The Cauchy problem

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1997 Bet
Precise formulation
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- Informal versions of Cosmic censorship:
  - Weak Cosmic Censorship Conjecture: An observer who has viewed a singularity is destined to fall into it.

Cosmic Censorship

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  - Strong Cosmic Censorship Conjecture: No singularity is ever visible to an observer

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Cosmic Censorship

- Informal versions of Cosmic censorship:
  - Weak Cosmic Censorship Conjecture: An observer who has viewed a singularity is destined to fall into it.
  - Strong Cosmic Censorship Conjecture: No singularity is ever visible to an observer
- Wrong! There are non-generic examples.

Cosmic Censorship Bet, 1997 version

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Whereas Stephen W. Hawking (having lost a previous bet on this subject by not demanding genericity) still firmly believes that naked singularities are an anathema and should be prohibited by the laws of classical physics, And whereas John Preskill and Kip Thorne (having won the previous bet) still regard naked singularities as quantum gravitational objects that might exist, unclothed by horizons, for all the Universe to see,
Therefore Hawking offers, and Preskill/Thorne accept, a wager that
When any form of classical matter or field that is incapable of becoming singular in flat spacetime is coupled to general relativity via the classical Einstein equations, then
A dynamical evolution from generic initial conditions (i.e., from an open set of initial data) can never produce a naked singularity (a past-incomplete null geodesic from scri-plus).
The loser will reward the winner with clothing to cover the winner's nakedness. The clothing is to be embroidered with a suitable, truly concessionary message.

Stephen W. Hawking, John P. Preskill, Kip S. Thorne Pasadena, California, 5

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Precise formulations of Cosmic Censorship

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- Let Σ be compact. Then for generic vacuum Cauchy data (g, K) on Σ, the maximal globally hyperbolic vacuum development of (Σ, g, K) is the maximal vacuum (say C^2) spacetime containing (Σ, g, K) as a hypersurface.

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- Let Σ be compact. Then for generic vacuum Cauchy data (g, K) on Σ, the maximal globally hyperbolic vacuum development of (Σ, g, K) is the maximal vacuum (say C^2) spacetime containing (Σ, g, K) as a hypersurface.
- Let Σ be a manifold which is a connected sum of $\mathbb{R}^3$ with a compact manifold. Then, for generic, asymptotically flat vacuum data (g, K) on Σ, the maximal globally hyperbolic development of (Σ, g, K) is asymptotically flat at future null infinity, with complete $\mathcal{I}^+$

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Singularity Theorems

Singularity theorems

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Let $(\mathbf{M}, \mathbf{g})$ be a 3+1 dimensions globally hyperbolic spacetime.

- Raychaudhuri equation for vorticity free (hypersurface orthogonal) null congruence generated by ξ:

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - |\sigma|^2 - \mathbf{R}_{\mu\nu}\xi^\mu\xi^\nu$$

where θ is the "expansion", σ is the "shear"

Singularity theorems

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$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - |\sigma|^2 - \mathbf{R}_{\mu\nu}\xi^\mu\xi^\nu$$

  where θ is the "expansion", σ is the "shear"
- If Null Energy Condition $\mathbf{R}_{\mu\nu}\ell^\mu\ell^\nu \geq 0$ for any null vector ℓ holds, then

$$\frac{d\theta}{d\lambda} + \frac{1}{2}\theta^2 \leq 0$$

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- If $\theta(0) < 0$, then $\theta \searrow -\infty$ at some $\lambda \leq 2/|\theta(0)|$

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Singularity Theorems

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- In 3+1 dimensions, let $S \in \mathbf{M}$ be spacelike 2-surface. Let $\theta^{\pm}$ be the null expansions for future null normals of S. Assume $\theta^{\pm} \leq -\theta_0 < 0$, for some $\theta_0 > 0$. Assume $\mathbf{M}$ satisfies null energy condition and has noncompact Cauchy surface.

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- Then there is a future null geodesic starting at S with finite affine length, i.e. $\mathbf{M}$ must be "singular"

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Singularity Theorems

- Suppose $\mathbf{M}$ satisfies the Strong Energy Condition $\mathbf{R}_{\mu\nu} V^\mu V^\nu \geq 0$ for all timelike vectors V. Assume $\mathbf{M}$ contains a Cauchy hypersurface where $\mathrm{tr} K > K_0$ for some $K_0 > 0$. Then $\mathbf{M}$ contains an incomplete timelike geodesic, i.e. is "singular".

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- Conditions can be weakened by adding "generic" condition

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- Message from singularity theorems:
  positive energy gives focussing effect, spacetimes tend to be incomplete

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- conjecture: suppose $(\mathbf{M}, \mathbf{g})$ maximally globally hyperbolic, is vacuum and has compact Cauchy surface. Then either $\mathbf{M}$ has an incomplete causal geodesic, or splits as a product.

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- The only positive results on censorship, which do not assume small data, are for systems with at least 2-dimensional isometry group: Gowdy or Bianchi.

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- The only positive results on censorship, which do not assume small data, are for systems with at least 2-dimensional isometry group: Gowdy or Bianchi.
- Programme on low regularity well posedness for the Einstein equations (Klainerman, Rodnianski) may lead to progress on the 3-dimensional problem

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- Friedrich: small hyperboloidal data, semi-global existence to the future

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- Christodoulou-Klainerman: small AF data, global existence, geodesically complete spacetime, weak fall-off

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- Andersson-Moncrief, Rieris: small data global existence to the future for spacetimes with compact Cauchy surface

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- *Belinskiĭ, Khalatnikov and Lifshitz (BKL) proposal:* heuristic scenario for generic cosmological singularities
- The singularity is *spacelike*: observers near the singularity can't have communicated in the past; *silence* holds — particle horizons shrink to zero.

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- The singularity is *spacelike*: observers near the singularity can't have communicated in the past; *silence* holds — particle horizons shrink to zero.
- The singularity is *local*: spatial derivatives are dynamically insignificant near the singularity
- For normal matter or spacetime dimension $D \leq 11$, the dynamics is *oscillatory* near the singularity.
- Caveat: There are spacetimes with null or partly null singularities (eg. spacetimes close to Reissner-Nordström)
- The observed nature of the singularity depends on the Cauchy slicing used to analyze it.

Silent, oscillatory

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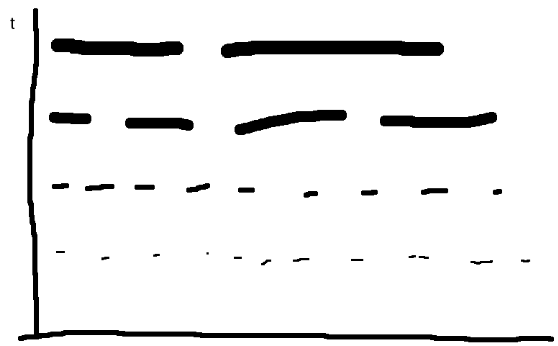

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BKL map

$$
\begin{aligned}
p &= \frac{1+u}{1+u+u^2} \\
q &= \frac{-u}{1+u+u^2} \\
r &= \frac{u+u^2}{1+u+u^2}
\end{aligned}
$$

BKL observed that the (chaotic!) map

$$
u \mapsto \begin{cases} u-1 & u>1 \\ 1/u & 0<u<1 \end{cases}
$$

is a good model for the asymptotic dynamics in generic spatially homogenous cosmologies (Bianchi IX)

BKL map image

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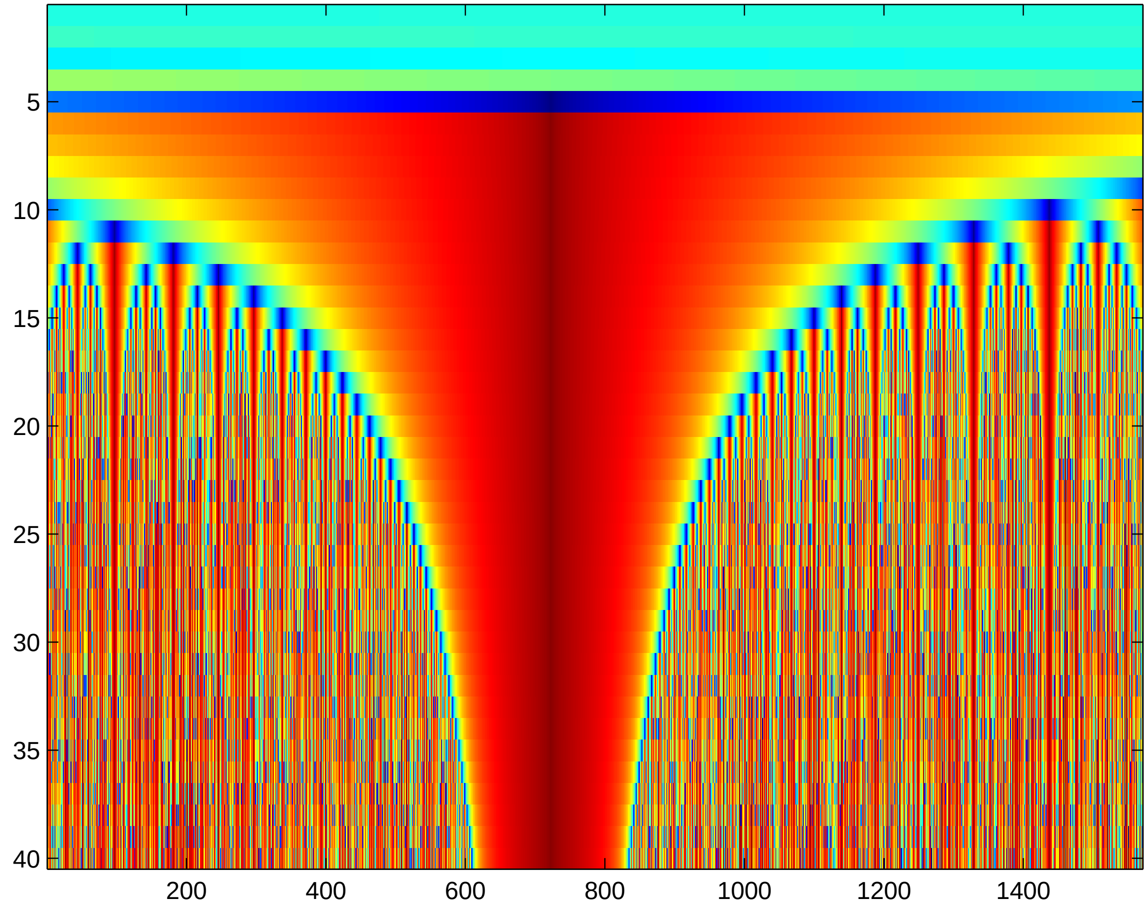

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Suppose $(\mathbf{M}, \mathbf{g})$ is vacuum, maximally globally hyperbolic
Σ Cauchy surface in $\mathbf{M}$
ξ vector field

- Lie derivative of $\mathbf{g}$ with respect to ξ:
  $\mathcal{L}_\xi \mathbf{g}_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$

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- Suppose data for ξ given at Σ and suppose $\mathcal{L}_\xi \mathbf{g} = 0$ at Σ. Then the TT and Ti components of $0 = \mathcal{L}_\xi \mathbf{g}$ at Σ determine $\partial_t \xi$ at Σ.

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- ξ is a Killing field if $\mathcal{L}_\xi \mathbf{g} \equiv 0$
- Suppose data for ξ given at Σ and suppose $\mathcal{L}_\xi \mathbf{g} = 0$ at Σ. Then the TT and Ti components of $0 = \mathcal{L}_\xi \mathbf{g}$ at Σ determine $\partial_t \xi$ at Σ.
- Killing's equation gives in vacuum

$$
\begin{aligned}
0 &= \nabla^\mu (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} \nabla_\gamma \xi^\gamma \mathbf{g}_{\mu\nu}) \\
&= \Box \xi_\nu + \mathbf{R}_\nu{}^\gamma \xi_\gamma = \Box \xi_\nu
\end{aligned}
$$

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- Since we have Cauchy data for ξ at Σ, there is a unique ξ defined globally on solving the above equation.

Symmetries

- Since we have Cauchy data for ξ at Σ, there is a unique ξ defined globally on solving the above equation.
- Suppose ξ is tangent to Σ. The structure equations imply that $\partial_t(\mathcal{L}_\xi \mathbf{g}) = 0$ at Σ, if ξ is Killing at Σ.

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- Suppose ξ is tangent to Σ. The structure equations imply that $\partial_t(\mathcal{L}_\xi \mathbf{g}) = 0$ at Σ, if ξ is Killing at Σ.
- Calculation gives

$$\Box \nabla_{(\mu}\xi_{\nu)} = 2\mathbf{R}^{\alpha}{}_{(\mu\nu)}{}^{\gamma}\nabla_{(\alpha}\xi_{\gamma)}$$

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- Suppose ξ is tangent to Σ. The structure equations imply that $\partial_t(\mathcal{L}_\xi \mathbf{g}) = 0$ at Σ, if ξ is Killing at Σ.
- Calculation gives

$$\Box \nabla_{(\mu}\xi_{\nu)} = 2\mathbf{R}^{\alpha}{}_{(\mu\nu)}{}^{\gamma}\nabla_{(\alpha}\xi_{\gamma)}$$

Since $\nabla_{(\mu}\xi_{\nu)}$ has trivial Cauchy data at Σ, $\nabla_{(\mu}\xi_{\nu)} \equiv 0$ on $\mathbf{M}$, so ξ is globally Killing

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- Suppose a compact group G acts on $\mathbf{M}$ by isometries.
- Then the action of G is generated by spacelike Killing fields.
- There is a foliation of $\mathbf{M}$ by Cauchy hypersurfaces invariant under the action of G

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- There is a foliation of $\mathbf{M}$ by Cauchy hypersurfaces invariant under the action of G
- Proof: average a timefunction

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Hierarchy of cosmological models

orbit dimension	system	type
3	Bianchi or K-S	ODE
2	Surface symmetry or G_2	1+1 PDE
1	G_1	2+1 PDE
0	G_0	3+1 PDE

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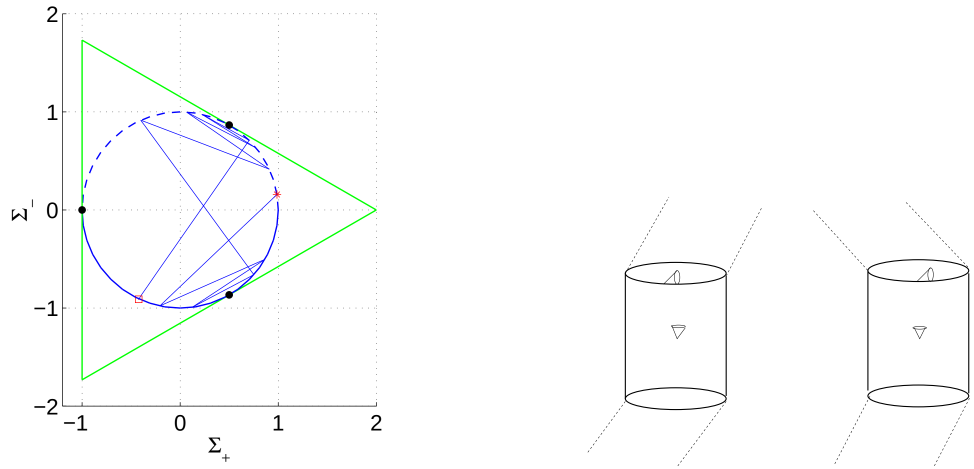
Bianchi

- spatially homogenous models $\Rightarrow$ Einstein equations become ODE's.
- Classify according to isometry group
- "generic" Bianchi models have oscillatory singularity (Ringström)



(a) (b)

(a) Kasner billiard — Bianchi IX — Mixmaster
(b) Taub-NUT has Cauchy horizon — extendible.

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Kaluza-Klein reduction

- (Σ^m, g) Lorentzian, (N^n, h) Riemannian, $\mathrm{Ric}_h = \lambda_0 h$
- Ansatz: $\mathbf{g} = e^{2\alpha\phi} g + e^{2\alpha\phi} h$

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Kaluza-Klein reduction

- (Σ^m, g) Lorentzian, (N^n, h) Riemannian, $\mathrm{Ric}_h = \lambda_0 h$
- Ansatz: $\mathbf{g} = e^{2\alpha\phi} g + e^{2\alpha\phi} h$

$$\mathcal{L} = \int_{\mathbf{M}} \mathbf{R}\sqrt{-\mathbf{g}}$$
$$= \int_N \sqrt{h} \int_\Sigma \sqrt{-g} e^{(m\alpha+n\beta)\phi} \left(e^{-2\alpha\phi} R_g + e^{-2\beta\phi} R_h\right)$$
$$+ \text{ terms with } \partial^2\phi \text{ and } (\partial\phi)^2$$

Kaluza-Klein reduction

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- Ansatz: $\mathbf{g} = e^{2\alpha\phi} g + e^{2\alpha\phi} h$

$$
\begin{aligned}
\mathcal{L} &= \int_{\mathbf{M}} \mathbf{R}\sqrt{-\mathbf{g}} \\
&= \int_N \sqrt{h} \int_\Sigma \sqrt{-g} e^{(m\alpha + n\beta)\phi} \left(e^{-2\alpha\phi} R_g + e^{-2\beta\phi} R_h\right) \\
&\quad + \text{terms with } \partial^2\phi \text{ and } (\partial\phi)^2
\end{aligned}
$$

- Condition for Einstein-Hilbert type action for g:

$$(m-2)\alpha + n\beta = 0$$

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Kaluza-Klein reduction

- Using conformal transformation rules for Ricci and scalar curvature, show that modulo total divergence the action takes the form

$$\mathcal{L} = \int_N \sqrt{h} \int_\Sigma \sqrt{-g}[R_g + e^{(2\alpha-2\beta)\phi} n\lambda_0 + (2-m-n)\alpha\beta|d\phi|^2]$$

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Kaluza-Klein reduction

- Special case: $N = S^1$, Σ 3-dimensional. Corresponds to the case of $\mathbf{M}$ 3+1 dimensional, with spacelike, hypersurface orthogonal Killing field, generating an S^1 action.

Kaluza-Klein reduction

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- Special case: $N = S^1$, Σ 3-dimensional. Corresponds to the case of $\mathbf{M}$ 3+1 dimensional, with spacelike, hypersurface orthogonal Killing field, generating an S^1 action.
- Reduced field equations:

$$0 = \Box\phi, \quad \text{Ric} = 2\nabla\phi \otimes \nabla\phi$$

i.e. 2+1 dimensional Einstein-scalar field equations (with $V = 0$).

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  i.e. 2+1 dimensional Einstein-scalar field equations (with $V = 0$).
- This is the polarized $U(1)$ problem

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The $U(1)$ problem

- Let J be a Killing field generating a spatial $U(1)$ action on $\mathbf{M}$.
- $\pi : \mathbf{M} \to \Sigma = \mathbf{M}/U(1)$ be the projection

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- Let J be a Killing field generating a spatial $U(1)$ action on $\mathbf{M}$.
- $\pi : \mathbf{M} \to \Sigma = \mathbf{M}/U(1)$ be the projection
- If $\mathbf{M}$ has compact Cauchy surface, $\Sigma \cong S \times \mathbb{R}$, with S a Riemann surface
- Let $\lambda = -\log(|J|)$.
- Calculate

$$dJ = \Theta + 2d\lambda \wedge J$$

where $i_J\Theta = 0$.

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- Then $\Theta = \pi^*(e^{2\lambda} F)$, where F two form on Σ

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- Then $\Theta = \pi^*(e^{2\lambda}F)$, where F two form on Σ
- Let $E = -\star(e^{4\lambda}F)$, 1-form on Σ
- Einstein equations $\Rightarrow dE = 0$

The $U(1)$ problem

- Then $\Theta = \pi^*(e^{2\lambda} F)$, where F two form on Σ
- Let $E = - \star (e^{4\lambda} F)$, 1-form on Σ
- Einstein equations $\Rightarrow dE = 0$
- assume $E = d\omega$ ($U(1)$ bundle is trivial in this case)
- Field equations become

$$\text{Ric} = \frac{1}{2}\langle \nabla u, \nabla u \rangle_h, \quad \Box u + {}^h\Gamma(\langle \nabla u, \nabla u \rangle) = 0$$

where $u = (\lambda, \omega)$ is a map to $\mathbb{H}^2$ with metric

$$h = 4d\lambda^2 + e^{-4\lambda} d\omega^2$$

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$$h = 4d\lambda^2 + e^{-4\lambda}d\omega^2$$

- This is the 2+1 dimensional Einstein-Wave maps equation, with hyperbolic target

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- small data, global existence result by Choquet-Bruhat and Moncrief for polarized case, spatially compact, and Choquet-Bruhat, trivial bundle case, spatially compact. Genus of $S > 1$ is needed for these results

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- small data, global existence result by Choquet-Bruhat and Moncrief for polarized case, spatially compact, and Choquet-Bruhat, trivial bundle case, spatially compact. Genus of $S > 1$ is needed for these results
- genus(S)=1 case open, nontrivial bundle case open

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- small data, global existence result by Choquet-Bruhat and Moncrief for polarized case, spatially compact, and Choquet-Bruhat, trivial bundle case, spatially compact. Genus of $S > 1$ is needed for these results
- genus(S)$=$1 case open, nontrivial bundle case open
- Heuristics and numerical studies indicate that polarized $U(1)$ has non-oscillatory behavior at the singularity while generic $U(1)$ has oscillatory behavior at the singularity

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- Assume $\mathbf{M}$ has spatial $U(1) \times U(1)$ action, generated by ξ_1, ξ_2
- Assume ξ_1, ξ_2 are hypersurface orthogonal

$$\xi_1 \wedge \xi_2 \wedge d\xi_1 = 0, \quad \xi_1 \wedge \xi_2 \wedge d\xi_2 = 0$$

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- Let (t, x) be coordinates on $\mathbf{M}/U(1) \times U(1)$
  $A(t, x)$ area of orbit

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  $A(t, x)$ area of orbit
  Gowdy time: $A = 4\pi^2 e^{-t}$ (this is harmonic time)

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- Orbit metric Ah, $h = h(t, x)$ unit determinant metric on T^2

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- Orbit metric Ah, $h = h(t, x)$ unit determinant metric on T^2
- y_1, y_2 coordinates on T^2. Unit determinant metric parametrized as

$$h = e^P dy_1^2 + 2e^P Q dy_1 dy_2 + (e^P Q^2 + e^{-P}) dy_2^2$$

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- P, Q coordinates on Teich$(T^2) \cong \mathbb{H}^2$ with metric $dP^2 + e^{2P} dQ^2$

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- P, Q coordinates on Teich$(T^2) \cong \mathbb{H}^2$ with metric $dP^2 + e^{2P} dQ^2$
- Gowdy metric on $T^3 \times \mathbb{R}$:

$$\begin{aligned}\ell_0^{-2}\,\mathrm{d}s^2 &= e^{(t-\lambda)/2}\left(-\,e^{-2t}\,\mathrm{d}t^2 + \mathrm{d}x^2\right)\\ &\quad + e^{-t}\left[\,e^P\left(\mathrm{d}y_1 + Q\,\mathrm{d}y_2\right)^2 + e^{-P}\,\mathrm{d}y_2^2\,\right],\end{aligned}$$

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- Einstein equations yield a wave maps type equation with metric $\eta = -d\tau^2 + e^{2\tau}dx^2$

$$\eta^{\alpha\beta}(\partial_\alpha\partial_\beta u^a + \Gamma^a_{bc}\partial_\alpha u^b \partial_\beta u^c) = 0$$

$+$ supplementary equations which allow one to reconstruct the metric

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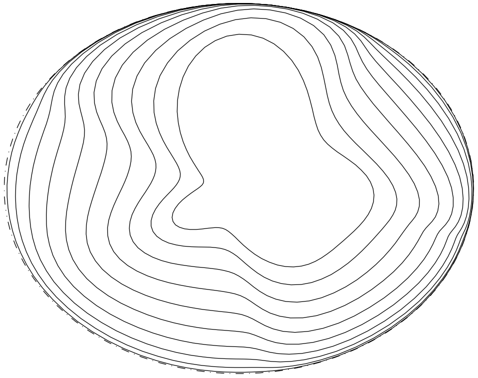
$$\eta^{\alpha\beta}(\partial_\alpha \partial_\beta u^a + \Gamma^a_{bc} \partial_\alpha u^b \partial_\beta u^c) = 0$$

  $+$ supplementary equations which allow one to reconstruct the metric
- The Gowdy Einstein equations can be viewed as equations for a loop in $\mathbb{H}^2$.



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Remarks

- global existence in holds (easy)

Remarks

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- global existence in holds (easy)
- As $t \to \infty$, $A \searrow 0$, so have cosmological singularity

Remarks

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- light cones in the η metric collapse as $t \to \infty \Rightarrow$ "asymptotic silence"

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- global existence in holds (easy)
- As $t \to \infty$, $A \searrow 0$, so have cosmological singularity
- light cones in the η metric collapse as $t \to \infty \Rightarrow$ "asymptotic silence"
- Cosmic Censorship holds. Proof (Ringström) involves showing the existence of an asymptotic velocity $\bar{v}$. $\bar{v} = 0$ corresponds to "flat Kasner" which is extendible. Curvature blowup for generic data is shown by perturbing away from "flat Kasner"

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- Energy

$$E = \frac{1}{2}\int_{S^1} \langle \partial_t u, \partial_t u\rangle + e^{-2t}\langle \partial_x u, \partial_x u\rangle = E_K + E_V$$

  E_K, E_V kinetic and potential energy terms

- $$\partial_\tau E = -2E_V$$

  $\Rightarrow$ potential energy tends to zero for some sequence of times $t_k \nearrow \infty$
  $\Rightarrow$ (heuristically) scale-free variables $(e^{-\tau}\partial_x P, e^{-\tau}\partial_x Q)$ become insignificant for the dynamics as $\tau \to \infty$

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Gowdy

- Generic Gowdy spacetimes have AVTD (Asymptotically Velocity Term Dominated) singularity
- Asymptotic behavior for (P, Q)

$$P(\tau, x) = k(x)\tau + \phi(x) + e^{-\epsilon\tau} u(t, x)$$
$$Q(\tau, x) = q(x) + e^{-2k(x)\tau}[\psi(x) + w(\tau, x)]$$

where $\epsilon > 0$, $u, w \to 0$ as $\tau \to \infty$ and $0 < k < 1$.

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Gowdy

- velocity in $\mathbb{H}^2$:

$$v_{\mathbb{H}^2}(t,x) = \sqrt{\langle \partial_t u, \partial_t u \rangle}$$

($v^2_{\mathbb{H}^2}$ is the kinetic energy density)

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- velocity in $\mathbb{H}^2$:

$$v_{\mathbb{H}^2}(t, x) = \sqrt{\langle \partial_t u, \partial_t u \rangle}$$

  ($v_{\mathbb{H}^2}^2$ is the kinetic energy density)
- $v_{\mathbb{H}^2}(t, x)$ has limit $\bar{v}$ as $t \to \infty$.

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- velocity in $\mathbb{H}^2$:

$$v_{\mathbb{H}^2}(t,x) = \sqrt{\langle \partial_t u, \partial_t u \rangle}$$

  ($v_{\mathbb{H}^2}^2$ is the kinetic energy density)
- $v_{\mathbb{H}^2}(t,x)$ has limit $\bar{v}$ as $t \to \infty$.
- Proof (Ringström) involves light cone energy estimates for quantities of the form

$$e^{\tau}[(\partial_t \partial_x^j P \pm e^{-t}\partial_x^{j+1} P)^2 + e^{2P}(\partial_t \partial_x^j Q \pm e^{-t}\partial_x^{j+1} Q)^2]$$

  note role of "null derivatives" $\partial_t \pm e^{-t}\partial_x$

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Spikes in Gowdy

- for generic data, $\bar{v}$ is continuous almost everywhere, and $0 < \bar{v} < 1$ at points of continuity, with jumps:

$$\bar{v}(x_*) = \lim_{x \to x_*} \bar{v}(x) + 1$$

at jump points x_* corresponding to the "spikes" in P

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Spikes in Gowdy

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$$\bar{v}(x_*) = \lim_{x \to x_*} \bar{v}(x) + 1$$

  at jump points x_* corresponding to the "spikes" in P
- $\bar{v} = 0$ corresponds to "flat Kasner".

Spikes in Gowdy

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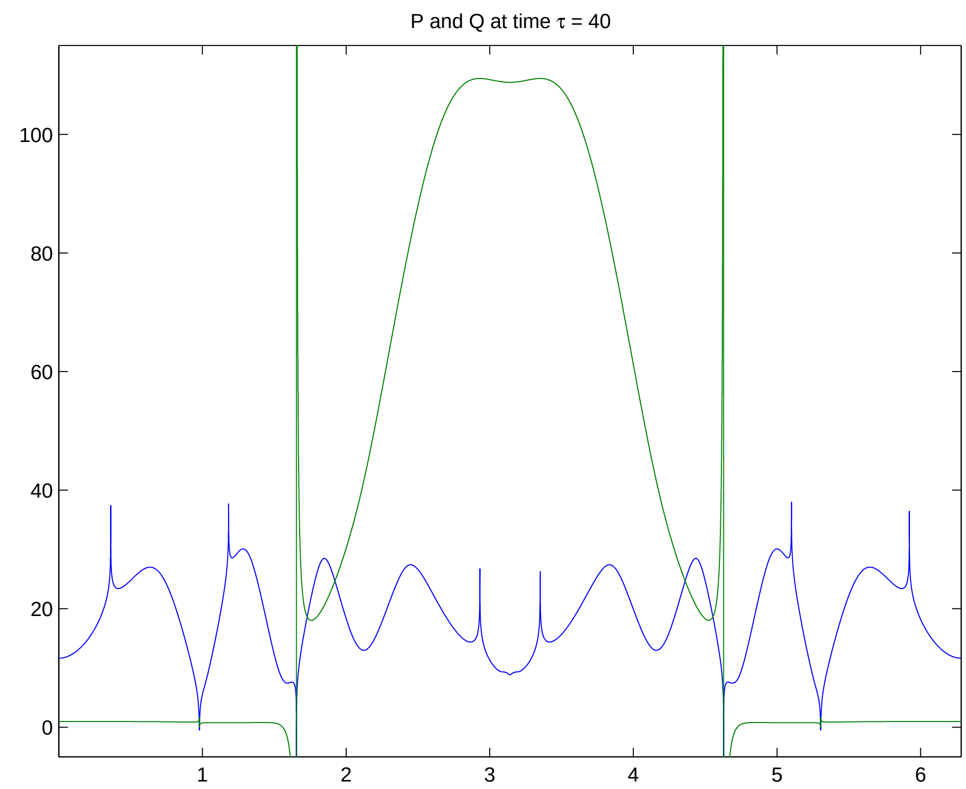

Figure 1: Spikes in P, Q. The very sharp spikes (in Q), so–called "false spikes" are coordinate effects.

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Snapshot of Gowdy, $t = 0$

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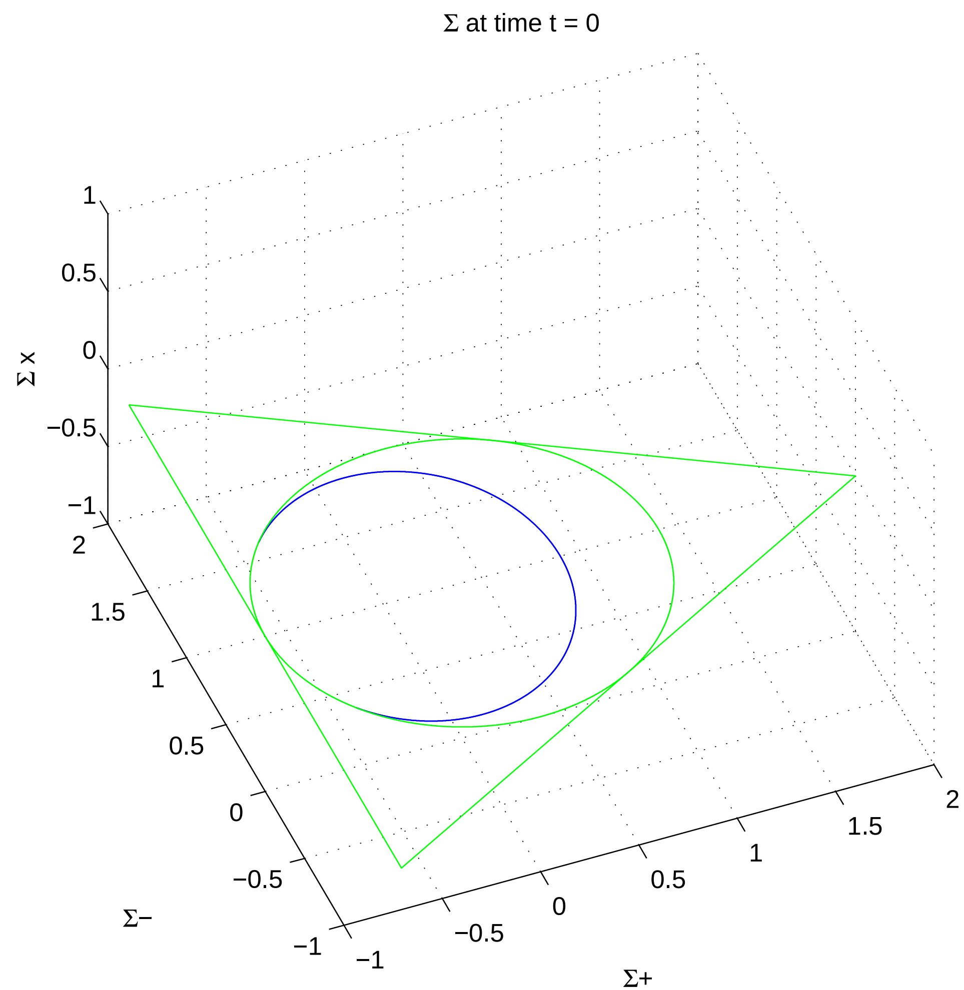

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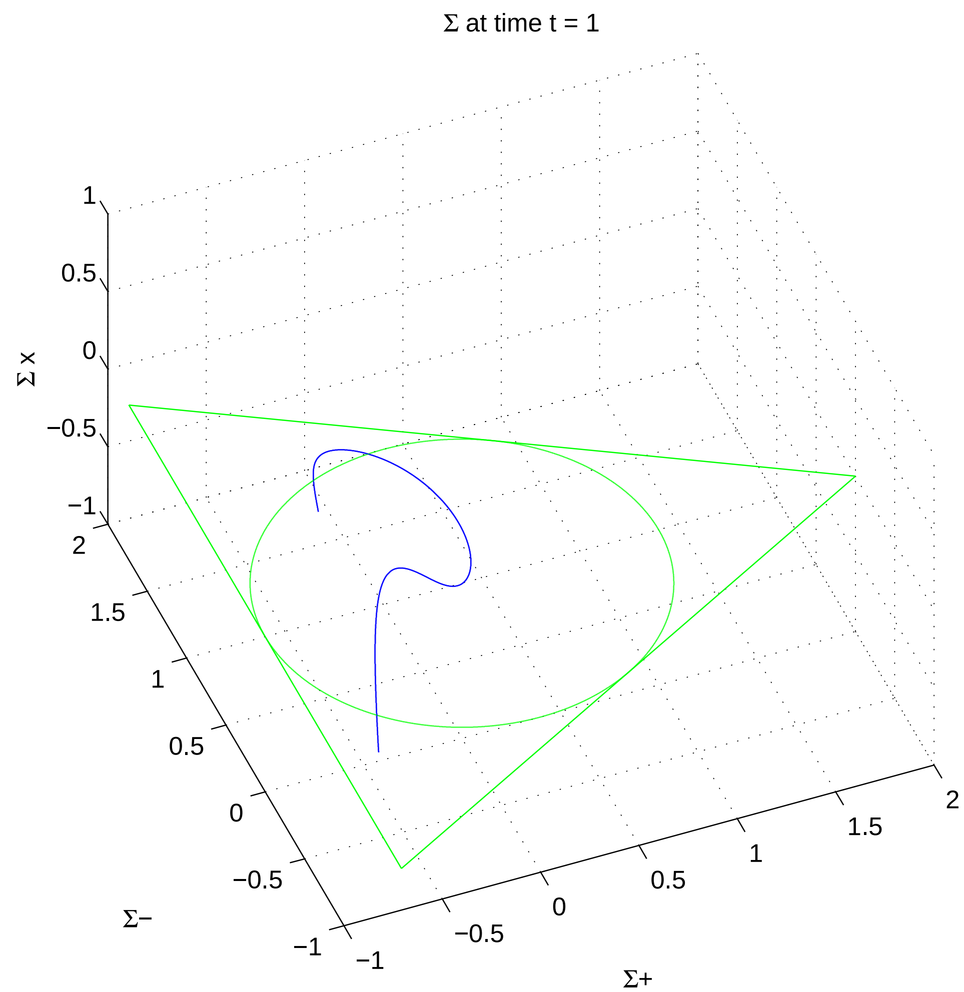
Snapshot of Gowdy, $t = 1$



Snapshot of Gowdy, $t = 2$

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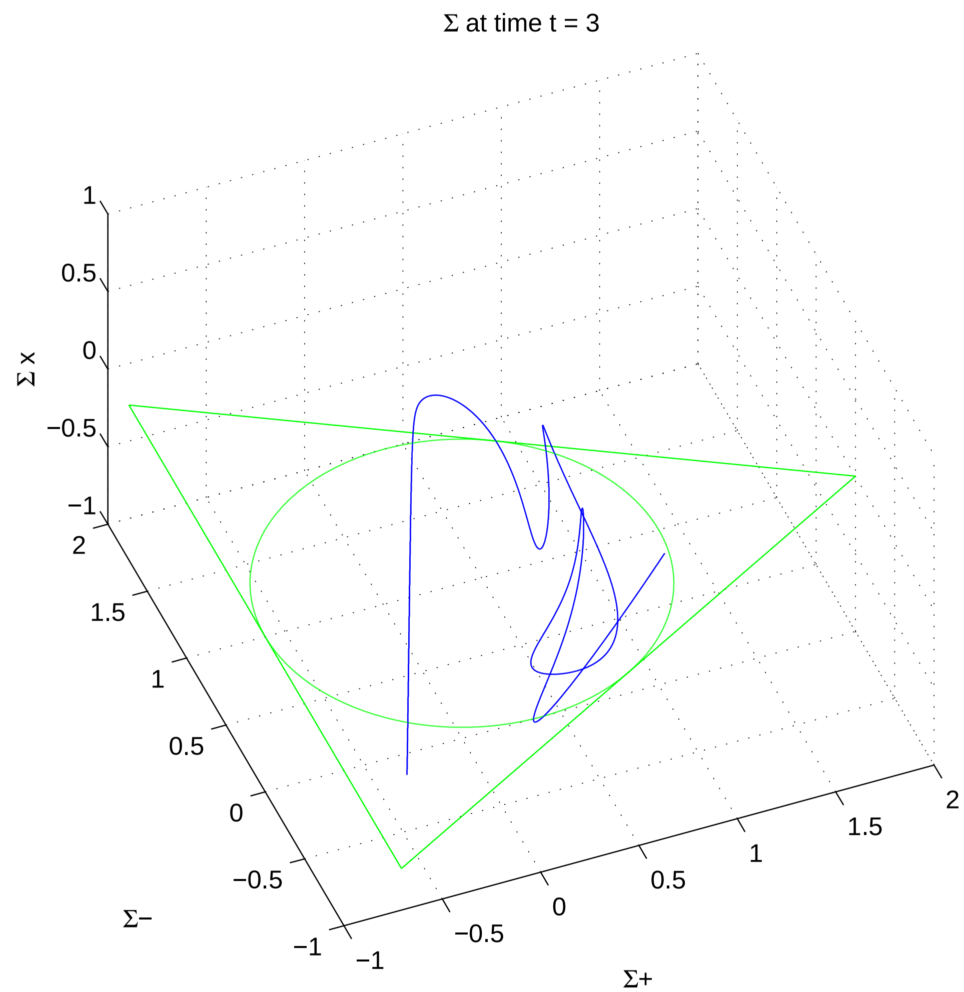

Snapshot of Gowdy, $t = 6$

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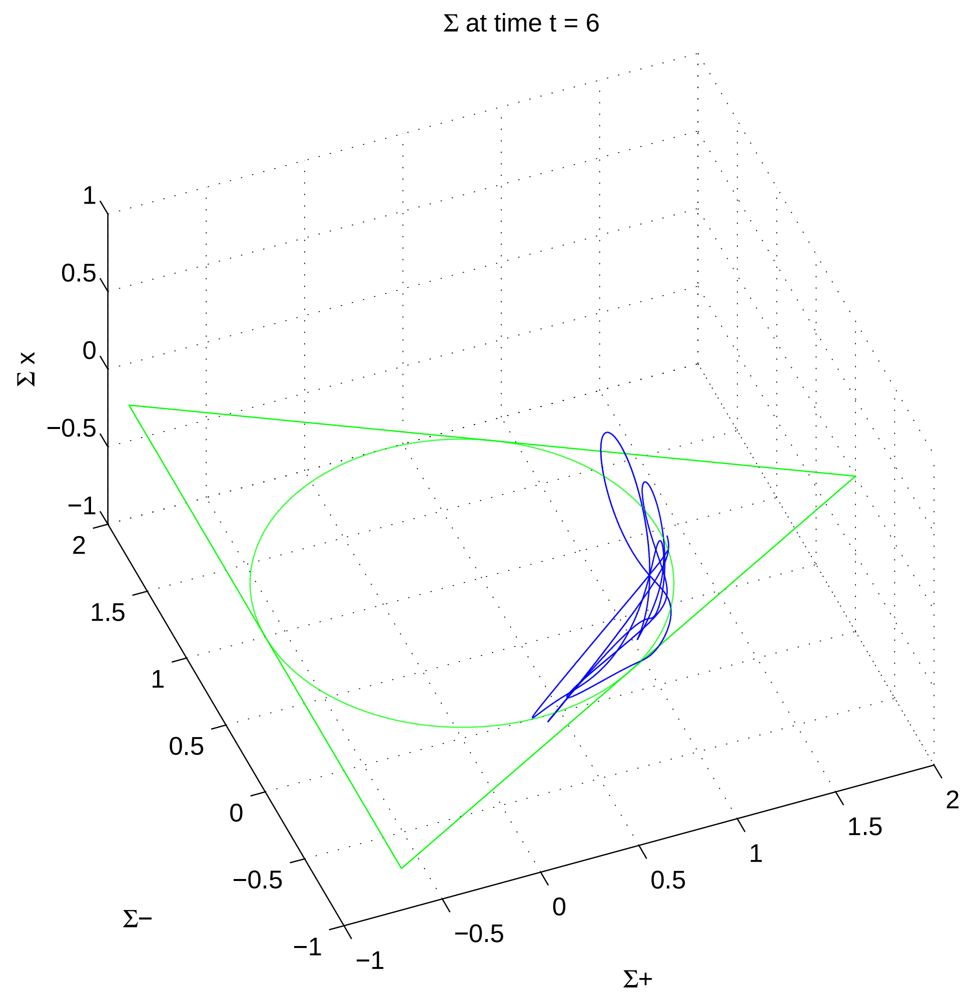

Snapshot of Gowdy, $t = 12$

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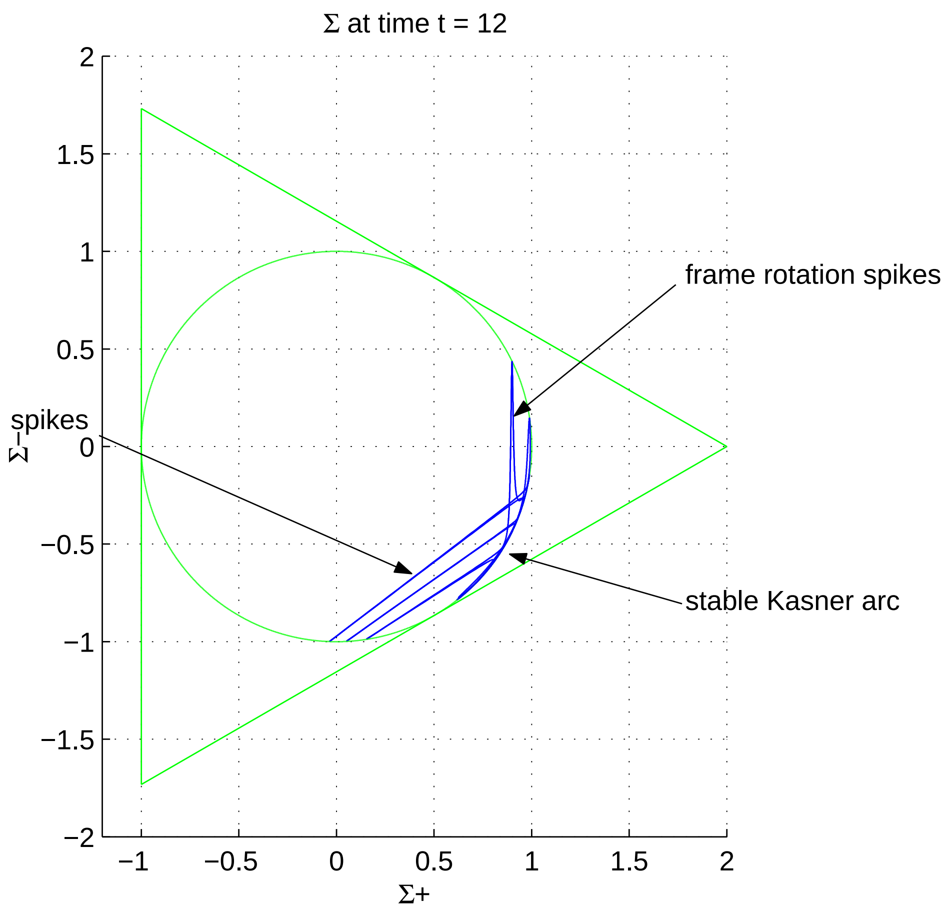

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- For generic data, the Kretschmann scalar $\kappa = \mathbf{R}_{\alpha\beta\gamma\delta}\mathbf{R}^{\alpha\beta\gamma\delta}$ blows up as $\tau \to \infty$, along generic timelines.

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- Proof involves using the genericity condition to perturb away from "flat Kasner".

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- Proof involves using the genericity condition to perturb away from "flat Kasner".
- This means: Cosmic Censorship holds for Gowdy

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If we relax the condition that the Killing fields generating the $U(1) \times U(1)$ action be hypersurface orthogonal, the general spacetimes metric takes the form

$$\ell_0^{-2}\, ds^2 = e^{(t+\lambda+\mu)/2}\left[- e^{-(2t+\mu/2)}\, dt^2 + dx^2 \right] + e^{-(t-P)}\left[(G_1 + QG_2)\, dx + dy_1 + Q\, dy_2 \right]^2 + e^{-(t+P)}\left[G_2\, dx + dy_2 \right]^2 ,$$

where G_1, G_2 are "twist potentials" ($= 0$ in the Gowdy case)

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- The reduced Einstein equations for $U(1) \times U(1)$ are a *quasi-linear* system, related to wave maps
- Global existence is well understood

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- The reduced Einstein equations for $U(1) \times U(1)$ are a *quasi-linear* system, related to wave maps
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- The behavior at the singularity is oscillatory (according to heuristics and numerical experiments)

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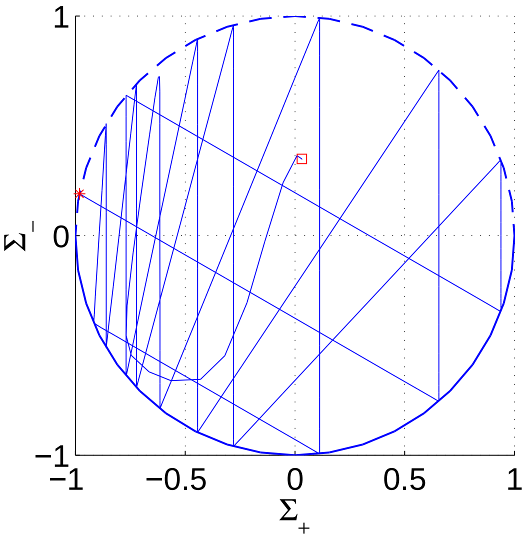

- Cosmic censorship is open

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