Hypersurfaces in the light cone and Minkowski-type problems

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Joint work with J.M. Espinar and J.A. Gálvez.

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The unit normal is no longer a Gauss map into $\mathbb{S}^n$.

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2. To solve as far as possible this Christoffel problem.

The approach to solve the Christoffel problem in $\mathbb{S}_1^{n+1}$ will illustrate how the geometry of spacelike hypersurfaces in the positive light cone is very helpful in order to study the geometry of spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$.

The Christoffel problem in $\mathbb{S}_1^{n+1}$

We need to find the appropriate extensions to spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ of

- The Gauss map of a hypersurface $M^n \subset \mathbb{R}^{n+1}$
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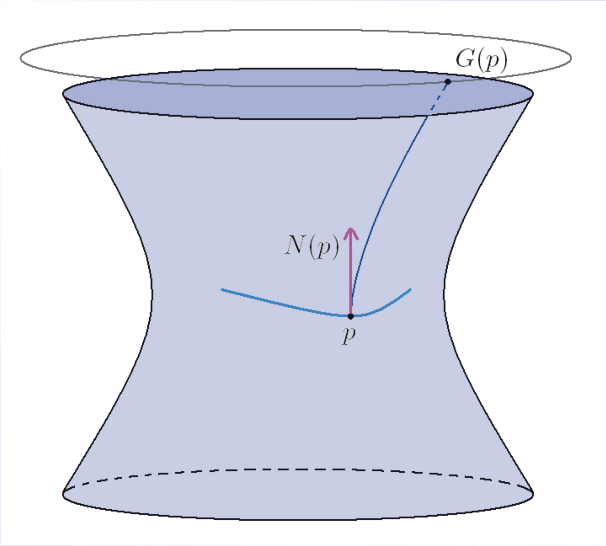
Specifically, for a spacelike hypersurface $M^n \subset \mathbb{S}_1^{n+1}$ we need ...

1. A Gauss map $G : M^n \to \mathbb{S}^n$.
2. A notion of curvature radii that make sense with the only hypothesis that the Gauss map G is a diffeomorphism.



The **Gauss map** G of a hypersurface in $\mathbb{S}^{n+1}_1$.

$$G : M^n \to \partial_{+,\infty}\mathbb{S}^{n+1}_1 \equiv \mathbb{S}^n_{+,\infty} \equiv \mathbb{S}^n.$$

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Therefore: $R_i := 1/\kappa_i$ do not serve anymore as curvature radii in $\mathbb{S}_1^{n+1}$ (they do not match the Gauss map properly).

The key observation:

If $M^n \subset \mathbb{R}^{n+1}$ with Gauss map N, and $p \in M^n$, then the principal curvature radii $R_i = 1/\kappa_i$ are linked to the Gauss map:

$$R_i(p) = \frac{1}{\kappa_i(p)} = \lim_{\epsilon \to 0} \frac{\text{arclength of } \alpha_i(-\epsilon, \epsilon)}{\text{arclength of } (N \circ \alpha_i)(-\epsilon, \epsilon)},$$

where α_i is the line of curvature of M^n associated to κ_i.

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Definition

The *curvature radii* of a spacelike hypersurface $M^n \subset \mathbb{S}_1^{n+1}$ with regular Gauss map G are

$$|\mathcal{R}_i| := \lim_{\epsilon\to 0} \frac{\text{arclength of } \alpha_i(-\epsilon,\epsilon)}{\text{arclength of } (G\circ\alpha_i)(-\epsilon,\epsilon)}.$$

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We need to find $f : \mathbb{S}^n \to \mathbb{S}^{n+1}_1$ with Gauss map $G(x) = x$ and mean of curvature radii

$$C(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\kappa_i(x) - 1}.$$

The associated light cone immersion

Let $f : M^n \longrightarrow \mathbb{S}_1^{n+1}$ be a spacelike hypersurface, with Gauss map $G : M^n \longrightarrow \mathbb{S}^n$ and unit normal $N : M^n \longrightarrow \mathbb{H}^{n+1}$.

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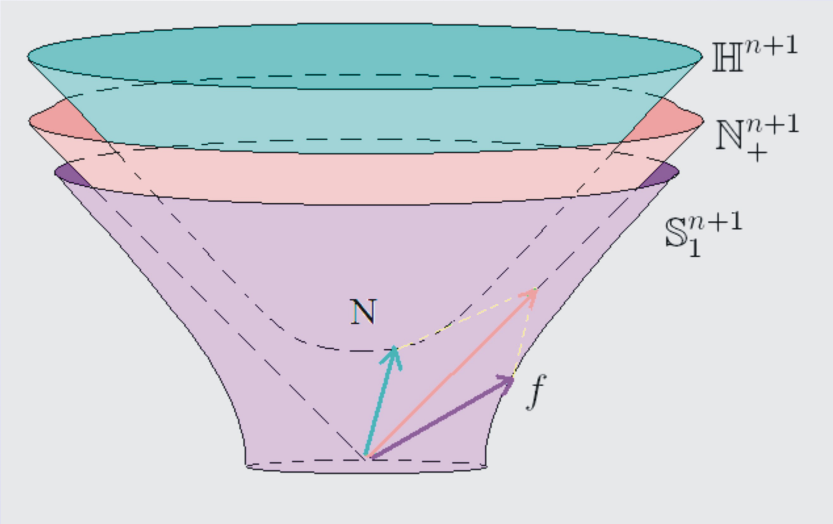
$\nu := f + N : M^n \to \mathbb{N}^{n+1}_+ \subset \mathbb{L}^{n+2}$ is the *associated light cone map*.



Properties of $\nu = f + N : M^n \to \mathbb{N}_+^{n+1}$

Let $f : \mathbb{S}^n \to \mathbb{S}_1^{n+1}$ denote a solution to the Christoffel problem. Then

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(A): ν is an immersion.

(B): It holds

$$g_\infty = \langle d\nu, d\nu \rangle = e^{2\rho} g_0,$$

where g_∞ is the Epstein metric of f, and g_0 is the $\mathbb{S}^n$-metric.

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(C): In particular, the Epstein metric g_∞ is globally conformal to g_0

Solving the Christoffel problem...

The Nirenberg problem

Let $S(x) : \mathbb{S}^n \longrightarrow \mathbb{R}$ denote a smooth function. Does it exist a conformal metric $g = e^{2u} g_0$ globally defined on $\mathbb{S}^n$ whose scalar curvature function is $S(x)$?

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Equivalently, we wish to solve globally on $\mathbb{S}^n$

$$\Delta^{g_0} u + \frac{n-2}{2} ||\nabla^{g_0} u||^2_{g_0} - \frac{n}{2} + \frac{e^{2u}}{2(n-1)} S = 0.$$

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OUR RESULT: The Christoffel problem in $\mathbb{S}^{n+1}_1$ and the Nirenberg problem in $\mathbb{S}^n$ are equivalent problems !!

The main theorem, Part I

Let $f : \mathbb{S}^n \to \mathbb{S}_1^{n+1}$ be a solution to the Christoffel problem for the smooth function $C(x)$. Then its Epstein metric $g_\infty = e^{2\rho} g_0$ is a solution to the Nirenberg problem for the function

$$S := -n(n-1)(2C+1).$$

And conversely?

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Let $g = e^{2\rho} g_0$ be a solution to the Nirenberg problem for the scalar curvature function $S(x)$. Is g the Epstein metric of some hypersurface $f : \mathbb{S}^n \to \mathbb{S}_1^{n+1}$?

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If YES : f is a solution to the Christoffel problem for $C(x)$ such that

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The answer... NO, BUT ALMOST YES

For $t > 0$ large enough the conformal metric $g_t := e^{2t} g$ is the Epstein metric of a hypersurface $f : \mathbb{S}^n \to \mathbb{S}_1^{n+1}$.

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But the Nirenberg problem is invariant under dilations OK!

The main theorem, Part II

Let $g = e^{2\rho} g_0$ be a solution to the Nirenberg problem for the scalar curvature function $S(x)$. Then for t large enough the Christoffel problem in $\mathbb{S}_1^{n+1}$ for the function

$$C_t(x) := -\frac{1}{2}\left(1 + \frac{e^{2t}}{n(n-1)}\right) S(x).$$

has a solution $f_t : \mathbb{S}^n \to \mathbb{S}_1^{n+1}$.

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has a solution $f_t : \mathbb{S}^n \to \mathbb{S}_1^{n+1}$. Moreover,

$$f_t = \frac{e^{\rho_t}}{2}\left(1 - e^{-2\rho_t}\left(1 + ||\nabla^{g_0}\rho||_{g_0}^2\right)\right)(1, x) + e^{-\rho_t}(0, \xi),$$

where $\xi := -x + \nabla^{g_0}\rho$ and $\rho_t := \rho + t$.

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This correspondence lets us translate the huge quantity of results on the Nirenberg problem into our Lorentzian setting problem.