Santiago de Compostela, 8 February 2007

Lorentzian metrics: prescribed scalar curvature and energy conditions

Marc Nardmann

Convention. All manifolds in this talk are **compact**, nonempty and connected and may have a boundary.

Recall a famous question in Riemannian geometry:

The Riemannian prescribed scalar curvature problem.
Given an n-manifold M and a function $s \in C^\infty(M,\mathbb{R})$, is there a Riemannian metric on M whose scalar curvature is s?

Well-known: The answer is yes if $n \geq 2$ and $\partial M \neq \varnothing$.

If $n \geq 3$ and $\partial M = \varnothing$, then one of the following statements holds:

$\boxed{+}$ Every $s \in C^\infty(M,\mathbb{R})$ on M is a scalar curvature.

$\boxed{-}$ $s \in C^\infty(M,\mathbb{R})$ is a scalar curv. iff it is somewhere negative.

$\boxed{0}$ s is a scalar curv. iff it is somewhere negative or identically 0.

One can generalise the problem to semi-Riemannian geometry. This is a conference on Lorentzian geometry, so we discuss only the *Lorentzian* $(-+++\dots)$ case.

There exist *several* natural generalisations. We mention only three:

The Lorentzian prescribed scalar curvature problem.
We are given an n-manifold M, a function $s \in C^\infty(M, \mathbb{R})$, and...

Version A: *... a connected component $\mathscr{C}$ of the space of Lorentzian metrics on M. Does $\mathscr{C}$ contain a metric with scalar curvature s?*

Version B: *... a line distribution V on M. Does M admit a Lorentzian metric with scalar curvature s which makes V timelike?*

Version C: *... an $(n-1)$-plane distribution H on M. Does M admit a Lorentzian metric with scalar curvature s which makes H spacelike?*

Theorem A1. Let M be an n-manifold with $n \geq 3$, or $n = 2$ and $\partial M \neq \varnothing$. *Then for **every** $s \in C^\infty(M, \mathbb{R})$, **every** connected component of the space of Lorentzian metrics on M contains a metric with scalar curvature s.*

So no obstructions to certain functions s here, in contrast to the Riemannian case!

When $n = 2$ and $\partial M = \varnothing$, an obstruction occurs:

The **Gauß/Bonnet formula for closed Lorentzian surfaces** [Avez 1962] says that

$$\int_M \mathrm{scal}_g \; d\mu_g = 0 \; .$$

So if $s = \mathrm{scal}_g$ for some metric g, then s is identically 0 or changes its sign.

Theorem A2. Let M be the 2-dimensional torus or the Klein bottle (these are the only closed 2-manifolds which admit a Lorentzian metric). *A function $s \in C^\infty(M, \mathbb{R})$ is the scalar curvature of a Lorentzian metric on M* ***if*** *and only if s is either identically 0 or changes its sign.*

I don't know yet whether for every sign-changing s *each* of the infinitely many connected components of the space of Lorentzian metrics on M contains a metric with scalar curvature s.

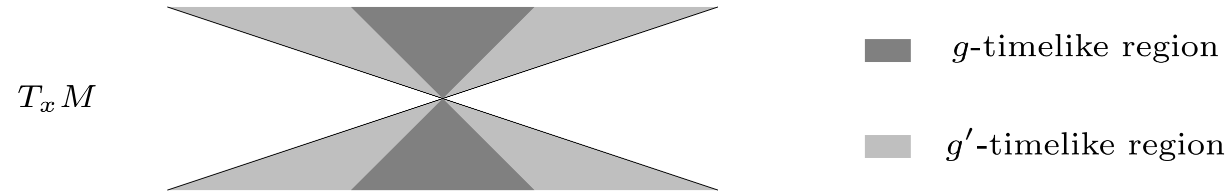
So much for Version A of the prescribed scalar curvature problem.
Let's move on to Version B.

If $n \geq 4$, then Version B of our problem has always a solution:

Theorem B. Let M be a manifold of dimension ≥ 4, let $s \in C^\infty(M, \mathbb{R})$, let V be a line distribution on M. *Then M admits a Lorentzian metric with scalar curvature s which makes V timelike.*

Even more is true:

Theorem B'. Let (M, g) be a Lorentzian manifold of dimension ≥ 4, let $s \in C^\infty(M, \mathbb{R})$. *Then M admits a Lorentzian metric g' with scalar curvature s such that every g-timelike vector in TM is g'-timelike.*



I don't know much about dimension 3 here.

Now we come to Version C of the prescribed scalar curvature problem, which is much harder.

Instead of stating the partial results I have obtained so far, let me just say what I expect the complete answers to be.

Recall that an $(n-1)$-plane distribution on an n-manifold M is called ***integrable*** iff it is the tangent distribution of an $(n-1)$-dimensional **foliation** of M.

Conjecture C1. Let M be a manifold of dimension $n \geq 4$, let $s \in C^\infty(M, \mathbb{R})$, let H be an $(n-1)$-plane distribution on M.

Assume that $\partial M \neq \varnothing$, or that H is not integrable. *Then M admits a Lorentzian metric with scalar curvature s which makes H spacelike.*

What happens when $\partial M = \varnothing$ and H is integrable? Here's my guess:

Conjecture C2. Let M be a closed manifold of dimension $n \geq 4$, let H be an integrable $(n-1)$-plane distribution on M.

Let $\mathscr{S}$ denote the set of all $s \in C^\infty(M, \mathbb{R})$ such that M admits a Lorentzian metric of scalar curvature s which makes H spacelike.

Then one of the following three statements holds:

$\boxed{+}$ $\mathscr{S} = C^\infty(M, \mathbb{R})$.

$\boxed{-}$ $s \in \mathscr{S}$ *iff s is somewhere negative.*

$\boxed{0}$ $s \in \mathscr{S}$ *iff s is somewhere negative or identically 0.*

Each of the three cases does indeed occur for suitable M and H.

Now let's go beyond scalar curvature and consider **Ricci curvature**.

Solving Ricci curvature ***equations*** (e.g. the Einstein equation) on arbitrary manifolds is too hard for current techniques, even in Riemannian geometry.

So let's construct metrics whose Ricci curvature solves an ***inequality***.

Definition. A Lorentzian manifold (M, g) satisfies
the strict causal convergence condition iff $\mathrm{Ric}_g(v, v) > 0$
holds for every timelike and every lightlike vector $v \in TM$.

Definition. Let (M, g) be a Lorentzian manifold, let $\Lambda \in \mathbb{R}$.
Consider the energy-momentum tensor $T := \mathrm{Ric}_g - \frac{1}{2}\,\mathrm{scal}_g\, g + \Lambda g$
with respect to the cosmological constant Λ.
We say that (M, g) satisfies
the strict dominant energy condition with respect to Λ iff
for every $x \in M$ and every timelike and every lightlike $v \in T_x M$,
the vector $-\sharp(T(v, .))$ is timelike and
contained in the same half of the full lightcone in $T_x M$ as v.
(Here $\sharp\colon T_x^* M \to T_x M$ is the isomorphism induced by g.)

Does every manifold which admits a Lorentzian metric at all admit one which satisfies the dominant energy condition?

The answer is *yes* in most cases:

Theorem R1. Let (M, g) be a Lorentzian manifold of dimension $n \geq 4$, let $\Lambda \in \mathbb{R}$.

If $n = 4$, assume that (M, g) is time- and space-orientable, and that either $\partial M \neq 0$, or M is closed with intersection form signature divisible by 4.

Then there exists a Lorentzian metric g' on M such that

- *g' satisfies the strict causal convergence condition;*
- *g' satisfies the strict dominant energy condition with respect to Λ;*
- *every g-timelike vector in TM is g'-timelike;*
- *M does not admit any spacelike $(n-1)$-dimensional foliation.*

A classical topic in General Relativity is the problem of ***topology change***.

Definition. Let S_0, S_1 be $(n-1)$-dimensional closed manifolds.

A ***weak Lorentz cobordism*** between S_0 and S_1 is a compact Lorentzian n-manifold (M, g) whose boundary is the disjoint union $S_0 \sqcup S_1$, such that M admits a timelike vector field which is inward-directed on S_0 and outward-directed on S_1.

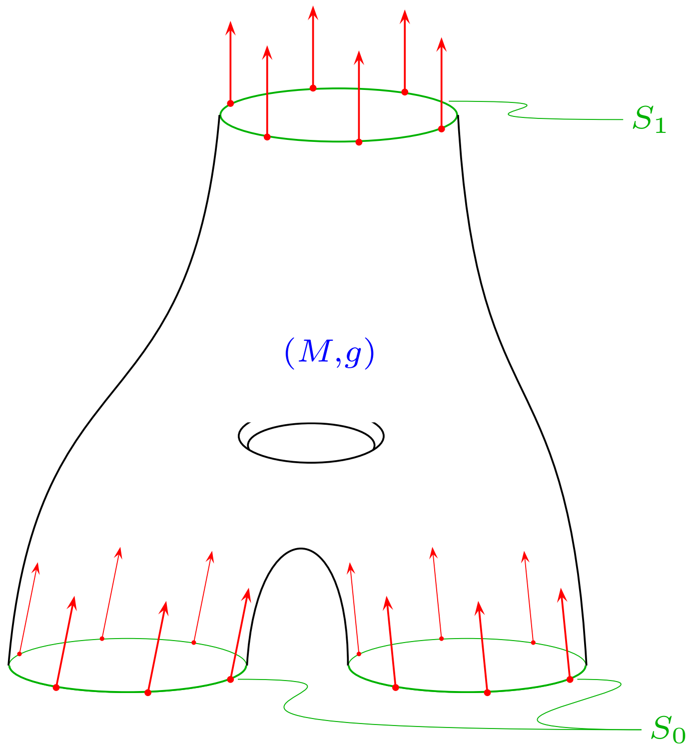
A ***Lorentz cobordism*** is a weak Lorentz cobordism (M, g) such that ∂M is g-spacelike.

Fact. S_0, S_1 are weakly Lorentz cobordant **iff** they are Lorentz cobordant.

Theorem [Tipler 1977]. *If there exists a Lorentz cobordism (M, g) between closed 3-manifolds S_0 and S_1 such that $\mathrm{Ric}_g(v, v) > 0$ holds for all lightlike $v \in TM$, then S_0 and S_1 are diffeomorphic.*

Theorem R2. Let S_0, S_1 be closed orientable 3-manifolds, let $\Lambda \in \mathbb{R}$. *Then there exists a* ***weak*** *Lorentz cobordism (M, g) between S_0 and S_1 such that $\mathrm{Ric}_g(v, v) > 0$ holds for all lightlike and all timelike $v \in TM$, and such that (M, g) satisfies the strict dominant energy condition with respect to Λ.*

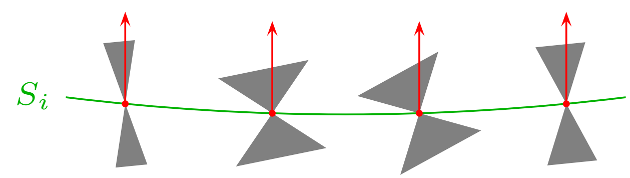
[weak] Lorentz cobordism:



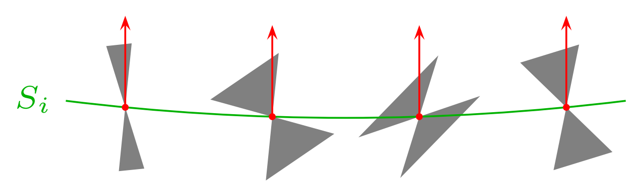
This picture is bad because the red vector field cannot be extended nonvanishingly to the whole cobordism. The viewer is supposed to imagine that it can. Unfortunately it is impossible to draw a good picture, because nontrivial 2-dimensional [weak] Lorentz cobordisms do not exist.

↑↑↑ timelike vector field on (M, g)

lightcones at the boundary:



Lorentz cobordism



weak Lorentz cobordism