New examples of Lorentzian metrics admitting timelike closed curves

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If there exist timelike closed curves then the time will be a periodic function of the parameter s of the curve. Hence there is a maximum and a minimum for $t(s)$ and there exist points for which $\frac{dt}{ds} = 0$ holds. Let $x^i(s)$ with $0 \leq s \leq 1$ and $x^i(0) = x^i(1)$ be a timelike closed curve on a chart of the underlying manifold, i.e.

$$g_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} > 0$$

holds for all s.

If s_0 is chosen such that $\frac{dt}{ds} = 0$ holds (which is always possible by the arguments above) then

$$g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} \,\Big|_{s=s_0} = g_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}$$

is negative if the matrix consisting of the $g_{\alpha\beta}$ is negative definite and this exactly is the condition for the absence of timelike closed curves.

The Field Equations

$$Ric - \frac{R}{2} g = \kappa T$$

There is one main possibility to handle with this system of PDEs: Find solutions and sort out the solutions with geometric or physical relevance.

Another possibility is to demand special (for example kinematical) properties and then to create the metric.

The Field Equations

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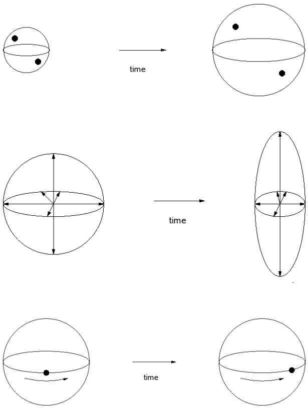
Spacetime kinematics

One can decompose the covariant derivative of the observer field into irreducible parts: the symmetric traceless part σ (corresponding to the shear), the antisymmetric part ω (corresponding to the rotation) and a part including the expansion Θ:

$$DX = \frac{\Theta}{3}P + \sigma + \omega + \dot{X} \otimes X.$$

Expansion, Shear, Rotation



Spacetime kinematics

Using the usual observer field $X^i = (1, 0, 0, 0)$ one has the following equations (here using local coordinates):

Spacetime kinematics

$$\sigma_{ik} = \frac{1}{2}(g_{0i,0}g_{0k} + g_{0k,0}g_{0i} - g_{ik,0}) - \frac{1}{3}\Theta P_{ik}$$

$$\omega_{ik} = \frac{1}{2}(g_{0i,0}g_{0k} - g_{0k,0}g_{0i} - g_{0i,k} + g_{0k,i})$$

$$\Theta = \frac{1}{2}\left(\log \det g\right)_{,0}$$

In addition we have

$$P_{ik} = g_{0i}g_{0k} - g_{ik}.$$

Metrics with vanishing shear tensor

In the case of vanishing shear (usually assumed in cosmology) we have for the spatial components:

$$0 = P_{\alpha\beta,0} - \frac{2}{3}\Theta P_{\alpha\beta}$$

Metrics with vanishing shear tensor

The solutions for vanishing shear are as follows:

Θ	Spatial metric $g_{\alpha\beta}(t, x^\gamma)$
const	$g_{0\alpha}g_{0\beta} + e^{\frac{2t\Theta}{3}}\left[g_{\alpha\beta}(0, x^\gamma) - g_{0\alpha}(0, x^\gamma)g_{0\beta}(0, x^\gamma)\right]$
$\Theta(t, x^\gamma)$	$\frac{1}{3}e^{\frac{2}{3}\int_0^t \Theta d\tau}\left(3g_{\alpha\beta} - \int_0^t \frac{2\Theta g_{0\alpha}g_{0\beta}}{\exp(\frac{2}{3}\int_0^t \Theta d\tau)} - 3\frac{(g_{0\beta}g_{0\alpha})_{,0}}{\exp(\frac{2}{3}\int_0^t \Theta d\tau)} d\tau\right)$

Remark:

The metrics given here describe *all* models without shear!

If the shear is not equal to zero the metric (with constant Θ) has the following form:

$$\begin{aligned} g_{\alpha\beta} &= \frac{e^{\frac{2}{3}\Theta t}}{3}(3g_{\alpha\beta}(0,x^\gamma) - \\ &\int_0^t e^{\frac{2}{3}\Theta\tau}[6\sigma_{\alpha\beta} - 3g_{0\beta}g_{0\alpha,0} + g_{0\alpha}(2\Theta g_{0\beta} - 3g_{0\beta,0})]d\tau) \end{aligned}$$

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\end{aligned}
$$

Gödel metric

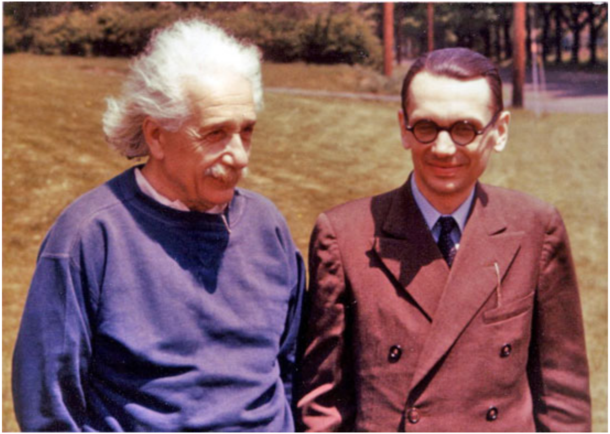
The components of the metric tensor are given by

$$(g_{ik}) = a^2 \begin{pmatrix} 1 & 0 & e^{x^1} & 0 \\ 0 & -1 & 0 & 0 \\ e^{x^1} & 0 & \frac{1}{2}e^{2x^1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Gödel metric

Here $a = const \neq 0$. This metric, invented by Gödel in 1949, describes a rotating spacetime without shear and expansion with respect to the given observer field with the components $X^i = (1, 0, 0, 0)$ and $1 = X_i X^i = g_{ik} X^k X^i = g_{00}$.

Two Friends



Gödel metric

Aspects:

- The kinematical properties.
- 5-dimensional group of isometries which is transitive (i.e. it is completely homogeneous).
- It is an exact solution of the field equations.
- The matter takes the form of a pressure-free perfect fluid.
- Timelike closed curves.

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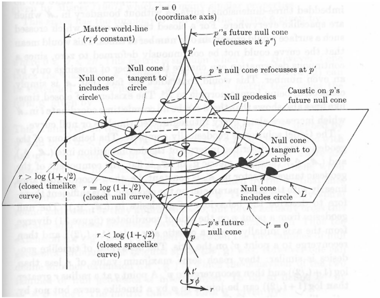
Gödel metric

The space (the coordinate x^3 is supressed) is rotationally symmetric about any point. The light cone opens out and tips over as r increases (see the line L) resulting in timelike closed curves.



Generalized Gödel Metric

The following geometric generalization of the Gödel model with expansion was constructed using a tool kit for the (geometric) construction of spacetime models:

Generalized Gödel Metric

$$(g_{ik}) = \begin{pmatrix} 1 & 0 & e^{x^1} & 0 \\ 0 & -e^{\frac{2}{3}\Theta t} & 0 & 0 \\ e^{x^1} & 0 & e^{2x^1} - \frac{1}{2}e^{\frac{2}{3}\Theta t + 2x^1} & 0 \\ 0 & 0 & 0 & -e^{\frac{2}{3}\Theta t} \end{pmatrix}$$

Generalized Gödel Metrics

The model begins non-causal, but from a particular point of the time interval, $\hat{t} = \frac{3 \ln 2}{2\Theta}$, one has causality (and $\Theta \rightarrow 0$ leads to $\hat{t} \rightarrow \infty$).

Related results, outlook

In this context there are other interesting investigations, for example for particular spacelike hypersurfaces in the de Sitter space. The importance here is related to the fact that *achronal spacelike hypersurfaces are acausal*. Important classification results for spacelike hypersurfaces with constant scalar curvature were given by Aledo-Alias-Romero (J. Geom. Phys. 31, 1999), Brasil-Colares-Palmas (J. Geom. Phys. 37, 2001) and Hu-Scherfner-Zhai (J. Diff. Geom. Appl. 25, 2007).