The Maslov index via semi-Riemannian submersions

Erasmo Caponio and Miguel Angel Javaloyes

Politecnico di Bari / Instituto de Matemática e Estatística, Universidade de São Paulo

IV International meeting in Lorentzian Geometry, Santiago
February 5-8 2007

The Maslov Index: an introduction

The Maslov Index: an introduction

- In semi-Riemannian geometry, when the metric g is non-positive, the index form of a geodesic $\gamma : [a, b] \to M$,

$$I(V, W) = \int_a^b (g(V', W') - g(R_{V\dot{\gamma}}W, \dot{\gamma}))ds$$

associated to the energy functional

$$f(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma})ds$$

is always strongly indefinite, its index and coindex are infinite.

The Maslov Index: an introduction

- In semi-Riemannian geometry, when the metric g is non-positive, the index form of a geodesic $\gamma : [a, b] \to M$,

$$I(V, W) = \int_a^b (g(V', W') - g(R_{V\dot{\gamma}} W, \dot{\gamma}))ds$$

associated to the energy functional

$$f(\gamma) = \int_a^b g(\dot{\gamma}, \dot{\gamma})ds$$

is always strongly indefinite, its index and coindex are infinite.

- There is no Morse Index Theorem in the classical sense, (index of the index form equals the number of conjugate points counted with multiplicity).

Some interpretations and applications

It is possible to establish a semi-Riemannian Morse Index Theorem making a splitting of

$$\mathcal{H} = \{E \text{ a vector field along } \gamma \text{ of class } H^1 \text{ and } E(a) = E(b) = 0\}$$

as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. Then the index and coindex of these subspaces are finite and its difference gives the Maslov index.

Some interpretations and applications

It is possible to establish a semi-Riemannian Morse Index Theorem making a splitting of

$$\mathcal{H} = \{E \text{ a vector field along } \gamma \text{ of class } H^1 \text{ and } E(a) = E(b) = 0\}$$

as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. Then the index and coindex of these subspaces are finite and its difference gives the Maslov index. See

P. Piccione and D. V. Tausk, *The Morse index theorem in semi-Riemannian geometry*, Topology, **41** (2002), no. 6, 1123–1159.

Some interpretations and applications

It is possible to establish a semi-Riemannian Morse Index Theorem making a splitting of

$$\mathcal{H} = \{E \text{ a vector field along } \gamma \text{ of class } H^1 \text{ and } E(a) = E(b) = 0\}$$

as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. Then the index and coindex of these subspaces are finite and its difference gives the Maslov index. See

P. Piccione and D. V. Tausk, *The Morse index theorem in semi-Riemannian geometry*, Topology, **41** (2002), no. 6, 1123–1159.

It is used in order to show a result of bifurcation when a non-degenerate conjugate point has non zero Maslov Index. It is also related with the spectral flow.

Some interpretations and applications

It is possible to establish a semi-Riemannian Morse Index Theorem making a splitting of

$$\mathcal{H} = \{E \text{ a vector field along } \gamma \text{ of class } H^1 \text{ and } E(a) = E(b) = 0\}$$

as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. Then the index and coindex of these subspaces are finite and its difference gives the Maslov index. See

P. Piccione and D. V. Tausk, *The Morse index theorem in semi-Riemannian geometry*, Topology, **41** (2002), no. 6, 1123–1159.

It is used in order to show a result of bifurcation when a non-degenerate conjugate point has non zero Maslov Index. It is also related with the spectral flow. See the reference:

P. Piccione, A. Portaluri, D. V. Tausk, *Spectral Flow, Maslov Index and Bifurcation of semi-Riemannian Geodesics*, Ann. Global Anal. Geom. **25** (2004), no. 2, 121–149.

The Maslov Index: abstract definition

The Maslov Index: abstract definition

- Given a symplectic space (V, ω), a Lagrangian L_0 and a path of Lagrangians in the Lagrangian Grasmannian Λ $t \to \Phi(t) \in \Lambda$, the Maslov index $\mu_{L_0}(\Phi)$ is an intersection number of Φ with

$$\Lambda_{\geq 1}(L_0) = \{L \in \Lambda \text{ such that } L \cap L_0 \neq \{0\}\}$$

The Maslov Index applied to geodesics

The Maslov Index applied to geodesics

Let (M, g) be a semi-Riemannian manifold and $\gamma : [a, b] \longrightarrow M$ a geodesic. Then

The Maslov Index applied to geodesics

Let (M, g) be a semi-Riemannian manifold and $\gamma : [a, b] \to M$ a geodesic. Then

- We associate to γ an orthonormal parallel frame, so that the vector fields are represented as $V : [a, b] \to \mathbb{R}^n$

The Maslov Index applied to geodesics

Let (M, g) be a semi-Riemannian manifold and $\gamma : [a, b] \longrightarrow M$ a geodesic. Then

- We associate to γ an orthonormal parallel frame, so that the vector fields are represented as $V : [a, b] \longrightarrow \mathbb{R}^n$
- Let $(\mathbb{R}^n \times \mathbb{R}^n, \omega)$ be the symplectic space given by

$$\omega_g((v_1, w_1), (v_2, w_2)) = g(v_1, w_2) - g(v_2, w_1)$$

The Maslov Index applied to geodesics

Let (M, g) be a semi-Riemannian manifold and $\gamma : [a, b] \longrightarrow M$ a geodesic. Then

- We associate to γ an orthonormal parallel frame, so that the vector fields are represented as $V : [a, b] \longrightarrow \mathbb{R}^n$
- Let $(\mathbb{R}^n \times \mathbb{R}^n, \omega)$ be the symplectic space given by

$$\omega_g((v_1, w_1), (v_2, w_2)) = g(v_1, w_2) - g(v_2, w_1)$$

- For every $t \in [a, b]$, the subspace

$$\Phi(t) = \{(J(t), J'(t)) \ : \ J \text{ is a Jacobi field such that } J(0) = 0\}$$

is a Lagrangian in $\mathbb{R}^n \times \mathbb{R}^n, \omega)$.

The Maslov Index applied to geodesics

Let (M, g) be a semi-Riemannian manifold and $\gamma : [a, b] \to M$ a geodesic. Then

- We associate to γ an orthonormal parallel frame, so that the vector fields are represented as $V : [a, b] \to \mathbb{R}^n$
- Let $(\mathbb{R}^n \times \mathbb{R}^n, \omega)$ be the symplectic space given by

  $$\omega_g((v_1, w_1), (v_2, w_2)) = g(v_1, w_2) - g(v_2, w_1)$$

- For every $t \in [a, b]$, the subspace

  $$\Phi(t) = \{(J(t), J'(t)) \; : \; J \text{ is a Jacobi field such that } J(0) = 0\}$$

  is a Lagrangian in $\mathbb{R}^n \times \mathbb{R}^n, \omega)$.
- Finally, we fix the Lagrangian $L_0 = \{0\} \times \mathbb{R}^n$. The Maslov Index of γ is $\mu_{L_0}(\Phi)$

Main properties of the Maslov Index

Main properties of the Maslov Index

- The curve $t \longrightarrow \Phi(t)$ intersects with L_0 only when t_0 is a conjugate instant.

Main properties of the Maslov Index

- The curve $t \rightarrow \Phi(t)$ intersects with L_0 only when t_0 is a conjugate instant.
- When t_0 is a non-degenerate conjugate instant, its contribution to the Maslov Index is given by the signature of the space $\mathbb{J}[t_0]^{\perp}$, where

$$\mathbb{J}[t_0] = \{J[t_0] : \ J \text{ is a Jacobi field along } \gamma \text{ with } J(0) = 0\}$$

Main properties of the Maslov Index

- The curve $t \to \Phi(t)$ intersects with L_0 only when t_0 is a conjugate instant.
- When t_0 is a non-degenerate conjugate instant, its contribution to the Maslov Index is given by the signature of the space $\mathbb{J}[t_0]^\perp$, where

$$\mathbb{J}[t_0] = \{J[t_0] : \ J \text{ is a Jacobi field along } \gamma \text{ with } J(0) = 0\}$$

- In the Riemannian case, the Maslov index coincides with the Morse Index

Main properties of the Maslov Index

- The curve $t \to \Phi(t)$ intersects with L_0 only when t_0 is a conjugate instant.
- When t_0 is a non-degenerate conjugate instant, its contribution to the Maslov Index is given by the signature of the space $\mathbb{J}[t_0]^\perp$, where

$$\mathbb{J}[t_0] = \{J[t_0] : \ J \text{ is a Jacobi field along } \gamma \text{ with } J(0) = 0\}$$

- In the Riemannian case, the Maslov index coincides with the Morse Index
- In semi-Riemannian geometry conjugate instants can accumulate

Semi-Riemannian submersions: definition

Let M and B be two semi-Riemannian manifolds. A semi-Riemannian submersions is a map $\pi : M \longrightarrow B$ such that

Semi-Riemannian submersions: definition

Let M and B be two semi-Riemannian manifolds. A semi-Riemannian submersions is a map $\pi : M \to B$ such that

(S1) It is a submersion (that is, $d_p\pi$ is surjective for every $p \in M$),

Semi-Riemannian submersions: definition

Let M and B be two semi-Riemannian manifolds. A semi-Riemannian submersions is a map $\pi : M \to B$ such that

(S1) It is a submersion (that is, $d_p\pi$ is surjective for every $p \in M$),

(S2) the fibers $\pi^{-1}(p)$ with $p \in B$ are non-degenerate,

Semi-Riemannian submersions: definition

Let M and B be two semi-Riemannian manifolds. A semi-Riemannian submersions is a map $\pi : M \to B$ such that

(S1) It is a submersion (that is, $d_p\pi$ is surjective for every $p \in M$),

(S2) the fibers $\pi^{-1}(p)$ with $p \in B$ are non-degenerate,

(S3) $d\pi$ is an isometry on the orthogonal space to the fibers (horizontal space).

Semi-Riemannian submersions: definition

Let M and B be two semi-Riemannian manifolds. A semi-Riemannian submersions is a map $\pi : M \to B$ such that

(S1) It is a submersion (that is, $d_p\pi$ is surjective for every $p \in M$),

(S2) the fibers $\pi^{-1}(p)$ with $p \in B$ are non-degenerate,

(S3) $d\pi$ is an isometry on the orthogonal space to the fibers (horizontal space).

For every point $p \in M$ we can split the tangent space as

$$T_pM = \mathcal{V}T_pM + \mathcal{H}T_pM.$$

Semi-Riemannian submersions: definition

Let M and B be two semi-Riemannian manifolds. A semi-Riemannian submersions is a map $\pi : M \to B$ such that

(S1) It is a submersion (that is, $d_p\pi$ is surjective for every $p \in M$),

(S2) the fibers $\pi^{-1}(p)$ with $p \in B$ are non-degenerate,

(S3) $d\pi$ is an isometry on the orthogonal space to the fibers (horizontal space).

For every point $p \in M$ we can split the tangent space as

$$T_pM = \mathcal{V}T_pM + \mathcal{H}T_pM.$$

B. O'Neill introduces the fundamental tensors T and A on M defined as follows. We denote by ∇ and ∇^* the Levi-Civita connections of M and B. Then for E and F of $\mathfrak{X}(M)$,

$$T_EF = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F) \quad \Rightarrow \quad T = 0 \Leftrightarrow \text{fibers are tot. geo..}$$

Semi-Riemannian submersions: definition

Let M and B be two semi-Riemannian manifolds. A semi-Riemannian submersions is a map $\pi : M \to B$ such that

(S1) It is a submersion (that is, $d_p\pi$ is surjective for every $p \in M$),

(S2) the fibers $\pi^{-1}(p)$ with $p \in B$ are non-degenerate,

(S3) $d\pi$ is an isometry on the orthogonal space to the fibers (horizontal space).

For every point $p \in M$ we can split the tangent space as

$$T_pM = \mathcal{V}T_pM + \mathcal{H}T_pM.$$

B. O'Neill introduces the fundamental tensors T and A on M defined as follows. We denote by ∇ and ∇^* the Levi-Civita connections of M and B. Then for E and F of $\mathfrak{X}(M)$,

$$T_EF = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F) \quad \Rightarrow \quad T = 0 \Leftrightarrow \text{fibers are tot. geo..}$$

and the dual tensor A:

$$A_EF = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F. \quad \Rightarrow \quad A = 0 \Leftrightarrow \text{Hor. distri. is integr..}$$

O'Neill's papers

O'Neill's papers

The fundamental equations were studied in the paper by B. O'Neill:

B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J., 13 (1966), pp. 459–469.

O'Neill's papers

The fundamental equations were studied in the paper by B. O'Neill:

B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J., 13 (1966), pp. 459–469.

A year later he pusblished another paper studying the relation between geodesics in the total space M and the projection in B:

——, *Submersions and geodesics*, Duke Math. J., 34 (1967), pp. 363–373.

O'Neill's papers

The fundamental equations were studied in the paper by B. O'Neill:

B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J., 13 (1966), pp. 459–469.

A year later he pusblished another paper studying the relation between geodesics in the total space M and the projection in B:

——, *Submersions and geodesics*, Duke Math. J., 34 (1967), pp. 363–373.

The results in the first paper are easily extended to the semi-Riemannian case, but there are some difficulties for the second paper.

The derivatives of a curve and its projection

The derivatives of a curve and its projection

Proposition

Let α be a curve in M with $X = \mathcal{H}\alpha'$ and $U = \mathcal{V}\alpha'$. Then

$$\begin{aligned}\mathcal{H}(\alpha'') &= \alpha''_* + 2A_X U + T_U U \\ \mathcal{V}(\alpha'') &= T_U X + \mathcal{V}(U')\end{aligned}$$

where α''_* is the horizontal lift to α of the acceleration of $\pi \circ \alpha$ in B.

The derivatives of a curve and its projection

Proposition

Let α be a curve in M with $X = \mathcal{H}\alpha'$ and $U = \mathcal{V}\alpha'$. Then

$$\mathcal{H}(\alpha'') = \alpha''_* + 2A_X U + T_U U$$
$$\mathcal{V}(\alpha'') = T_U X + \mathcal{V}(U')$$

where α''_* is the horizontal lift to α of the acceleration of $\pi \circ \alpha$ in B.

Theorem

Let $\pi : M \to B$ a semi-Riemannian submersion. If γ is a geodesic of M that is horizontal at some point, then it is always horizontal (hence $\pi \circ \gamma$ is a geodesic of B).

The derived vector field

The derived vector field

Definition

If $E = H + V$ is a vector field on a horizontal curve γ and $X = \gamma'$, then

$$D(E) = \mathcal{V}(V') - T_V X + 2A_X H$$

is the derived vector field of E.

The derived vector field

Definition

If $E = H + V$ is a vector field on a horizontal curve γ and $X = \gamma'$, then

$$D(E) = \mathcal{V}(V') - T_V X + 2A_X H$$

is the derived vector field of E.

The geometric interpretation follows from

Proposition

The derived vector field of E is zero if and only if it can be obtained as the variational field of a variation by horizontal curves.

The derived vector field

Definition

If $E = H + V$ is a vector field on a horizontal curve γ and $X = \gamma'$, then

$$D(E) = \mathcal{V}(V') - T_V X + 2A_X H$$

is the derived vector field of E.

The geometric interpretation follows from

Proposition

The derived vector field of E is zero if and only if it can be obtained as the variational field of a variation by horizontal curves.

We can interpret $D(E)$ as a measure of how far from being horizontal the variation is.

Lifting variations

Proposition

Let $\beta : [a, b] \to B$ be a curve in B and fix $p \in \pi^{-1}(\beta(a))$.

Lifting variations

Proposition

Let $\beta : [a, b] \to B$ be a curve in B and fix $p \in \pi^{-1}(\beta(a))$.

- Then there exists a horizontal lift $\gamma : [a, c| \to M$ ($c \leq b$) that projects into β and it is the maximal horizontal lift through $p = \gamma(a)$ projecting to β.

Lifting variations

Proposition

Let $\beta : [a, b] \to B$ be a curve in B and fix $p \in \pi^{-1}(\beta(a))$.

- Then there exists a horizontal lift $\gamma : [a, c| \to M$ ($c \leq b$) that projects into β and it is the maximal horizontal lift through $p = \gamma(a)$ projecting to β.
- Furthermore, let $\Psi : [a, b] \times (-\epsilon, \epsilon) \to B$ be a variation of the curve β and $\eta : (-\epsilon, \epsilon) \to M$ such that $\pi \circ \eta(s) = \Psi(a, s)$, then there exists a variation $\Gamma : [a, c| \times (-\delta, \delta) \to M$ with $\Gamma(a, s) = \eta(s)$ and such that $\Gamma(t, s)$ is the horizontal lift of $\Psi(t, s)$ departing from $\eta(s)$ for every $s \in (-\delta, \delta)$.

Derived vector field and Jacobi fields

Derived vector field and Jacobi fields

Theorem

Let E be a vector field on a horizontal geodesic γ in M. Then

$$\mathcal{H}(E'' - R_{EX}X) = (E'_* - R^*_{E^*X}X) + 2A_X D$$
$$\mathcal{V}(E'' - R_{EX}X) = \mathcal{V}(D') + T_D X$$

where $D = D(E)$ is the derived vector field of E, $X = \gamma'$ and R and R^* are the curvature tensors of M and B.

Derived vector field and Jacobi fields

Theorem

Let E be a vector field on a horizontal geodesic γ in M. Then

$$\mathcal{H}(E'' - R_{EX}X) = (E'_* - R^*_{E^*X}X) + 2A_X D$$
$$\mathcal{V}(E'' - R_{EX}X) = \mathcal{V}(D') + T_D X$$

where $D = D(E)$ is the derived vector field of E, $X = \gamma'$ and R and R^* are the curvature tensors of M and B.

Corollary

A field E on a horizontal geodesic in M with derived vector field $D(E) = 0$ is Jacobi if and only if $P = d\pi(E)$ is a Jacobi field of $\pi \circ \gamma$ in B.

The main result

Theorem

Let $\pi : M \to B$ be a semi-Riemannian submersion and $\gamma : [a, b] \to M$ a horizontal geodesic segment. Then

The main result

Theorem

Let $\pi : M \to B$ be a semi-Riemannian submersion and $\gamma : [a, b] \to M$ a horizontal geodesic segment. Then

- an instant t_0 is a focal point of γ related to the fibre $\mathcal{F}(a)$ in $\gamma(a)$ if and only if is a conjugate instant of the curve $\pi \circ \gamma$.

The main result

Theorem

Let $\pi : M \to B$ be a semi-Riemannian submersion and $\gamma : [a, b] \to M$ a horizontal geodesic segment. Then

- an instant t_0 is a focal point of γ related to the fibre $\mathcal{F}(a)$ in $\gamma(a)$ if and only if is a conjugate instant of the curve $\pi \circ \gamma$.
- Furthermore, if t_0 is a non-degenerate focal instant of γ, then so is of $\pi \circ \gamma$ and the contribution to the Maslov index given by $\gamma(t_0)$ coincides with the one of $\pi \circ \gamma(t_0)$ as a conjugate instant.

Scheme of the proof

Lemma

Let γ be a horizontal geodesic in M and let $\mathcal{F}(a)$ be the fibre passing through $\gamma(a)$.
Then a field E on γ is a $\mathcal{F}(a)$-Jacobi field iff E is Jacobi, $D(E) = 0$ and $E(a)$ is vertical.

Scheme of the proof

Lemma

Let γ be a horizontal geodesic in M and let $\mathcal{F}(a)$ be the fibre passing through $\gamma(a)$.
Then a field E on γ is a $\mathcal{F}(a)$-Jacobi field iff E is Jacobi, $D(E) = 0$ and $E(a)$ is vertical.

"$\Rightarrow$"

$\mathcal{F}(a)$-Jacobi field means that it is the variational vector field of a variation by geodesics orthogonal to the fiber $\mathcal{F}(a)$. Then these geodesics are horizontal, so that $D(E) = 0$ and $E(a)$ is vertical.

Scheme of the proof

Lemma

Let γ be a horizontal geodesic in M and let $\mathcal{F}(a)$ be the fibre passing through $\gamma(a)$.
Then a field E on γ is a $\mathcal{F}(a)$-Jacobi field iff E is Jacobi, $D(E) = 0$ and $E(a)$ is vertical.

"$\Rightarrow$"

$\mathcal{F}(a)$-Jacobi field means that it is the variational vector field of a variation by geodesics orthogonal to the fiber $\mathcal{F}(a)$. Then these geodesics are horizontal, so that $D(E) = 0$ and $E(a)$ is vertical.

"$\Leftarrow$"

$D(E)(a) = 0$ is just the condition to E be $\mathcal{F}(a)$-Jacobi.

Scheme of the proof

Scheme of the proof

$$\mathcal{J} = \{\text{Jacobi fields along } \gamma\}$$

$$\mathcal{J}_*^0 = \{E \quad J \text{ Jacobi fields along } \pi \circ \gamma \text{ such that } J(a) = 0\}$$

$$\mathcal{J}_\delta^v = \{E \in \mathcal{J} \ : \ E(a) \text{ is vertical and } D(E) = 0\}$$

Scheme of the proof

$$\mathcal{J} = \{\text{Jacobi fields along } \gamma\}$$
$$\mathcal{J}^0_* = \{E \quad J \text{ Jacobi fields along } \pi \circ \gamma \text{ such that } J(a) = 0\}$$
$$\mathcal{J}^v_\delta = \{E \in \mathcal{J} \ : \ E(a) \text{ is vertical and } D(E) = 0\}$$

It is enough to show that $\mathcal{J}^v_\delta[t_0]^\perp$ and $\mathcal{J}^0_*[t_0]^\perp$ are isometric subspaces.

Scheme of the proof

$$\mathcal{J} = \{\text{Jacobi fields along } \gamma\}$$

$$\mathcal{J}_*^0 = \{E \quad J \text{ Jacobi fields along } \pi \circ \gamma \text{ such that } J(a) = 0\}$$

$$\mathcal{J}_\delta^v = \{E \in \mathcal{J} \ : \ E(a) \text{ is vertical and } D(E) = 0\}$$

It is enough to show that $\mathcal{J}_\delta^v[t_0]^\perp$ and $\mathcal{J}_*^0[t_0]^\perp$ are isometric subspaces. In fact we will see that $\mathcal{J}_\delta^v[t_0]$ is the lifting of $\mathcal{J}_*^0[t_0]$ (so that contains the vertical subspace).

Scheme of the proof

$$\mathcal{J} = \{\text{Jacobi fields along } \gamma\}$$
$$\mathcal{J}_*^0 = \{E \quad J \text{ Jacobi fields along } \pi \circ \gamma \text{ such that } J(a) = 0\}$$
$$\mathcal{J}_\delta^v = \{E \in \mathcal{J} \;:\; E(a) \text{ is vertical and } D(E) = 0\}$$

It is enough to show that $\mathcal{J}_\delta^v[t_0]^\perp$ and $\mathcal{J}_*^0[t_0]^\perp$ are isometric subspaces. In fact we will see that $\mathcal{J}_\delta^v[t_0]$ is the lifting of $\mathcal{J}_*^0[t_0]$ (so that contains the vertical subspace).

- If $F \in \mathcal{J}_*^0$ then it is the variation vector field of a variation by geodesics $\psi : [a, b] \times (-\epsilon, \epsilon) \to B$.

Scheme of the proof

$$\mathcal{J} = \{\text{Jacobi fields along } \gamma\}$$
$$\mathcal{J}_*^0 = \{E \quad J \text{ Jacobi fields along } \pi \circ \gamma \text{ such that } J(a) = 0\}$$
$$\mathcal{J}_\delta^v = \{E \in \mathcal{J} \ : \ E(a) \text{ is vertical and } D(E) = 0\}$$

It is enough to show that $\mathcal{J}_\delta^v[t_0]^\perp$ and $\mathcal{J}_*^0[t_0]^\perp$ are isometric subspaces. In fact we will see that $\mathcal{J}_\delta^v[t_0]$ is the lifting of $\mathcal{J}_*^0[t_0]$ (so that contains the vertical subspace).

- If $F \in \mathcal{J}_*^0$ then it is the variation vector field of a variation by geodesics $\psi : [a, b] \times (-\epsilon, \epsilon) \to B$.
- Given v a vertical vector tangent to $\gamma(t_0)$, we can choose a curve $s \to \beta(s) \in M$ such that $\beta'(0) = v + F[t_0]$.

Scheme of the proof

$$\mathcal{J} = \{\text{Jacobi fields along } \gamma\}$$
$$\mathcal{J}_*^0 = \{E \quad J \text{ Jacobi fields along } \pi\circ\gamma \text{ such that } J(a) = 0\}$$
$$\mathcal{J}_\delta^v = \{E \in \mathcal{J} \;:\; E(a) \text{ is vertical and } D(E) = 0\}$$

It is enough to show that $\mathcal{J}_\delta^v[t_0]^\perp$ and $\mathcal{J}_*^0[t_0]^\perp$ are isometric subspaces. In fact we will see that $\mathcal{J}_\delta^v[t_0]$ is the lifting of $\mathcal{J}_*^0[t_0]$ (so that contains the vertical subspace).

- If $F \in \mathcal{J}_*^0$ then it is the variation vector field of a variation by geodesics $\psi : [a,b]\times(-\epsilon,\epsilon) \to B$.
- Given v a vertical vector tangent to $\gamma(t_0)$, we can choose a curve $s \to \beta(s) \in M$ such that $\beta'(0) = v + F[t_0]$.
- The variation $\eta(t,s) = \exp_{\beta(s)}((t-t_0)\psi_t(t_0,s))$ is a lift of $\psi(t,s)$ by horizontal geodesics. Then $\eta_s(t,0) \in \mathcal{J}_\delta^v$ and $v + F[t_0] \in \mathcal{J}_\delta^v[t_0]$

Applications to static spacetimes

Applications to static spacetimes

- A standard static spacetime is a product manifold $M \times \mathbb{R}$ endowed of a Lorentzian metric

$$l((\xi,\tau),(\xi,\tau)) = g(\xi,\xi) - \beta(x)\tau^2$$

where g is a Riemannian metric in M and β is a positive fucntion in M.

Applications to static spacetimes

- A standard static spacetime is a product manifold $M \times \mathbb{R}$ endowed of a Lorentzian metric

$$l((\xi, \tau), (\xi, \tau)) = g(\xi, \xi) - \beta(x)\tau^2$$

  where g is a Riemannian metric in M and β is a positive fucntion in M.
- The projection $\pi : M \times \mathbb{R} \to M$ is a semi-Riemannian submersion. Thus, if we consider a horizontal geodesic in $M \times \mathbb{R}$ we can compute the Maslov index respect to the integral curve of the Killing filed through the origin.

Applications to static spacetimes

- A standard static spacetime is a product manifold $M \times \mathbb{R}$ endowed of a Lorentzian metric

$$l((\xi,\tau),(\xi,\tau)) = g(\xi,\xi) - \beta(x)\tau^2$$

  where g is a Riemannian metric in M and β is a positive fucntion in M.
- The projection $\pi : M \times \mathbb{R} \to M$ is a semi-Riemannian submersion. Thus, if we consider a horizontal geodesic in $M \times \mathbb{R}$ we can compute the Maslov index respect to the integral curve of the Killing filed through the origin.
- It is just the geometric index of the projected geodesic in M.