Cauchy problem for Quasi-linear wave equations

Overview

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

II. Cauchy problem for the Einstein equation

- Variational formulation
- Einstein-matter equations
- Constraints
- Harmonic coordinates
- Local well-posedness
- Global uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

III. Survey

- Cosmic censorship
- Singularity theorems
- Review of small data results
- BKL proposal
- Spacetimes with symmetries
- Kaluza-Klein reduction
- The $U(1)$ problem
- Gowdy
- G_2

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

The Wave equation

Wave equation in Minkowski space

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- Minkowski space $\mathbb{R}^{n,1}$, metric $\mathbf{m} = -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2$,
- wave operator $\Box = -\partial_t^2 + \Delta$.

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

Wave equation in Minkowski space

- Minkowski space $\mathbb{R}^{n,1}$, metric $\mathbf{m} = -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2$,
- wave operator $\Box = -\partial_t^2 + \Delta$.
- The Cauchy problem for the wave equation

$$\Box u = F$$
$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

has unique solution for "nice" initial data f, g and sources F.

Wave equation in Minkowski space

Overview of topics

The Wave equation

Minkowski

Huygen's principle

Energy

Fundamental solution

Wave maps

Null condition

Equivariant

Yang-Mills

Gauge fixing

Spacetimes

Wave equation

The Einstein equations

Global Uniqueness

- Minkowski space $\mathbb{R}^{n,1}$, metric $\mathbf{m} = -dt^2 + (dx^1)^2 + \cdots + (dx^n)^2$,
- wave operator $\Box = -\partial_t^2 + \Delta$.
- The Cauchy problem for the wave equation

$$\Box u = F$$

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

  has unique solution for "nice" initial data f, g and sources F.
- Finite speed of propagation:
  - Data in the ball $|x| \leq R$ has no influence outside the domain of influence $\{(t,x) : |x| \leq |t| + R\}$. Data in $|x| \leq R$ determines the solution in the domain of dependence $\{(t,x) : |x| \leq R - |t|\}$

Huygen's principle

Overview of topics

The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing

Spacetimes

Wave equation

The Einstein equations

Global Uniqueness

- In $1+1$ dimensions
$$\Box u = 0$$
has general solution $u = \phi(x+t) + \psi(x-t)$ for functions ϕ, ψ.
- Information propagates along null curves $t + r =$ const, $t - r =$ const.

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

Huygen's principle

- In $1+1$ dimensions
$$\Box u = 0$$
has general solution $u = \phi(x+t) + \psi(x-t)$ for functions ϕ, ψ.
- Information propagates along null curves $t+r =$ const, $t-r =$ const.
- If $n+1$ is even, Huygen's principle holds in flat $\mathbb{R}^{n+1}$: Information propagates along null curves $=$ characteristics.
- If $n+1$ is odd, or in non-flat spacetimes Huygen's principle fails to hold in general.
- In general, fundamental solution has support in the solid lightcone.

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

Energy

- Lagrangian $\mathcal{L} = \int_{\mathbf{M}} \nabla_\mu u \nabla_\nu u \mathbf{m}^{\mu\nu} \sqrt{-\mathbf{m}}$
- Stress energy tensor $S_{\mu\nu} = \nabla_\mu u \nabla_\nu u - \frac{1}{2} \nabla_\gamma u \nabla^\gamma u \mathbf{m}_{\mu\nu}$
- $\nabla^\mu S_{\mu\nu} = \Box u \nabla_\nu u$

Energy

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- Lagrangian $\mathcal{L} = \int_{\mathbf{M}} \nabla_\mu u \nabla_\nu u \mathbf{m}^{\mu\nu} \sqrt{-\mathbf{m}}$
- Stress energy tensor $S_{\mu\nu} = \nabla_\mu u \nabla_\nu u - \frac{1}{2} \nabla_\gamma u \nabla^\gamma u \mathbf{m}_{\mu\nu}$
- $\nabla^\mu S_{\mu\nu} = \Box u \nabla_\nu u$
- Energy density $S_{tt} = \frac{1}{2}(u_t^2 + |\nabla_x u|^2)$
- Dominant energy condition: $-S^\mu_\nu V^\nu$ is future causal for any future causal V^μ
  $\Leftrightarrow S_{tt} \geq |S_{ti}|$

Fundamental solution

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- The solution to $\Box u = F$ with trivial initial data in $\mathbb{R}^{3+1}$ is

$$u(t,x) = \frac{1}{4\pi} \int_0^t \int_{|x-y|=t-s} |x-y|^{-1} F(s,y) ds d\sigma(y)$$

Fundamental solution

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- The solution to $\Box u = F$ with trivial initial data in $\mathbb{R}^{3+1}$ is

$$u(t,x) = \frac{1}{4\pi}\int_0^t \int_{|x-y|=t-s} |x-y|^{-1} F(s,y)\, ds\, d\sigma(y)$$

- Typical decay in 3+1 dimensions is $|u(t,x)| \le C(t+|x|)^{-1}$, with faster decay away from the null cone.
- Proof uses vector fields method. Idea: Consider the conformal Killing fields: translations, Lorentz rotations, scaling, inverted translations. Apply powers of these to both sides of wave eq. Use energy estimate to get weighted estimates for u and its derivatives.

Wave maps equation

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- Wave maps equation: $u : (\mathbb{R}^{n,1}, \mathbf{m}_{\mu\nu}) \to (N, h_{AB})$, where (N, h_{AB}) Riemannian manifold
- Lagrangian $\int \nabla_\mu u^A, \nabla_\nu u^B h_{AB} \mathbf{m}^{\mu\nu} \sqrt{-\mathbf{m}}$

Wave maps equation

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- Wave maps equation: $u : (\mathbb{R}^{n,1}, \mathbf{m}_{\mu\nu}) \longrightarrow (N, h_{AB})$, where (N, h_{AB}) Riemannian manifold
- Lagrangian $\int \nabla_\mu u^A, \nabla_\nu u^B h_{AB} \mathbf{m}^{\mu\nu} \sqrt{-\mathbf{m}}$
- Stress energy tensor

$$S_{\mu\nu} = \nabla_\mu u^A \nabla_\nu u^B h_{AB} - \frac{1}{2} \nabla_\gamma u^A \nabla^\gamma u^B h_{AB} \mathbf{m}_{\mu\nu}$$

Wave maps equation

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- Wave maps equation: $u : (\mathbb{R}^{n,1}, \mathbf{m}_{\mu\nu}) \longrightarrow (N, h_{AB})$, where (N, h_{AB}) Riemannian manifold
- Lagrangian $\int \nabla_\mu u^A, \nabla_\nu u^B h_{AB} \mathbf{m}^{\mu\nu} \sqrt{-\mathbf{m}}$
- Stress energy tensor
$$S_{\mu\nu} = \nabla_\mu u^A \nabla_\nu u^B h_{AB} - \frac{1}{2} \nabla_\gamma u^A \nabla^\gamma u^B h_{AB} \mathbf{m}_{\mu\nu}$$
- Cauchy problem:

$$\Box u^A + \Gamma^A_{BC}(u) \nabla_\mu u^B \nabla_\nu u^C \mathbf{m}^{\mu\nu} = 0,$$
$$u^A(0,x) = u^A_0(x), \partial_t u^A(0,x) = u^A_1(x)$$

Null condition

Overview of topics

The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing

Spacetimes

Wave equation

The Einstein equations

Global Uniqueness

- The semilinear equation

$$\Box u = Q(\partial u, \partial u)$$

  satisfies the null condition if $Q(\xi, \xi) = 0$ for any null vector ξ
- Wave maps equation satisfies null condition

Null condition

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- The semilinear equation

$$\Box u = Q(\partial u, \partial u)$$

  satisfies the null condition if $Q(\xi, \xi) = 0$ for any null vector ξ
- Wave maps equation satisfies null condition
- If null condition holds, have global existence for small data in $R^{3,1}$.

Null condition

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- The semilinear equation

$$\Box u = Q(\partial u, \partial u)$$

  satisfies the null condition if $Q(\xi, \xi) = 0$ for any null vector ξ
- Wave maps equation satisfies null condition
- If null condition holds, have global existence for small data in $R^{3,1}$.
- For $n \geq 4$, have global existence for small data for quadratic semi-linear equations.
- Without null condition, expect blowup for small data in $R^{3,1}$. Example: $\Box u = |\partial_t u|^2$ does not satisfy the null condition. Have finite time blowup for small data.

Equivariant wave maps equation

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- Consider $n = 2$. Assume SO(2) acts isometrically on N, so that $h = d\rho^2 + f(\rho)^2 d\Omega^2$
- Consider equivariant maps $u = (\phi(r), k\omega)$, where the integer k is the rotation number. Then wave maps equation takes the form

$$-\partial_t^2\phi + \frac{1}{r}\partial_r\phi + \partial_r^2\phi = k\frac{f(\phi)f'(\phi)}{r^2}$$

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

Equivariant wave maps equation

- Consider $n = 2$. Assume $\mathrm{SO}(2)$ acts isometrically on N, so that $h = d\rho^2 + f(\rho)^2 d\Omega^2$
- Consider equivariant maps $u = (\phi(r), k\omega)$, where the integer k is the rotation number. Then wave maps equation takes the form

$$-\partial_t^2 \phi + \frac{1}{r}\partial_r \phi + \partial_r^2 \phi = k\frac{f(\phi)f'(\phi)}{r^2}$$

- For $N = S^2$, $k \geq 1$, there are blowup solutions.

Equivariant wave maps equation

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- Consider $n = 2$. Assume SO(2) acts isometrically on N, so that $h = d\rho^2 + f(\rho)^2 d\Omega^2$
- Consider equivariant maps $u = (\phi(r), k\omega)$, where the integer k is the rotation number. Then wave maps equation takes the form

$$-\partial_t^2\phi + \frac{1}{r}\partial_r\phi + \partial_r^2\phi = k\frac{f(\phi)f'(\phi)}{r^2}$$

- For $N = S^2$, $k \geq 1$, there are blowup solutions.
- Can have blowup only if there is a harmonic map from $\mathbb{R}^2 \to N$. Global existence for $N = \mathcal{H}^2$
- Detailed asymptotics of the blowup is known for $N = S^2$

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

Yang-Mills

- Principal $SU(2)$ bundle over $\mathbb{R}^{3,1}$.
  - A connection, $F = dA + [A, A]$ curvature
  - gauge transformation: $A \to U^{-1}dU + U^{-1}AU$, $F \to U^{-1}FU$

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

Yang-Mills

- Principal $SU(2)$ bundle over $\mathbb{R}^{3,1}$.
  - A connection, $F = dA + [A, A]$ curvature
  - gauge transformation: $A \to U^{-1}dU + U^{-1}AU$, $F \to U^{-1}FU$
  - Lagrangian $\mathcal{L} = \int F_{\alpha\beta}F^{\alpha\beta}$
  - $\mathcal{L}$ gauge invariant

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

Yang-Mills

- Principal $SU(2)$ bundle over $\mathbb{R}^{3,1}$.
  - A connection, $F = dA + [A, A]$ curvature
  - gauge transformation: $A \to U^{-1}dU + U^{-1}AU$, $F \to U^{-1}FU$
  - Lagrangian $\mathcal{L} = \int F_{\alpha\beta}F^{\alpha\beta}$
  - $\mathcal{L}$ gauge invariant
- Euler-Lagrange equation $\partial^\alpha F_{\alpha\beta} + [A^\alpha, F_{\alpha\beta}] = 0$

Gauge fixing for Yang-Mills

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- $\partial^\alpha F_{\alpha\beta} = \partial^\alpha(\partial_\alpha A_\beta - \partial_\beta A_\alpha) +$ lower order terms
  $= \Box A_\beta + \partial_\beta \partial^\alpha A_\alpha +$ lower order terms

Gauge fixing for Yang-Mills

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- $\partial^\alpha F_{\alpha\beta} = \partial^\alpha(\partial_\alpha A_\beta - \partial_\beta A_\alpha) +$ lower order terms
  $= \Box A_\beta + \partial_\beta \partial^\alpha A_\alpha +$ lower order terms
- term $\partial_\beta \partial^\alpha A_\alpha$ due to gauge symmetry of $\mathcal{L}$ – ruins well posedness
- need gauge fixing to have well posed Cauchy problem

Gauge fixing for Yang-Mills

Overview of topics
The Wave equation
Minkowski
Huygen's principle
Energy
Fundamental solution
Wave maps
Null condition
Equivariant
Yang-Mills
Gauge fixing
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness

- $\partial^\alpha F_{\alpha\beta} = \partial^\alpha(\partial_\alpha A_\beta - \partial_\beta A_\alpha) +$ lower order terms
  $= \Box A_\beta + \partial_\beta \partial^\alpha A_\alpha +$ lower order terms
- term $\partial_\beta \partial^\alpha A_\alpha$ due to gauge symmetry of $\mathcal{L}$ – ruins well posedness
- need gauge fixing to have well posed Cauchy problem
- Gauge conditions:
  $\partial^\alpha A_\alpha$ Lorentz gauge
  $A_0 = 0$ temporal gauge
  $\partial^i A_i$ Coloumb gauge

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacetimes

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacetimes

- $(\mathbf{M}, \mathbf{g})$ spacetime, signature $- + + \cdots +$, coordinates x^α
- ∇ connection, $\mathbf{R}$ curvature

Spacetimes

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

- $(\mathbf{M}, \mathbf{g})$ spacetime, signature $- + + \cdots +$, coordinates x^α
- ∇ connection, $\mathbf{R}$ curvature
- Causality notions:
  - causal curve

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacetimes

- $(\mathbf{M}, \mathbf{g})$ spacetime, signature $- + + \cdots +$, coordinates x^α
- ∇ connection, $\mathbf{R}$ curvature
- Causality notions:
  - causal curve
  - time orientation, time function

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacetimes

- $(\mathbf{M}, \mathbf{g})$ spacetime, signature $- + + \cdots +$, coordinates x^α
- ∇ connection, $\mathbf{R}$ curvature
- Causality notions:
  - causal curve
  - time orientation, time function
  - Cauchy surface, globally hyperbolic, domain of dependence

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacetimes

- $(\mathbf{M}, \mathbf{g})$ spacetime, signature $- + + \cdots +$, coordinates x^α
- ∇ connection, $\mathbf{R}$ curvature
- Causality notions:
  - causal curve
  - time orientation, time function
  - Cauchy surface, globally hyperbolic, domain of dependence
  - Cauchy horizon

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacetimes

- $(\mathbf{M}, \mathbf{g})$ spacetime, signature $- + + \cdots +$, coordinates x^α
- ∇ connection, $\mathbf{R}$ curvature
- Causality notions:
  - causal curve
  - time orientation, time function
  - Cauchy surface, globally hyperbolic, domain of dependence
  - Cauchy horizon

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacelike hypersurface

- $i : M \to \mathbf{M}$ spacelike hypersurface, coordinates x^i so that $(x^\alpha) = (t, x^i)$

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacelike hypersurface

- $i : M \to \mathbf{M}$ spacelike hypersurface, coordinates x^i so that $(x^\alpha) = (t, x^i)$
- timelike normal T, induced metric $g = i^*\mathbf{g}$, connection ∇

Spacelike hypersurface

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

- $i : M \to \mathbf{M}$ spacelike hypersurface, coordinates x^i so that $(x^\alpha) = (t, x^i)$
- timelike normal T, induced metric $g = i^* \mathbf{g}$, connection ∇
- $\mathbf{g} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$
  $N = (-g_{00})^{\frac{1}{2}}$, $X = g_{0i}$

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacelike hypersurface

- $i : M \to \mathbf{M}$ spacelike hypersurface, coordinates x^i so that $(x^\alpha) = (t, x^i)$
- timelike normal T, induced metric $g = i^* \mathbf{g}$, connection ∇
- $\mathbf{g} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$
  $N = (-g_{00})^{\frac{1}{2}}$, $X = g_{0i}$
- second fundamental form $K = -i^*(\nabla T)$,
  $K_{ij} = -\frac{1}{2}(\mathcal{L}_T \mathbf{g})_{ij}$

Spacelike hypersurface

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

- $i : M \to \mathbf{M}$ spacelike hypersurface, coordinates x^i so that $(x^\alpha) = (t, x^i)$
- timelike normal T, induced metric $g = i^* \mathbf{g}$, connection ∇
- $\mathbf{g} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$
  $N = (-g_{00})^{\frac{1}{2}}$, $X = g_{0i}$
- second fundamental form $K = -i^*(\nabla T)$,
  $K_{ij} = -\frac{1}{2}(\mathcal{L}_T \mathbf{g})_{ij}$
- structure equations

$$\nabla_j k_{im} - \nabla_i k_{jm} = \mathbf{R}_{mTij} \qquad \text{Codazzi}$$

$$R_{ij} - k_{ia} k^a{}_j + k_{ij} \mathrm{tr} k = \mathbf{R}_{iTjT} + \mathbf{R}_{ij} \qquad \text{Gauss}$$

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacelike foliation

- $i_t : M \to \mathbf{M}$ foliation

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacelike foliation

- $i_t : M \to \mathbf{M}$ foliation
- $T = T^\alpha \partial_\alpha$, $\partial_t = NT + X$
  where N is the lapse function and $X = X^i \partial_i$ is the shift vectorfield
  N, X *embedding parameters*

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacelike foliation

- $i_t : M \to \mathbf{M}$ foliation
- $T = T^\alpha \partial_\alpha$, $\partial_t = NT + X$
  where N is the lapse function and $X = X^i \partial_i$ is the shift vectorfield
  N, X *embedding parameters*
- Defining relation for K
  $\mathcal{L}_{\partial_t} g_{ij} = -2NK_{ij} + \mathcal{L}_X g_{ij}$

Overview of topics
The Wave equation
Spacetimes
Spacetimes
Spacelike hypersurface
Spacelike foliation
Wave equation
The Einstein equations
Global Uniqueness

Spacelike foliation

- $i_t : M \to \mathbf{M}$ foliation
- $T = T^\alpha \partial_\alpha$, $\partial_t = NT + X$
  where N is the lapse function and $X = X^i \partial_i$ is the shift vectorfield
  N, X *embedding parameters*
- Defining relation for K
  $\mathcal{L}_{\partial_t} g_{ij} = -2NK_{ij} + \mathcal{L}_X g_{ij}$
- structure equation (from second variation equation and Gauss)

$$\mathcal{L}_{\partial_t} K_{ij} = -\nabla_i \nabla_j N + N(-\mathbf{R}_{ij} + R_{ij} + \mathrm{tr} K K_{ij} - 2K_{im}K^m{}_j) + \mathcal{L}_X K_{ij}$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Wave equation in curved spacetime

Basic facts

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Lagrangian $\mathcal{L}[u] = \int_{\mathbf{M}} \nabla_\mu u \nabla_\nu u \mathbf{g}^{\mu\nu} \sqrt{-\mathbf{g}}$
  where $\sqrt{-\mathbf{g}} = \sqrt{-\det(\mathbf{g}_{\mu\nu})}$ is the volume element.

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Basic facts

- Lagrangian $\mathcal{L}[u] = \int_{\mathbf{M}} \nabla_\mu u \nabla_\nu u \mathbf{g}^{\mu\nu} \sqrt{-\mathbf{g}}$
  where $\sqrt{-\mathbf{g}} = \sqrt{-\det(\mathbf{g}_{\mu\nu})}$ is the volume element.
- wave operator
  $\Box u = \frac{1}{\sqrt{-\mathbf{g}}} \partial_\mu (\mathbf{g}^{\mu\nu} \sqrt{-\mathbf{g}} \partial_\nu) u = \mathbf{g}^{\mu\nu} \nabla_\mu \nabla_\nu u$

Basic facts

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Lagrangian $\mathcal{L}[u] = \int_{\mathbf{M}} \nabla_\mu u \nabla_\nu u \mathbf{g}^{\mu\nu} \sqrt{-\mathbf{g}}$
  where $\sqrt{-\mathbf{g}} = \sqrt{-\det(\mathbf{g}_{\mu\nu})}$ is the volume element.

- wave operator
  $\Box u = \frac{1}{\sqrt{-\mathbf{g}}} \partial_\mu (\mathbf{g}^{\mu\nu} \sqrt{-\mathbf{g}} \partial_\nu) u = \mathbf{g}^{\mu\nu} \nabla_\mu \nabla_\nu u$
- stress energy tensor $S = \delta_{\mathbf{g}} \mathcal{L}$:

$$S_{\mu\nu} = \nabla_\mu u \nabla_\nu u - \frac{1}{2} \nabla_\gamma u \nabla^\gamma u \mathbf{g}_{\mu\nu}$$

Basic facts

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Lagrangian $\mathcal{L}[u] = \int_{\mathbf{M}} \nabla_\mu u \nabla_\nu u \mathbf{g}^{\mu\nu} \sqrt{-\mathbf{g}}$ where $\sqrt{-\mathbf{g}} = \sqrt{-\det(\mathbf{g}_{\mu\nu})}$ is the volume element.
- wave operator $\Box u = \dfrac{1}{\sqrt{-\mathbf{g}}} \partial_\mu (\mathbf{g}^{\mu\nu} \sqrt{-\mathbf{g}} \partial_\nu) u = \mathbf{g}^{\mu\nu} \nabla_\mu \nabla_\nu u$
- stress energy tensor $S = \delta_{\mathbf{g}} \mathcal{L}$:

$$S_{\mu\nu} = \nabla_\mu u \nabla_\nu u - \frac{1}{2} \nabla_\gamma u \nabla^\gamma u \mathbf{g}_{\mu\nu}$$

- $\Box u = 0 \Rightarrow \nabla^\mu S_{\mu\nu} = 0$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Cauchy problem for the Wave equation

- Suppose $(\mathbf{M}, \mathbf{g})$ globally hyperbolic, M Cauchy surface, F given function on $\mathbf{M}$

Cauchy problem for the Wave equation

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Suppose $(\mathbf{M}, \mathbf{g})$ globally hyperbolic, M Cauchy surface, F given function on $\mathbf{M}$
- The Cauchy problem
  $\Box u = F$,
  $u\big|_M = u_0$, $\partial_t u\big|_M = u_1$
  has unique global solution u for reasonable F, u_0, u_1

Cauchy problem for the Wave equation

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Suppose $(\mathbf{M}, \mathbf{g})$ globally hyperbolic, M Cauchy surface, F given function on $\mathbf{M}$
- The Cauchy problem
  $\Box u = F$,
  $u\big|_M = u_0$, $\partial_t u\big|_M = u_1$
  has unique global solution u for reasonable F, u_0, u_1
- Analogous statements hold for nonlinear equations. Example: Yang-Mills in 3+1 dimensional, globally hyperbolic spacetimes.

Energy

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- energy density

$$e(u) = S_{TT} = \frac{1}{2}(T(u)^2 + |\nabla_x u|)$$

- energy $\mathcal{E}[u,t] = \int_M e(u)\sqrt{g}$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Energy

- energy density

$$e(u) = S_{TT} = \frac{1}{2}(T(u)^2 + |\nabla_x u|)$$

- energy $\mathcal{E}[u,t] = \int_M e(u)\sqrt{g}$
- propagation law

$$\partial_t \mathcal{E}[u,t] = \int_M \left(T(u)\Box u + S_{\mu\nu}\nabla^\mu T^\nu\right)\sqrt{g}$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Energy

- energy density

$$e(u) = S_{TT} = \frac{1}{2}(T(u)^2 + |\nabla_x u|)$$

- energy $\mathcal{E}[u,t] = \int_M e(u)\sqrt{g}$
- propagation law

$$\partial_t \mathcal{E}[u,t] = \int_M \left(T(u)\Box u + S_{\mu\nu}\nabla^\mu T^\nu\right)\sqrt{g}$$

- basic energy inequality

$$|\,\partial_t E[u]\,| \le C\,(||f||_{L^2} + E[u]||\partial \mathbf{g}||_{L^\infty})$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Energy estimate

- Gronwall: assume f, a, b in $L^\infty([0,T])$, a nondecreasing and

$$f(t) \leq a(t) + \int_0^t b(\tau) f(\tau) d\tau$$

Then

$$f(T) \leq a(T) e^{\int_0^T b(\tau) d\tau}$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Energy estimate

- Gronwall: assume f, a, b in $L^\infty([0,T])$, a nondecreasing and
$$f(t) \le a(t) + \int_0^t b(\tau) f(\tau) d\tau$$
Then
$$f(T) \le a(T) e^{\int_0^T b(\tau) d\tau}$$

- energy estimate
$$E[u, t_1] \le C e^{\int_{t_0}^{t_1} ||\partial \mathbf{g}||_{L^\infty}} \left(E[u, t_0] + \int_{t_0}^{t_1} ||f||_{L^2_x} dt \right)$$

Finite speed of propagation, Uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Let u be C^2 solution of

$$\Box u = F(u, \partial u, \partial^2 u)$$

in $\Lambda^-_{p,M} = I^-(p) \cap J^+(M)$

Finite speed of propagation, Uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Let u be C^2 solution of

$$\Box u = F(u, \partial u, \partial^2 u)$$

in $\Lambda^-_{p,M} = I^-(p) \cap J^+(M)$

- Assume $F(0, 0, \partial^2 u) \equiv 0$
Then $F(u, \partial u, \partial^2 u) \leq C(|u| + |\partial u|)$ for small $u \in C^2$.

Finite speed of propagation, Uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Let u be C^2 solution of

$$\Box u = F(u, \partial u, \partial^2 u)$$

  in $\Lambda^-_{p,M} = I^-(p) \cap J^+(M)$

- Assume $F(0, 0, \partial^2 u) \equiv 0$
  Then $F(u, \partial u, \partial^2 u) \leq C(|u| + |\partial u|)$ for small $u \in C^2$.
- Assume $u = 0$, $\partial_t u = 0$ in $B_p = I^-(p) \cap M$

Finite speed of propagation, Uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Let u be C^2 solution of

$$\Box u = F(u, \partial u, \partial^2 u)$$

  in $\Lambda^-_{p,M} = I^-(p) \cap J^+(M)$

- Assume $F(0, 0, \partial^2 u) \equiv 0$
  Then $F(u, \partial u, \partial^2 u) \leq C(|u| + |\partial u|)$ for small $u \in C^2$.
- Assume $u = 0$, $\partial_t u = 0$ in $B_p = I^-(p) \cap M$
  Then $u \equiv 0$ in $\Lambda^-_{p,M}$.

Finite speed of propagation, Uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Let u be C^2 solution of

$$\Box u = F(u, \partial u, \partial^2 u)$$

in $\Lambda^-_{p,M} = I^-(p) \cap J^+(M)$

- Assume $F(0, 0, \partial^2 u) \equiv 0$
Then $F(u, \partial u, \partial^2 u) \leq C(|u| + |\partial u|)$ for small $u \in C^2$.
- Assume $u = 0$, $\partial_t u = 0$ in $B_p = I^-(p) \cap M$
Then $u \equiv 0$ in $\Lambda^-_{p,M}$.
- Proof: Energy estimate + Gronwall.

Finite speed of propagation, Uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Let u be C^2 solution of

$$\Box u = F(u, \partial u, \partial^2 u)$$

  in $\Lambda^-_{p,M} = I^-(p) \cap J^+(M)$

- Assume $F(0, 0, \partial^2 u) \equiv 0$
  Then $F(u, \partial u, \partial^2 u) \leq C(|u| + |\partial u|)$ for small $u \in C^2$.
- Assume $u = 0$, $\partial_t u = 0$ in $B_p = I^-(p) \cap M$
  Then $u \equiv 0$ in $\Lambda^-_{p,M}$.
- Proof: Energy estimate + Gronwall.
- Uniquenss of regular solutions to the Cauchy problem proved similarly

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Sobolev spaces

Let M be n-dimensional, $s \geq 0$ integer.

- L^2-Sobolev norm $||f||_{H^s} = \sum_{j \leq s} ||\partial^j f||_{L^2}$

  H^s = closure of C_0^∞ with respect to $||\cdot||_{H^s}$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Sobolev spaces

Let M be n-dimensional, $s \geq 0$ integer.

- L^2-Sobolev norm $||f||_{H^s} = \sum_{j \leq s} ||\partial^j f||_{L^2}$
  H^s = closure of C_0^∞ with respect to $|| \cdot ||_{H^s}$
- Sobolev imbedding:
  For $s > n/2$, $H^s \subset L^\infty$
  For $0 \leq s < n/2$, $H^s \subset L^p$, $p = 2n/(n-2s)$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Sobolev spaces

Let M be n-dimensional, $s \geq 0$ integer.

- L^2-Sobolev norm $||f||_{H^s} = \sum_{j \leq s} ||\partial^j f||_{L^2}$

  H^s = closure of C_0^∞ with respect to $|| \cdot ||_{H^s}$
- Sobolev imbedding:

  For $s > n/2$, $H^s \subset L^\infty$

  For $0 \leq s < n/2$, $H^s \subset L^p$, $p = 2n/(n-2s)$
- product estimate. For $s \geq 0$,

$$||fg||_{H^s} \leq ||f||_{L^\infty}||g||_{H^s} + ||f||_{H^s}||g||_{L^\infty}$$

Sobolev spaces

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- commutator estimate (D^s some s-th order operator)

$$||[D^s, u]v||_{L^2} \leq C(||\partial u||_{L^\infty}||v||_{H^{s-1}} + ||u||_{H^s}||v||_{L^\infty})$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Sobolev spaces

- commutator estimate (D^s some s-th order operator)

$$||[D^s, u]v||_{L^2} \le C(||\partial u||_{L^\infty}||v||_{H^{s-1}} + ||u||_{H^s}||v||_{L^\infty})$$

- for $s > n/2$,

$$||fg||_{H^s} \le ||f||_{H^s}||g||_{H^s}$$

- If $F \in C^\infty(\mathbb{R})$, $s > n/2$,

$$||F(u)||_{H^s} \le C(F, ||u||_{L^\infty})||u||_{H^s}$$

Quasilinear wave equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- $L[u]u = \Box_{\mathbf{g}(u)} u$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Quasilinear wave equations

- $L[u]u = \Box_{\mathbf{g}(u)} u$
- Consider the Cauchy problem

$$L[u]u = F(u, \partial u)$$

$$u\Big|_{t=0} = u_0, \quad \partial_t\Big|_{t=0} = u_1$$

Quasilinear wave equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- $L[u]u = \Box_{\mathbf{g}(u)} u$
- Consider the Cauchy problem

$$L[u]u = F(u, \partial u)$$

$$u\Big|_{t=0} = u_0, \quad \partial_t\Big|_{t=0} = u_1$$

- Higher order energies. $E_s[u] = \sum_{j \le s-1} E[D^j u]$, $D = \partial_x$

Quasilinear wave equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- $L[u]u = \Box_{\mathbf{g}(u)} u$
- Consider the Cauchy problem

$$L[u]u = F(u, \partial u)$$

$$u\Big|_{t=0} = u_0, \quad \partial_t\Big|_{t=0} = u_1$$

- Higher order energies. $E_s[u] = \sum_{j \leq s-1} E[D^j u]$, $D = \partial_x$
- Prove local existence and well-posedness by contraction mapping principle

Quasilinear wave equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Using commutator and product estimates, get

$$||L[u]D^{s-1}u||_{L^2} \leq C(||f||_{H^{s-1}} + E_s[u])$$

- Energy estimate gives

$$E_s[u,T] \leq C(E_s[u,0] + ||f||_{L^1([0,T];H^{s-1})})$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Quasilinear wave equations

- Let X be the space of $u \in L^2[0,T] \times \Sigma$, with

$$\partial u \in L^\infty([0,T]; H^s) \cap C([0,T]; L^2)$$

with norm

$$|||u|||_s = \sup_{t \in [0,T]} E_s[u,t]$$

- For $\delta > 0$, let B_δ be the set

$$B_\rho = \{u \in X, u(0) = u_0, \partial_t u(0) = u_1, |||u|||_s \leq \rho\}$$

- The apriori bounds show for $s > n/2 + 1$, for δ, T sufficiently small B_δ is invariant under $\mathcal{F}$.

Quasilinear wave equations

Overview of topics

The Wave equation

Spacetimes

Wave equation

Basic facts

Energy

Causality, Uniqueness

Sobolev spaces

Quasilinear wave equations

Properties of solutions

The Einstein equations

Global Uniqueness

- Define a metric

$$\rho(u, v) = |||u - v|||_1$$

on B_δ. Then (B_δ, ρ) is a complete metric space.

- The map $\mathcal{F} : v \longrightarrow u$ defined by solving $L[u]u = F(v, \partial v)$ is a contraction T sufficiently small,

$$\rho(\mathcal{F}(v_1), \mathcal{F}(v_2)) \leq \frac{1}{2}\rho(v_1, v_2)$$

Proof: energy estimate

- By the contraction mapping principle, there is a unique solution to the equation $\mathcal{F}(u) = u$ in B_δ

Quasilinear wave equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- We have proved

  Theorem 1. *Local well-posedness holds for quasilinear wave equations in H^s, $s > n/2 + 1$.*

- More explicitly:
  Consider the Cauchy problem for the QL wave equation
  $L[u]u = F(u, \partial u)$
  with data
  $u\big|_\Sigma = u_0$, $\partial_t u\big|_\Sigma = u_1$
  $(u_0, u_1) \in H^s \times H^{s-1}$, $s > n/2 + 1$

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

Quasilinear wave equations

- We have proved

  Theorem 2. *Local well-posedness holds for quasilinear wave equations in H^s, $s > n/2 + 1$.*

- More explicitely:
  Consider the Cauchy problem for the QL wave equation
  $L[u]u = F(u, \partial u)$
  with data
  $u\big|_\Sigma = u_0,\ \partial_t u\big|_\Sigma = u_1$
  $(u_0, u_1) \in H^s \times H^{s-1},\ s > n/2 + 1$
- then there is $T > 0$ so that there is a unique solution for $t \in [0, T]$
  (this statement refers to a given foliation)

Properties of solutions

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Strong well-posedness: the solution curve depends continuously on the data

Properties of solutions

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Strong well-posedness: the solution curve depends continuously on the data
- Continuation principle: solution can be continued as long as $||\partial u||_{L^\infty}$ is in L^1_t.

Properties of solutions

Overview of topics
The Wave equation
Spacetimes
Wave equation
Basic facts
Energy
Causality, Uniqueness
Sobolev spaces
Quasilinear wave equations
Properties of solutions
The Einstein equations
Global Uniqueness

- Strong well-posedness: the solution curve depends continuously on the data
- Continuation principle: solution can be continued as long as $||\partial u||_{L^\infty}$ is in L^1_t.
- smoothness propagates: C^∞ initial data gives C^∞ solution until blowup.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

The Einstein equations

Variational formulation

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{geom}} + \mathcal{L}_{\text{matter}}$$

$$\mathcal{L}_{\text{geom}} = \frac{1}{G_n} \int \mathbf{R}\sqrt{-\mathbf{g}}$$

$\mathcal{L}_{\text{geom}}$ is known as the Einstein-Hilbert Lagrangian

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Variational formulation

- Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{geom}} + \mathcal{L}_{\text{matter}}$$

$$\mathcal{L}_{\text{geom}} = \frac{1}{G_n} \int \mathbf{R} \sqrt{-\mathbf{g}}$$

$\mathcal{L}_{\text{geom}}$ is known as the Einstein-Hilbert Lagrangian

- variations: $\delta \mathbf{g}^{\mu\nu} = h^{\mu\nu}$

$$\delta \sqrt{-\mathbf{g}} = -\frac{1}{2} \mathbf{g}_{\mu\nu} h^{\mu\nu} \sqrt{-\mathbf{g}}$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Variational formulation

- Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{geom}} + \mathcal{L}_{\text{matter}}$$

$$\mathcal{L}_{\text{geom}} = \frac{1}{G_n} \int \mathbf{R}\sqrt{-\mathbf{g}}$$

$\mathcal{L}_{\text{geom}}$ is known as the Einstein-Hilbert Lagrangian

- variations: $\delta \mathbf{g}^{\mu\nu} = h^{\mu\nu}$

$$\delta\sqrt{-\mathbf{g}} = -\frac{1}{2}\mathbf{g}_{\mu\nu} h^{\mu\nu}\sqrt{-\mathbf{g}}$$

$$\delta[\mathbf{R}\sqrt{-\mathbf{g}}] = [(\delta\mathbf{R}_{\mu\nu})\mathbf{g}^{\mu\nu} + (\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R} g_{\mu\nu})h^{\mu\nu}]\sqrt{-\mathbf{g}}$$

where $(\delta\mathbf{R}_{\mu\nu})\mathbf{g}^{\mu\nu}$ is a total divergence

Variational formulation

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathsf{geom}} + \mathcal{L}_{\mathsf{matter}}$$

$$\mathcal{L}_{\mathsf{geom}} = \frac{1}{G_n} \int \mathbf{R}\sqrt{-\mathbf{g}}$$

$\mathcal{L}_{\mathsf{geom}}$ is known as the Einstein-Hilbert Lagrangian

- variations: $\delta \mathbf{g}^{\mu\nu} = h^{\mu\nu}$

$$\delta\sqrt{-\mathbf{g}} = -\frac{1}{2}\mathbf{g}_{\mu\nu}h^{\mu\nu}\sqrt{-\mathbf{g}}$$

$$\delta[\mathbf{R}\sqrt{-\mathbf{g}}] = [(\delta\mathbf{R}_{\mu\nu})\mathbf{g}^{\mu\nu} + (\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R}g_{\mu\nu})h^{\mu\nu}]\sqrt{-\mathbf{g}}$$

where $(\delta\mathbf{R}_{\mu\nu})\mathbf{g}^{\mu\nu}$ is a total divergence

Variational formulation

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Einstein vacuum equations:
  $\delta_{\mathbf{g}}\mathcal{L}_{\text{geom}} = 0 \Rightarrow G_{\mu\nu} = 0$
  where
  $G_{\mu\nu} = \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R}g_{\mu\nu}$ is the *Einstein tensor*.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Variational formulation

- Einstein vacuum equations:
  $\delta_{\mathbf{g}} \mathcal{L}_{\text{geom}} = 0 \Rightarrow G_{\mu\nu} = 0$
  where
  $G_{\mu\nu} = \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R} g_{\mu\nu}$ is the *Einstein tensor*.
- Bianchi $\Rightarrow \nabla^\mu G_{\mu\nu} = 0$.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Variational formulation

- Einstein vacuum equations:
  $\delta_{\mathbf{g}} \mathcal{L}_{\text{geom}} = 0 \Rightarrow G_{\mu\nu} = 0$
  where
  $G_{\mu\nu} = \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R} g_{\mu\nu}$ is the *Einstein tensor*.
- Bianchi $\Rightarrow \nabla^\mu G_{\mu\nu} = 0$.
- Einstein-matter equations:
  $\delta_{\mathbf{g}} \mathcal{L} = 0 \Rightarrow G_{\mu\nu} = G_n S_{\mu\nu}$
  where $S_{\mu\nu} = \delta_{\mathbf{g}} \mathcal{L}_{\text{matter}}$ is the *stress energy tensor*

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Variational formulation

- Einstein vacuum equations:
  $\delta_{\mathbf{g}} \mathcal{L}_{\text{geom}} = 0 \Rightarrow G_{\mu\nu} = 0$
  where
  $G_{\mu\nu} = \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{R} g_{\mu\nu}$ is the *Einstein tensor*.
- Bianchi $\Rightarrow \nabla^\mu G_{\mu\nu} = 0$.
- Einstein-matter equations:
  $\delta_{\mathbf{g}} \mathcal{L} = 0 \Rightarrow G_{\mu\nu} = G_n S_{\mu\nu}$
  where $S_{\mu\nu} = \delta_{\mathbf{g}} \mathcal{L}_{\text{matter}}$ is the *stress energy tensor*
- $G_3 = 8\pi G$.

Hamiltonian formulation

- $\pi^{ij} = \sqrt{g}(K^{ij} - \mathrm{tr} K g^{ij})$

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Hamiltonian formulation

- $\pi^{ij} = \sqrt{g}(K^{ij} - \mathrm{tr}K g^{ij})$

  $$\mathcal{L}_{\mathrm{geom}} = \int dt \int_\Sigma \pi \dot{g} - N\mathcal{H} - X^i \mathcal{J}_i + \text{ total divergence}$$

  where

  $$\mathcal{H} = \sqrt{g}R + \frac{1}{2}(\mathrm{tr}\pi)^2/\sqrt{g} - |\pi|^2/\sqrt{g} = \sqrt{g}(R + (\mathrm{tr}K)^2 - |K|^2)$$

  $$\mathcal{J}_i = \nabla^j \pi_{ij} = \sqrt{g}(\nabla^j K_{ij} - \nabla_i \mathrm{tr}K)$$

Hamiltonian formulation

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- $\pi^{ij} = \sqrt{g}(K^{ij} - \text{tr}K g^{ij})$
  $$\mathcal{L}_{\text{geom}} = \int dt \int_\Sigma \pi \dot{g} - N\mathcal{H} - X^i \mathcal{J}_i + \text{ total divergence}$$
  where
  $$\mathcal{H} = \sqrt{g}R + \frac{1}{2}(\text{tr}\pi)^2/\sqrt{g} - |\pi|^2/\sqrt{g} = \sqrt{g}(R + (\text{tr}K)^2 - |K|^2)$$
  $$\mathcal{J}_i = \nabla^j \pi_{ij} = \sqrt{g}(\nabla^j K_{ij} - \nabla_i \text{tr}K)$$
- N, X^i "Lagrange multipliers", no E-L eq's for N, X^i variatons w.r.t. N, X^i give vacuum constraints $\mathcal{H} = 0, \mathcal{J} = 0$

Vacuum Constraints

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Contracted Gauss and Codazzi equations give

$$G_{TT} = R - |K|^2 + (\mathrm{tr}K)^2$$
$$G_{Ti} = \nabla^j K_{ij} - \nabla_i \mathrm{tr}K$$

  Thus, the vacuum Einstein constraint equations are $G_{TT} = 0, G_{Ti} = 0$.

- Remarkable fact: In order to construct a vacuum Cauchy development from a set of data (Σ, g, K), it is sufficient that (Σ, g, K) solves the vacuum constraint equations. We call such (Σ, g, K) a *vacuum Cauchy data set.*

Vacuum Cauchy problem

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Vacuum constraint equations

$$R - |K|^2 + (\mathrm{tr} K)^2 = 0, \qquad \nabla^i K_{ij} - \nabla_j \mathrm{tr} K = 0$$

- (Σ, g, K) where (g, K) solve vacuum constraints on Σ is a vacuum *Cauchy data set*

Vacuum Cauchy problem

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Vacuum constraint equations

$$R - |K|^2 + (\mathrm{tr}K)^2 = 0, \qquad \nabla^i K_{ij} - \nabla_j \mathrm{tr}K = 0$$

- (Σ, g, K) where (g, K) solve vacuum constraints on Σ is a vacuum *Cauchy data set*
- **Cauchy problem:** Given vacuum Cauchy data set (Σ, g, K), construct a (maximal) globally hyperbolic vacuum spacetime $(\mathbf{M}, \mathbf{g})$ containing (Σ, g, K) as a Cauchy hypersurface.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints,
Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Vacuum Cauchy problem

- Vacuum constraint equations

$$R - |K|^2 + (\mathrm{tr}K)^2 = 0, \qquad \nabla^i K_{ij} - \nabla_j \mathrm{tr}K = 0$$

- (Σ, g, K) where (g, K) solve vacuum constraints on Σ is a vacuum *Cauchy data set*
- **Cauchy problem:** Given vacuum Cauchy data set (Σ, g, K), construct a (maximal) globally hyperbolic vacuum spacetime $(\mathbf{M}, \mathbf{g})$ containing (Σ, g, K) as a Cauchy hypersurface.
- $\mathbf{M}$ is called a *Cauchy development*, of (Σ, g, K)

Solving the constraints, Conformal method

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- (Σ^3, g) Riemannian.
  $g \to \phi^4 g$ conformal deformation. Scalar curvature transforms according to
  $R[\phi^4 g] = \phi^{-5}(-8\Delta\phi + R\phi)$

Solving the constraints, Conformal method

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- (Σ^3, g) Riemannian.
  $g \to \phi^4 g$ conformal deformation. Scalar curvature transforms according to
  $R[\phi^4 g] = \phi^{-5}(-8\Delta\phi + R\phi)$

Solving the constraints, Conformal method

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- (Σ^3, g) Riemannian.
  $g \to \phi^4 g$ conformal deformation. Scalar curvature transforms according to
  $R[\phi^4 g] = \phi^{-5}(-8\Delta\phi + R\phi)$
- Yamabe theorem: For any compact (Σ, g), g can be conformally deformed to constant scalar curvature

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Solving the constraints, Conformal method

- (Σ^3, g) Riemannian.
  $g \to \phi^4 g$ conformal deformation. Scalar curvature transforms according to
  $R[\phi^4 g] = \phi^{-5}(-8\Delta\phi + R\phi)$
- Yamabe theorem: For any compact (Σ, g), g can be conformally deformed to constant scalar curvature
- Given conformal background metric $\hat{g}$ on Σ^3, σ symmetric 2-tensor so that $\hat{g}^{ij}\sigma_{ij} = 0$, $\hat{\nabla}^i\sigma_{ij} = 0$, $H =$ constant, solve

$$-8\Delta_{\hat{g}}\phi + R[\hat{g}]\phi - |\sigma|^2_{\hat{g}}\phi^{-7} + 6H^2\phi^{-5} = 0$$

Solving the constraints, Conformal method

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- (Σ^3, g) Riemannian.
  $g \to \phi^4 g$ conformal deformation. Scalar curvature transforms according to
  $R[\phi^4 g] = \phi^{-5}(-8\Delta\phi + R\phi)$
- Yamabe theorem: For any compact (Σ, g), g can be conformally deformed to constant scalar curvature
- Given conformal background metric $\hat{g}$ on Σ^3, σ symmetric 2-tensor so that $\hat{g}^{ij}\sigma_{ij} = 0$, $\hat{\nabla}^i\sigma_{ij} = 0$, $H =$ constant, solve

$$-8\Delta_{\hat{g}}\phi + R[\hat{g}]\phi - |\sigma|^2_{\hat{g}}\phi^{-7} + 6H^2\phi^{-5} = 0$$

- Define $g = \phi^4\hat{g} \qquad K = \phi^{-2}\sigma + Hg$
  Then (g, K) solve the vacuum constraint equations with $\mathrm{tr}K = 3H$

Remarks

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Conformal method gives constant mean curvature data, does not work in general
- constructs solutions to constraints on all compact Σ
- asymptotic conditions: AE, AH etc. can be handled

Remarks

Overview of topics

The Wave equation

Spacetimes

Wave equation

The Einstein equations

Variational formulation

Hamiltonian formulation

Vacuum Constraints

Cauchy problem

Solving the constraints, Conformal method

Remarks

Matter

Harmonic coordinates

Reduced Einstein

Gauge Propagation

Global Uniqueness

- Conformal method gives constant mean curvature data, does not work in general
- constructs solutions to constraints on all compact Σ
- asymptotic conditions: AE, AH etc. can be handled
- Linearization of
  $(g, K) \to (R - |K|^2 + (\mathrm{tr}K)^2, \nabla^i K_{ij} - \nabla_j \mathrm{tr}K)$
  has surjective but not injective symbol, i.e. degenerate elliptic
  solutions of constraints have "local" freedom, inspide of elliptic nature
- Gluing: data sets can be glued at non-KID points
  AF data can be made asymptotically Schwarzschild
  Constructs example of spacetime with no CMC slice

Einstein-matter equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Example:

$$\mathcal{L}_{\mathrm{matter}} = \int \langle \nabla\phi, \nabla\phi \rangle + V(\phi)$$

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Einstein-matter equations

- Example:

$$\mathcal{L}_{\text{matter}} = \int \langle \nabla\phi, \nabla\phi \rangle + V(\phi)$$

- stress energy $S_{\mu\nu} = \delta_{\mathbf{g}} \mathcal{L}_{\text{matter}}$

$$S_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}(\nabla_\gamma \phi \nabla^\gamma \phi + V(\phi))\mathbf{g}_{\mu\nu}$$

Einstein-matter equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Example:

$$\mathcal{L}_{\text{matter}} = \int \langle \nabla\phi, \nabla\phi \rangle + V(\phi)$$

- stress energy $S_{\mu\nu} = \delta_{\mathbf{g}} \mathcal{L}_{\text{matter}}$

$$S_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2}(\nabla_\gamma \phi \nabla^\gamma \phi + V(\phi))\mathbf{g}_{\mu\nu}$$

- Einstein equation $\mathbf{R}_{\mu\nu} - \mathbf{R}\mathbf{g}_{\mu\nu} = G_n S_{\mu\nu}$
- matter equation $\Box\phi = V'(\phi)$

Einstein-Matter Constraints

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Einstein-matter constraints:
  $G_{TT} = G_n S_{TT}$, $G_{Ti} = G_n S_{Ti}$

Einstein-Matter Constraints

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Einstein-matter constraints:
  $G_{TT} = G_n S_{TT}$, $G_{Ti} = G_n S_{Ti}$
- For Einstein scalar field in 3+1 dimensions:

$$R - |K|^2 + (\mathrm{tr}K)^2 = 8\pi G(T(\phi)^2 + |\nabla\phi|^2 + V(\phi)) =: 2\rho$$

$$\nabla^i K_{ij} - \nabla_j \mathrm{tr}K = 8\pi G T(\phi)\nabla_j\phi =: \mu_j$$

Einstein-Matter Constraints

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Einstein-matter constraints:
  $G_{TT} = G_n S_{TT}$, $G_{Ti} = G_n S_{Ti}$
- For Einstein scalar field in 3+1 dimensions:

$$R - |K|^2 + (\mathrm{tr}K)^2 = 8\pi G(T(\phi)^2 + |\nabla\phi|^2 + V(\phi)) =: 2\rho$$

$$\nabla^i K_{ij} - \nabla_j \mathrm{tr}K = 8\pi G T(\phi)\nabla_j\phi =: \mu_j$$

- Dominant energy condition (DEC): $\rho \geq |\mu|$
  holds for Einstein-scalar field, if $V \geq 0$.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Harmonic coordinates

- $\Box x^\alpha = \frac{1}{\sqrt{-\mathbf{g}}}\partial_\mu(\mathbf{g}^{\mu\alpha}\sqrt{-\mathbf{g}}) = -\mathbf{g}^{\mu\nu}\mathbf{\Gamma}^\alpha_{\mu\nu}$

Harmonic coordinates

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- $\Box x^\alpha = \frac{1}{\sqrt{-\mathbf{g}}}\partial_\mu(\mathbf{g}^{\mu\alpha}\sqrt{-\mathbf{g}}) = -\mathbf{g}^{\mu\nu}\mathbf{\Gamma}^\alpha_{\mu\nu}$
- Gauge source function:
  Given $F^\alpha = F^\alpha(x, \mathbf{g})$, let
  $V^\alpha = \mathbf{g}^{\mu\nu}\mathbf{\Gamma}^\alpha_{\mu\nu} - F^\alpha$

Harmonic coordinates

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- $\Box x^\alpha = \frac{1}{\sqrt{-\mathbf{g}}}\partial_\mu(\mathbf{g}^{\mu\alpha}\sqrt{-\mathbf{g}}) = -\mathbf{g}^{\mu\nu}\mathbf{\Gamma}^\alpha_{\mu\nu}$
- Gauge source function:
  Given $F^\alpha = F^\alpha(x, \mathbf{g})$, let
  $V^\alpha = \mathbf{g}^{\mu\nu}\mathbf{\Gamma}^\alpha_{\mu\nu} - F^\alpha$
- Example:
  Fix a background spacetime $(\hat{M}, \hat{g})$, $\hat{\nabla}$ covariant derivative, $\hat{\Gamma}$ Christoffel's of $\hat{g}$

Harmonic coordinates

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- $\Box x^\alpha = \frac{1}{\sqrt{-\mathbf{g}}}\partial_\mu(\mathbf{g}^{\mu\alpha}\sqrt{-\mathbf{g}}) = -\mathbf{g}^{\mu\nu}\mathbf{\Gamma}^\alpha_{\mu\nu}$
- Gauge source function:
  Given $F^\alpha = F^\alpha(x, \mathbf{g})$, let
  $V^\alpha = \mathbf{g}^{\mu\nu}\mathbf{\Gamma}^\alpha_{\mu\nu} - F^\alpha$
- Example:
  Fix a background spacetime $(\hat{M}, \hat{g})$, $\hat{\nabla}$ covariant derivative, $\hat{\Gamma}$ Christoffel's of $\hat{g}$
- tension field
  $$V^\alpha = \mathbf{g}^{\mu\nu}(\Gamma^\alpha_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\nu})$$
- $V = 0 \Leftrightarrow \mathbf{Id} : \mathbf{M} \to \hat{M}$ is harmonic
  (cf. DeTurck's trick for Ricci flow)
- Simplest case: $\Box x^\alpha = 0$

Harmonic coordinates

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Computation gives

$$\mathbf{R}_{\mu\nu} = -\frac{1}{2}\mathbf{g}^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\mathbf{g}_{\mu\nu} + J(\mathbf{g},\partial\mathbf{g})_{\mu\nu} + \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu)$$

- $J(\mathbf{g},\partial\mathbf{g})_{\mu\nu}$ is quadratic in $\partial\mathbf{g}$

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Harmonic coordinates

- Computation gives

$$\mathbf{R}_{\mu\nu} = -\frac{1}{2}\mathbf{g}^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\mathbf{g}_{\mu\nu} + J(\mathbf{g}, \partial\mathbf{g})_{\mu\nu} + \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu)$$

- $J(\mathbf{g}, \partial\mathbf{g})_{\mu\nu}$ is quadratic in $\partial\mathbf{g}$
- $$\mathbf{R}^{\text{red}}_{\mu\nu} := \mathbf{R}_{\mu\nu} - \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu)$$

  is a quasilinear hyperbolic operator $\Rightarrow$ has well posed Cauchy problem

Reduced vacuum Einstein equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Suppose we have vacuum Cauchy data (Σ, g, K) solving the Einstein constraints, and given lapse, shift N, X on Σ
- By choosing $\partial_t N, \partial_t X$, we can always get tension field $V = 0$ on Σ

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Reduced vacuum Einstein equations

- Suppose we have vacuum Cauchy data (Σ, g, K) solving the Einstein constraints, and given lapse, shift N, X on Σ
- By choosing $\partial_t N, \partial_t X$, we can always get tension field $V = 0$ on Σ
- Use result about Cauchy problem for QL wave equations to construct a solution $\mathbf{g}$ to the reduced Einstein equations
$$\mathbf{R}^{\mathsf{red}}_{\mu\nu} = 0$$
with data $\mathbf{g}_{\mu\nu}, \partial_t \mathbf{g}_{\mu\nu}$ corresponding to (g, N, X) and $(K, \partial_t N, \partial_t X)$

Gauge Propagation

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Assume we have solved $\mathbf{R}^{\text{red}}_{\mu\nu} = 0$

Gauge Propagation

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Assume we have solved $\mathbf{R}^{\text{red}}_{\mu\nu} = 0$
- Then

$$G_{\mu\nu} = \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu - \nabla_\gamma V^\gamma \mathbf{g}_{\mu\nu})$$

- The Einstein constraints $G_{TT} = 0$, $G_{Ti} = 0$ hold on Σ by assumption,
this forces $\partial_t V = 0 \Rightarrow V$ has trivial Cauchy data

Gauge Propagation

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Assume we have solved $\mathbf{R}^{\text{red}}_{\mu\nu} = 0$
- Then

$$G_{\mu\nu} = \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu - \nabla_\gamma V^\gamma \mathbf{g}_{\mu\nu})$$

- The Einstein constraints $G_{TT} = 0$, $G_{Ti} = 0$ hold on Σ by assumption,
  this forces $\partial_t V = 0 \Rightarrow V$ has trivial Cauchy data
- From $\nabla^\mu G_{\mu\nu} = 0$ get

$$\nabla^\mu(\nabla_\mu V_\nu + \nabla_\nu V_\mu - \nabla_\gamma V^\gamma \mathbf{g}_{\mu\nu}) = 0$$

which gives

$$\Box V_\nu + \mathbf{R}_\nu{}^\delta V_\delta = 0$$

Cauchy problem for the Einstein equations

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

- Since V has trivial initial data, $V \equiv 0$ by uniqueness, so $\mathbf{g}$ solves the full vacuum Einstein equation $\mathbf{R}_{\mu\nu} = 0$.
- Important point: For general data (Σ, g, K) with noncompact Σ, *existence time may be* $= 0$
- However, due to causality (finite propagation speed) we can solve the Cauchy problem locally and patch together.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Variational formulation
Hamiltonian formulation
Vacuum Constraints
Cauchy problem
Solving the constraints, Conformal method
Remarks
Matter
Harmonic coordinates
Reduced Einstein
Gauge Propagation
Global Uniqueness

Cauchy problem for the Einstein equations

- Since V has trivial initial data, $V \equiv 0$ by uniqueness, so $\mathbf{g}$ solves the full vacuum Einstein equation $\mathbf{R}_{\mu\nu} = 0$.
- Important point: For general data (Σ, g, K) with noncompact Σ, *existence time may be* $= 0$
- However, due to causality (finite propagation speed) we can solve the Cauchy problem locally and patch together.

Theorem 4. *Given vacuum Cauchy data set $\mathcal{S} = (\Sigma, g, K)$, there is a vacuum Cauchy development of $\mathcal{S}$*

Remark. This result does not claim any maximality or uniqueness for the development of $\mathcal{S}$
For $n = 3$, this proof requires data in H^s, $s \geq 4$.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Global Uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Global uniqueness

Theorem 5 (Choquet-Bruhat, Geroch, 1969). *Let $\mathcal{S} = (\Sigma, g, K)$ be a vacuum Cauchy data set. Then there is a maximal vacuum Cauchy development of $\mathcal{S}$, which is unique up to diffeomorphism.*

Global uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Theorem 6 (Choquet-Bruhat, Geroch, 1969). *Let $\mathcal{S} = (\Sigma, g, K)$ be a vacuum Cauchy data set. Then there is a maximal vacuum Cauchy development of $\mathcal{S}$, which is unique up to diffeomorphism.*

Proof: In the proof we will use the term *extension* both for a vacuum Cauchy development of a Cauchy data set $\mathcal{S}$ and for an extension in the following sense:
If M, M' are globally hyperbolic, vacuum and there is diffeo $\psi : M \to M'$, then we call M' an *extension* of M

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Global uniqueness

Theorem 7 (Choquet-Bruhat, Geroch, 1969). *Let $\mathcal{S} = (\Sigma, g, K)$ be a vacuum Cauchy data set. Then there is a maximal vacuum Cauchy development of $\mathcal{S}$, which is unique up to diffeomorphism.*

Proof: In the proof we will use the term *extension* both for a vacuum Cauchy development of a Cauchy data set $\mathcal{S}$ and for an extension in the following sense:
If M, M' are globally hyperbolic, vacuum and there is diffeo $\psi : M \to M'$, then we call M' an *extension* of M

- Recall: A binary relation $\leq$ on a set X is a partial order if it is reflexive, transitive and antisymmetric

Global uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Theorem 8 (Choquet-Bruhat, Geroch, 1969). *Let $\mathcal{S} = (\Sigma, g, K)$ be a vacuum Cauchy data set. Then there is a maximal vacuum Cauchy development of $\mathcal{S}$, which is unique up to diffeomorphism.*

Proof: In the proof we will use the term *extension* both for a vacuum Cauchy development of a Cauchy data set $\mathcal{S}$ and for an extension in the following sense:
If M, M' are globally hyperbolic, vacuum and there is diffeo $\psi : M \to M'$, then we call M' an *extension* of M

- Recall: A binary relation $\le$ on a set X is a partial order if it is reflexive, transitive and antisymmetric
- Zorn's lemma: A partially ordered set such that each totally ordered subset has an upper bound, has a maximal element. Zorn $\Leftrightarrow$ axiom of choice.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Global uniqueness

Theorem 9 (Choquet-Bruhat, Geroch, 1969). *Let $\mathcal{S} = (\Sigma, g, K)$ be a vacuum Cauchy data set. Then there is a maximal vacuum Cauchy development of $\mathcal{S}$, which is unique up to diffeomorphism.*

Proof: In the proof we will use the term *extension* both for a vacuum Cauchy development of a Cauchy data set $\mathcal{S}$ and for an extension in the following sense:
If M, M' are globally hyperbolic, vacuum and there is diffeo $\psi : M \to M'$, then we call M' an *extension* of M

- Recall: A binary relation $\leq$ on a set X is a partial order if it is reflexive, transitive and antisymmetric
- Zorn's lemma: A partially ordered set such that each totally ordered subset has an upper bound, has a maximal element. Zorn $\Leftrightarrow$ axiom of choice.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Global uniqueness

- Let $\mathcal{M} = \{$ extensions of $\mathcal{S}\}$

Global uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

- Let $\mathcal{M} = \{$ extensions of $\mathcal{S}\}$
- Let $N, N' \in \mathcal{M}$. (U, ψ) is a common part if $U \subset N$, $\psi : U \to N'$ diffeo. U is nonempty by local well-posedness.

Global uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

- Let $\mathcal{M} = \{$ extensions of $\mathcal{S}\}$
- Let $N, N' \in \mathcal{M}$. (U, ψ) is a common part if $U \subset N$, $\psi : U \to N'$ diffeo. U is nonempty by local well-posedness.
- $N, N' \in \mathcal{M}$. $N \leq N'$ if the maximal common part is $U = N$. $\mathcal{M}$ is partially ordered by $\leq$.

Global uniqueness

- Let $\mathcal{M} = \{$ extensions of $\mathcal{S}\}$
- Let $N, N' \in \mathcal{M}$. (U, ψ) is a common part if $U \subset N$, $\psi : U \to N'$ diffeo. U is nonempty by local well-posedness.
- $N, N' \in \mathcal{M}$. $N \leq N'$ if the maximal common part is $U = N$. $\mathcal{M}$ is partially ordered by $\leq$.
- Consider totally ordered subset $\{N_\alpha\}$ of $\mathcal{M}$. Common parts (U, ψ) induce equivalence relation $\sim$. Let $K = (\cup_\alpha N_\alpha)/\sim$. This defined a topology on K, as well as a natural metric and differentiable structure.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Global uniqueness

- Let $\mathcal{M} = \{$ extensions of $\mathcal{S}\}$
- Let $N, N' \in \mathcal{M}$. (U, ψ) is a common part if $U \subset N$, $\psi : U \to N'$ diffeo. U is nonempty by local well-posedness.
- $N, N' \in \mathcal{M}$. $N \leq N'$ if the maximal common part is $U = N$. $\mathcal{M}$ is partially ordered by $\leq$.
- Consider totally ordered subset $\{N_\alpha\}$ of $\mathcal{M}$. Common parts (U, ψ) induce equivalence relation $\sim$. Let $K = (\cup_\alpha N_\alpha)/\sim$. This defined a topology on K, as well as a natural metric and differentiable structure.
- From the definitions, K is a development of each N_α and hence K is and upper bound to $\{N_\alpha\}$. Therefore by Zorn, $\mathcal{M}$ has a maximal element M.

Global uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

- Remains to show: M is an extension of each extension of $\mathcal{S}$. Let M' be some other extension. Let $\tilde{M} = M \cup M' / \sim$. If $\tilde{M} \in \mathcal{M}$, then $\tilde{M} = M$ and we are done. Suppose not, then can argue $\tilde{M}$ is fails to be Hausdorff.

Global uniqueness

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

- Remains to show: M is an extension of each extension of $\mathcal{S}$. Let M' be some other extension. Let $\tilde{M} = M \cup M'/\sim$. If $\tilde{M} \in \mathcal{M}$, then $\tilde{M} = M$ and we are done. Suppose not, then can argue $\tilde{M}$ is fails to be Hausdorff.
- Final steps in the proof:
  Let (U, ψ) be maximal common part of M, M', $\psi : U \to M$. Take a non-Haussdorff point p', consider $\psi(T' - p') \cup \{p\}$ for some Cauchy surface T' in M'.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Global uniqueness

- Remains to show: M is an extension of each extension of $\mathcal{S}$. Let M' be some other extension. Let $\tilde{M} = M \cup M'/\sim$. If $\tilde{M} \in \mathcal{M}$, then $\tilde{M} = M$ and we are done. Suppose not, then can argue $\tilde{M}$ is fails to be Hausdorff.
- Final steps in the proof:
  Let (U, ψ) be maximal common part of M, M', $\psi : U \to M$. Take a non-Haussdorff point p', consider $\psi(T' - p') \cup \{p\}$ for some Cauchy surface T' in M'.
- By local well posedness, get extension, which contradicts maximality of U. Thus, $\tilde{M}$ is Haussdorff, and $\tilde{M} \in \mathcal{M}$. Thus $\tilde{M} = M$, and hence M is unique.

Examples

Maximal globally hyperbolic spacetimes, which *do not have any* (even non-globally hyperbolic) C^2 extensions

Examples

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Maximal globally hyperbolic spacetimes, which *do not have any* (even non-globally hyperbolic) C^2 extensions

- Minkowski space $\mathbb{R}^{n+1}$

Examples

Maximal globally hyperbolic spacetimes, which *do not have any* (even non-globally hyperbolic) C^2 extensions

- Minkowski space $\mathbb{R}^{n+1}$
- Lorentz cone $-dt^2 + t^2\gamma^n_{\mathbb{H}}$ for $n \geq 2$, where $\gamma^n_{\mathbb{H}}$ is an n-dimensional compact hyperbolic manifold

Examples

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Maximal globally hyperbolic spacetimes, which *do not have any* (even non-globally hyperbolic) C^2 extensions

- Minkowski space $\mathbb{R}^{n+1}$
- Lorentz cone $-dt^2 + t^2\gamma^n_{\mathbb{H}}$ for $n \geq 2$, where $\gamma^n_{\mathbb{H}}$ is an n-dimensional compact hyperbolic manifold
- non-flat Kasner

Examples

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Maximal globally hyperbolic spacetimes, which *do not have any* (even non-globally hyperbolic) C^2 extensions

- Minkowski space $\mathbb{R}^{n+1}$
- Lorentz cone $-dt^2 + t^2\gamma^n_{\mathbb{H}}$ for $n \geq 2$, where $\gamma^n_{\mathbb{H}}$ is an n-dimensional compact hyperbolic manifold
- non-flat Kasner
- Generic T^3 Gowdy spacetimes

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Examples

Maximal globally hyperbolic spacetimes, which *do not have any* (even non-globally hyperbolic) C^2 extensions

- Minkowski space $\mathbb{R}^{n+1}$
- Lorentz cone $-dt^2 + t^2\gamma^n_{\mathbb{H}}$ for $n \geq 2$, where $\gamma^n_{\mathbb{H}}$ is an n-dimensional compact hyperbolic manifold
- non-flat Kasner
- Generic T^3 Gowdy spacetimes
- Non-Taub Bianchi IX

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Examples

Maximal globally hyperbolic spacetimes, which *do not have any* (even non-globally hyperbolic) C^2 extensions

- Minkowski space $\mathbb{R}^{n+1}$
- Lorentz cone $-dt^2 + t^2\gamma^n_{\mathbb{H}}$ for $n \geq 2$, where $\gamma^n_{\mathbb{H}}$ is an n-dimensional compact hyperbolic manifold
- non-flat Kasner
- Generic T^3 Gowdy spacetimes
- Non-Taub Bianchi IX

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Examples

Maximal globally hyperbolic spacetimes which *have* non-globally hyperbolic vacuum extensions

- $I^+(\{0\}))$ in $\mathbb{R}^{n+1}$

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Examples

Maximal globally hyperbolic spacetimes which *have* non-globally hyperbolic vacuum extensions

- $I^+(\{0\})$) in $\mathbb{R}^{n+1}$
- 1+1 dimensional Misner: Lorentz cone $-dt^2 + t^2 S^1$

Examples

Overview of topics

The Wave equation

Spacetimes

Wave equation

The Einstein equations

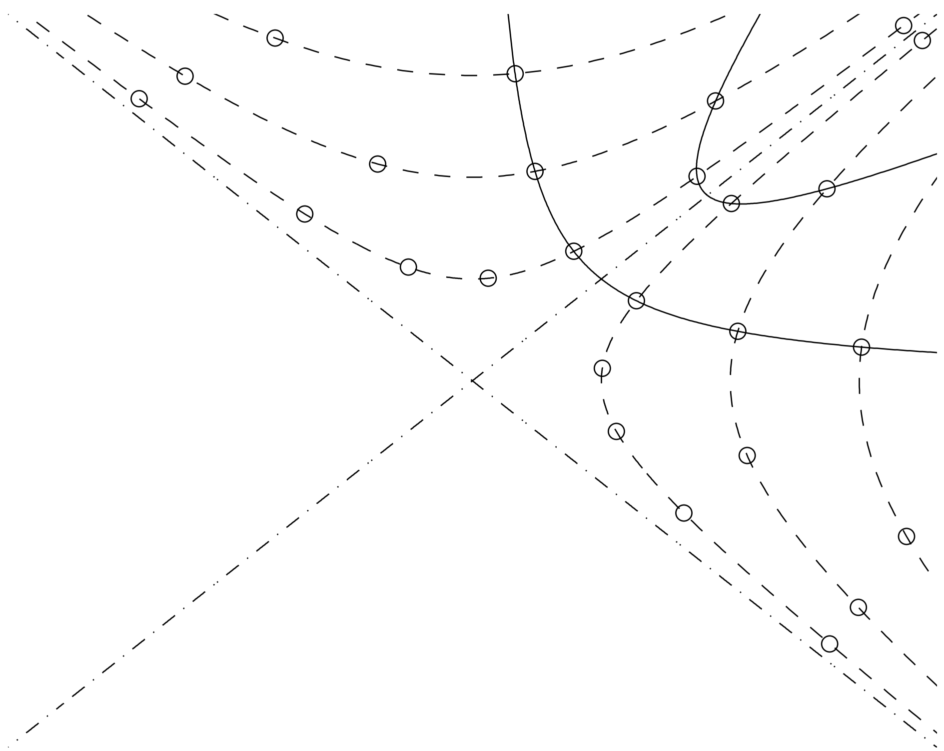
Global Uniqueness

Global uniqueness

Examples

Maximal globally hyperbolic spacetimes which *have* non-globally hyperbolic vacuum extensions

- $I^+(\{0\}))$ in $\mathbb{R}^{n+1}$
- 1+1 dimensional Misner: Lorentz cone $-dt^2 + t^2 S^1$



Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Examples

- The Taub-NUT spacetime $\mathbf{M} \cong \mathbb{R} \times S^3$, is a spatially homogenous solution of the Einstein equations.
- The Cauchy surfaces are S^3 collapsing along the Hopf-fibration.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Examples

- The Taub-NUT spacetime $\mathbf{M} \cong \mathbb{R} \times S^3$, is a spatially homogenous solution of the Einstein equations.
- The Cauchy surfaces are S^3 collapsing along the Hopf-fibration.
- The Cauchy horizon $\mathcal{H}$ is a pair of null S^3, and the spacetime metric is smooth at $\mathcal{H}$.

Overview of topics
The Wave equation
Spacetimes
Wave equation
The Einstein equations
Global Uniqueness
Global uniqueness
Examples

Examples

- The Taub-NUT spacetime $\mathbf{M} \cong \mathbb{R} \times S^3$, is a spatially homogenous solution of the Einstein equations.
- The Cauchy surfaces are S^3 collapsing along the Hopf-fibration.
- The Cauchy horizon $\mathcal{H}$ is a pair of null S^3, and the spacetime metric is smooth at $\mathcal{H}$.
- There are inequivalent (non-globally hyperbolic) vacuum extensions of Taub-NUT