Stationary surfaces in $\mathbb{L}^3$

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ABOUT:

a) Lorentz-Minkowski space $\mathbb{L}^3$
b) Spacelike surfaces
c) Some condition on the mean curvature

MOTIVATION:

Spacelike constant mean curvature surfaces in $\mathbb{L}^3$:

$\Downarrow$

generalization from a variational problem

- ✓ The variational problem
- ✓ The settings
- ✓ The results

THE VARIATIONAL PROBLEM

Consider M^2 a surface and an immersion

$$x : M \longrightarrow \mathbb{L}^3$$

Definition. The immersion $\mathbf{x}$ is spacelike if $\mathbf{x}^*\langle , \rangle$ is a Riemannian metric.

$$\langle d\mathbf{x}_p(v), d\mathbf{x}_p(v)\rangle > 0, \qquad v \in T_pM$$

Theory of Riemannian submanifolds

$$\nabla^0_X Y = \nabla_X Y + \sigma(X, Y)$$ (Gauss equation)

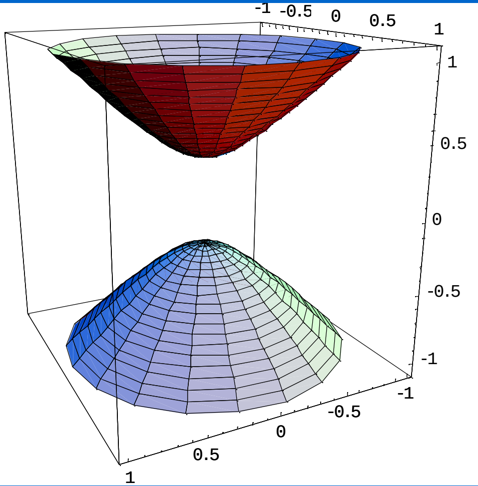
$$\nabla^0_X N = -A_N(X)$$ (Weingarten equation)

$$\overrightarrow{H} = \frac{1}{2}\,\mathrm{trace}(\sigma) = -\frac{1}{2}\,\mathrm{trace}(A)N$$

$$H = -\frac{1}{2}\,\mathrm{trace}(A)$$

Hyperbolic planes

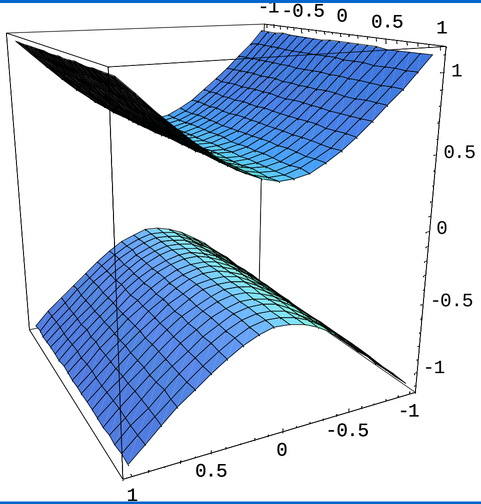
$$x \in \mathbb{L}^3; \langle x, x \rangle = -\rho^2$$



$$H = \frac{1}{\rho}$$

Hyperbolic cylinders

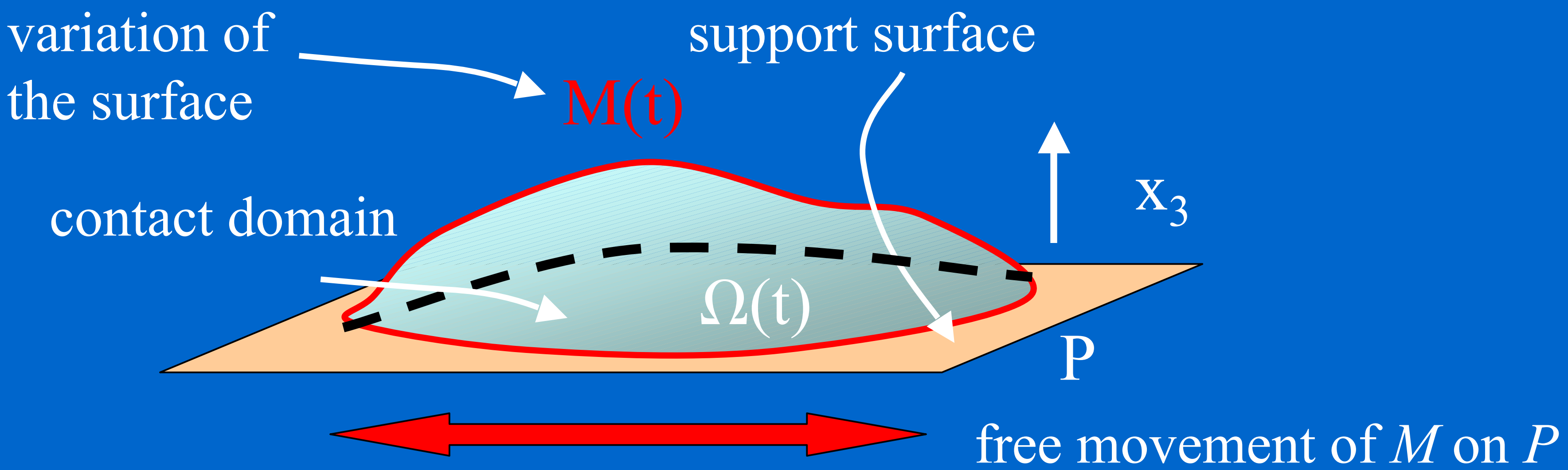
$$x \in \mathbb{R}^3; \langle \pi(x), \pi(x) \rangle = -\rho^2$$



$$H = \frac{1}{2\rho}$$

A variation of x, $X : M \times (-\varepsilon,\varepsilon) \longrightarrow \mathbb{L}^3$

- $X(- , t) : M \longrightarrow \mathbb{L}^3$ is a spacelike inmersion; $X(-,t)(M) := M(t)$ and $X(- , 0) = x$.
- $\partial M(t)$ is included in P and M(t) lies in one side of P.



A(t) = area M(t), V(t) = volume M(t)

S(t) = area Ω(t), Y(t) = $\int_V x_3 \, dV$

Energy = A(t) – cosh(β) S(t) + Y(t)
V(t) = constant

M is stationary
E'(0)=0

⟺

1) H = $\kappa x_3 + \lambda$

2) M meets P in a constant angle β.

Theorem [Alías&Pastor, 1999]. Planar discs and hyperbolic caps are the only <u>CMC</u> <u>stationary discs</u> in $\mathbb{L}^3$ supported in a plane.

The same result holds if the support surface is a hyperbolic plane.

Let M be a stationary compact embedded surface in $\mathbb{L}^3$ supported on a plane P. Then

- ❑ M is rotational symmetric.
- ❑ M is a topological disc.

The proof

1. The maximum principle
2. The Alexandrov reflection method

1. The maximum principle

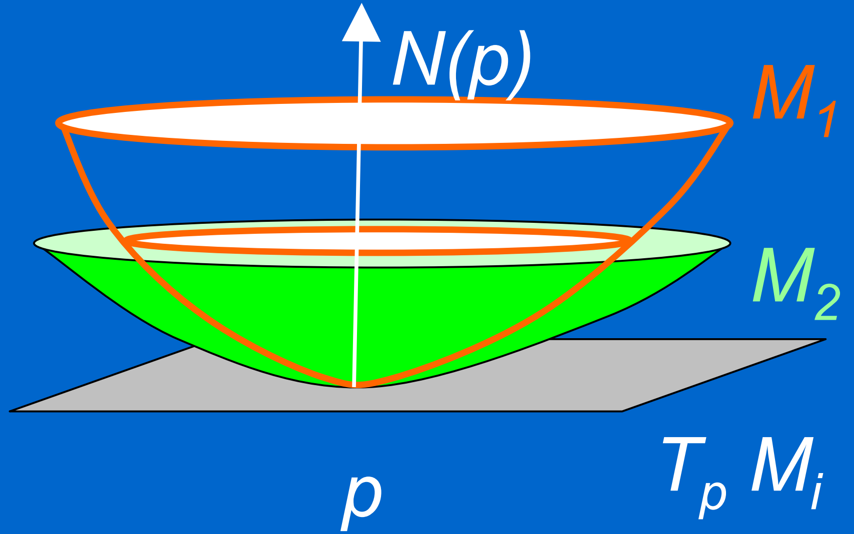
$$Q(u) := \mathrm{div}\,\frac{Du}{\sqrt{1-|Du|^2}} = 2H$$

Q(u)-Q(v) is a linear elliptic operator

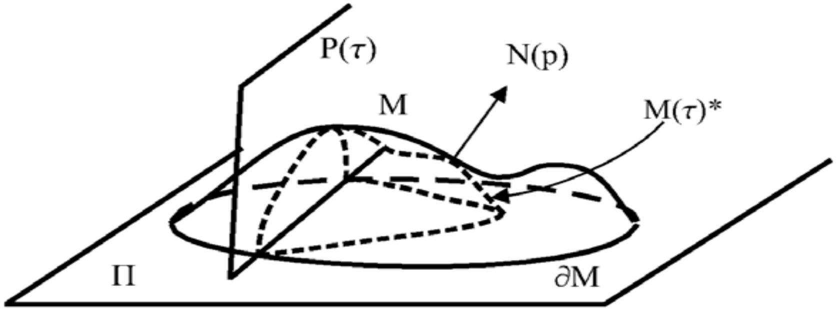
Linear elliptic equation ⇒ Maximum principle

$$M_1 \geq M_2 \Downarrow H_1 \geq H_2$$

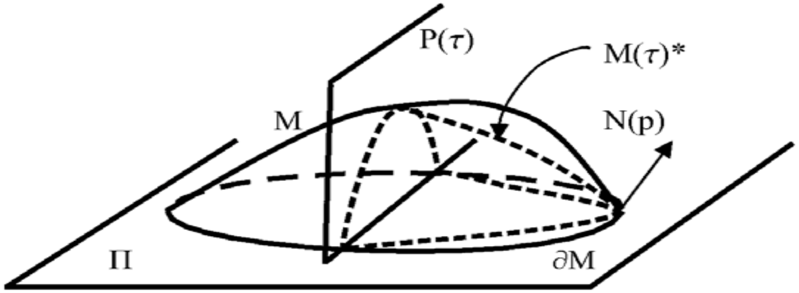
$$H_1 = H_2 \Downarrow M_1 = M_2$$



2. The Alexandrov reflection method



(a)

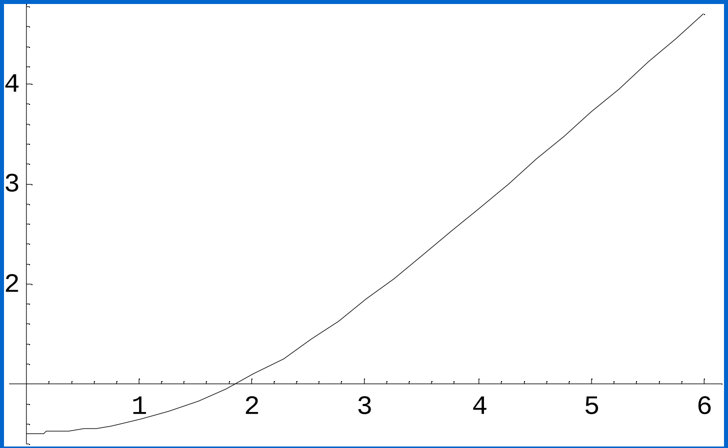


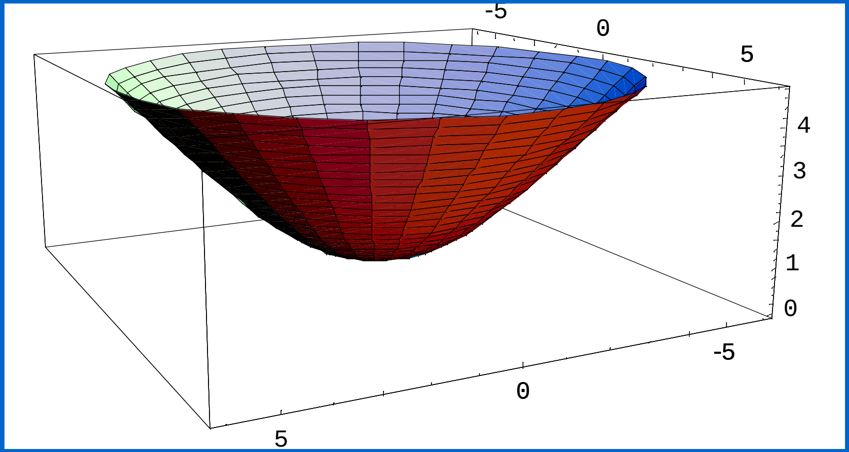
(b)

THE SETTINGS

- Rotational compact stationary surfaces
- Rotational non-compact stationary surfaces
- Cylindrical stationary surfaces

➢ Rotational compact stationary surfaces



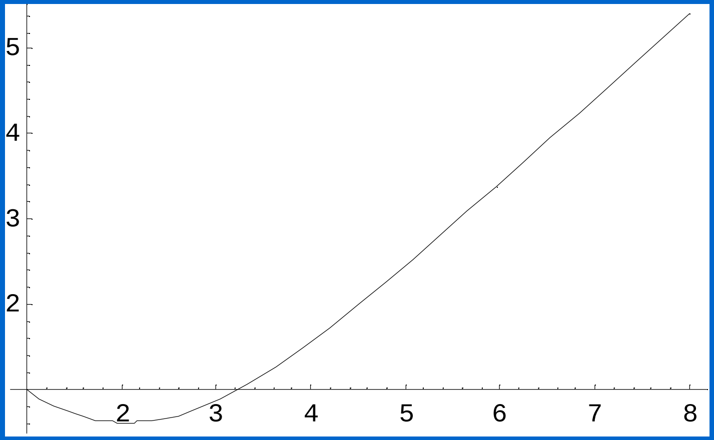


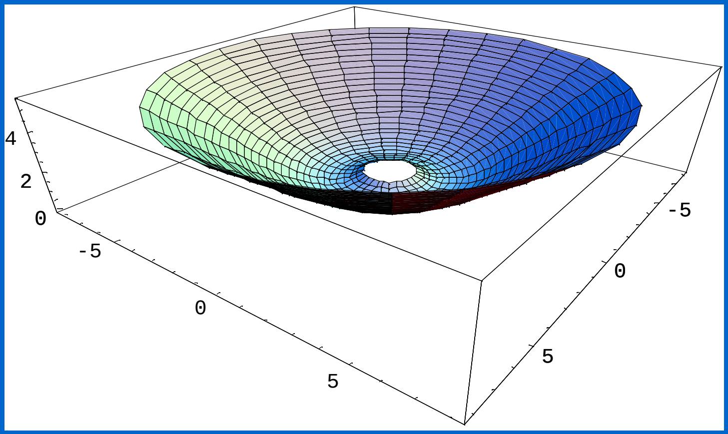
$$\frac{1}{r}\,\frac{d}{dr}\left(\frac{r u'(r)}{\sqrt{1-u'(r)^2}}\right) = \kappa\, u(r), \quad 0 \le r \le R$$

$$u'(0) = 0$$

$$u'(R) = \tanh(\beta)$$

➢ Rotational non-compact stationary surfaces



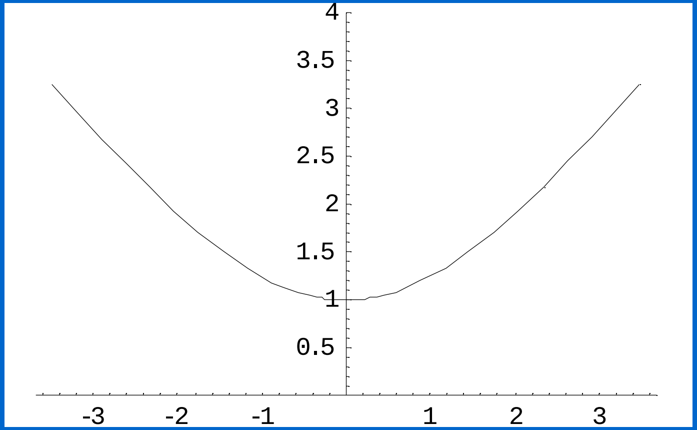


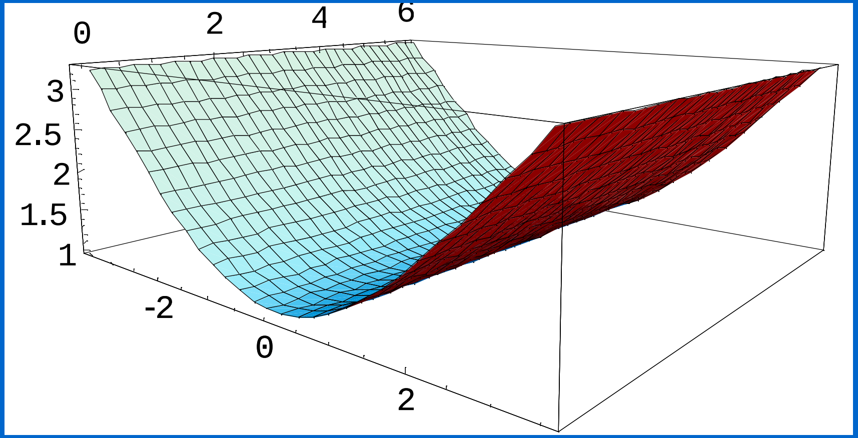
$$\frac{1}{r}\frac{d}{dr}\left(\frac{ru'(r)}{\sqrt{1-u'(r)^2}}\right) = \kappa\, u(r), \quad R \leq r < \infty$$

$$u(R) = b$$

$$u'(R) = \tanh(\beta)$$

➢ Cylindrical stationary surfaces





$$\frac{d}{dr}\left(\frac{r u'(r)}{\sqrt{1-u'(r)^2}}\right) = \kappa\, u(r), \quad 0 \le r \le R$$

$$u'(0) = 0$$

$$u'(R) = \tanh(\beta)$$

THE RESULTS

Description of the surfaces

- existence and uniqueness
- sign of κ

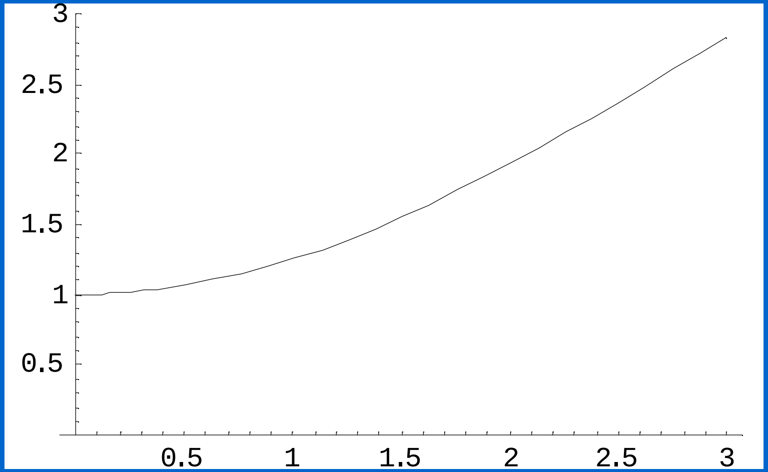
Estimates of the size

- height, volume, area,...
- in terms of the initial conditions

➢ Rotational compact stationary surfaces

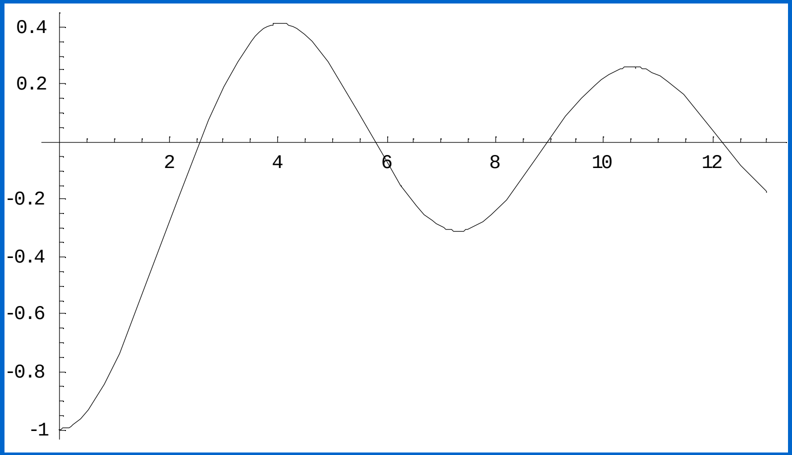
Case $\kappa > 0$.

- u can extend to $\mathbb{R}$.
- u is strictly increasing.
- u is a convex function with $u'(r) \to 1$.



Case $\kappa < 0$.

- u extends to $\mathbb{R}$ and $u(r) \to 0$.
- u has infinite zeroes.
- between maximum and minimum there is one inflection.

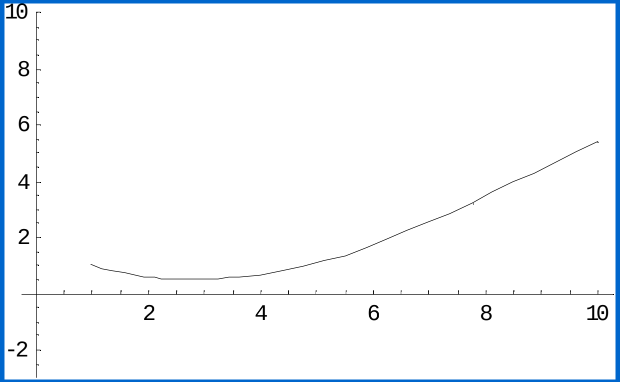


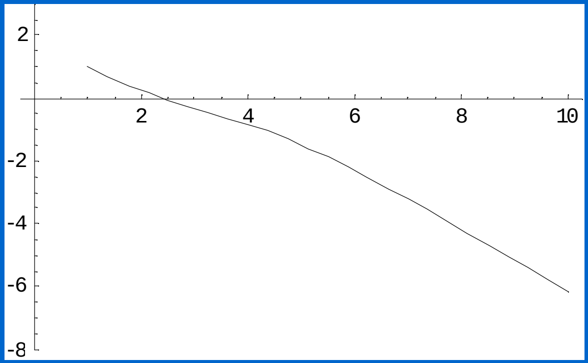
➢ Rotational non-compact stationary surfaces

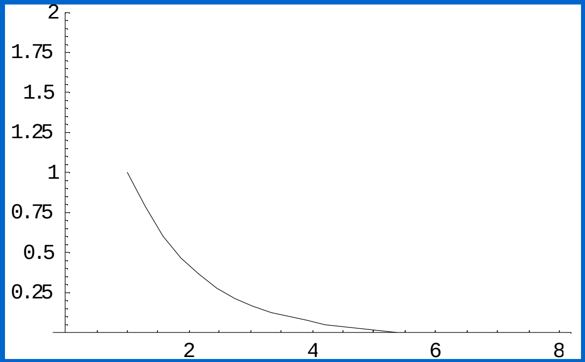
Solutions asymptotic to a spacelike plane

Given $\kappa > 0$ and R>0, there exists an angle β such that the solution u=u(r) satisfies

$$\lim_{r\to\infty} u(r) = 0$$



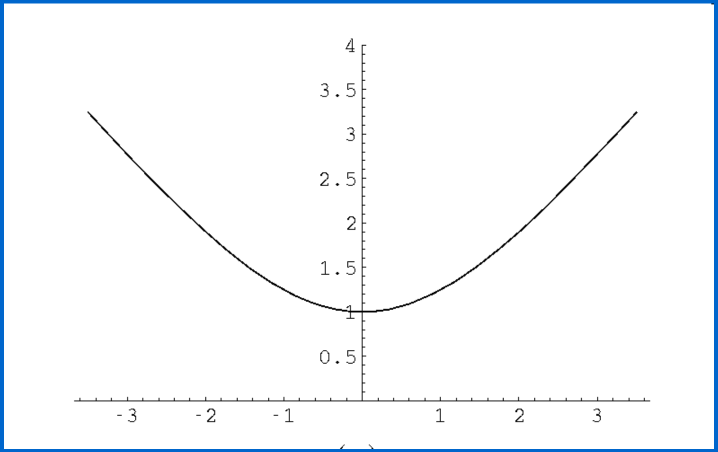




➢ Cylindrical stationary surfaces

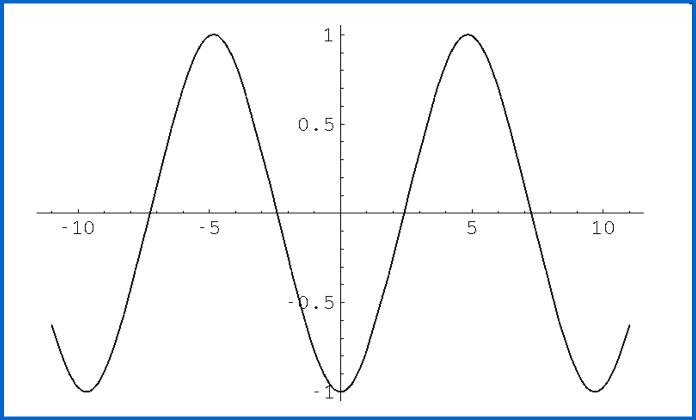
Case $\kappa > 0$.

- u can extend to $\mathbb{R}$.
- u is strictly increasing.
- u is a convex function with $u'(r) \to 1$.

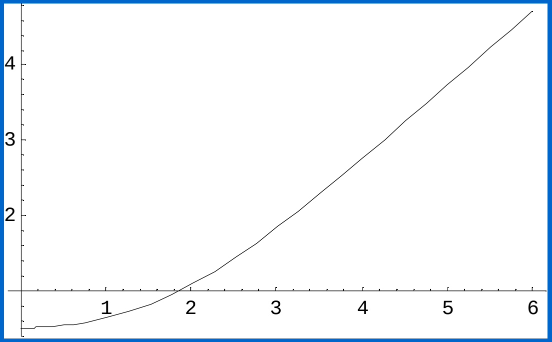


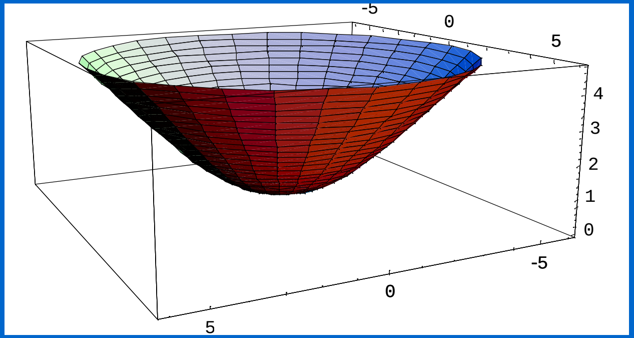
Case $\kappa < 0$.

- u can extend to $\mathbb{R}$.
- u is a periodic function.
- u has infinite zeroes = inflections.



➢ Estimates of rotational compact stationary surfaces: case $\kappa > 0$.





$$\frac{2\sinh\beta}{R\kappa} + \frac{R}{\sinh\beta} + \frac{2R}{3}\;\frac{1-\cosh^3\beta}{\sinh^3\beta} < u_0 < \frac{2\sinh\beta}{R\kappa}$$

$$u(R) < \frac{2\sinh\beta}{R\kappa} + R\;\frac{\cosh\beta}{\sinh\beta} + \frac{2R}{3}\;\frac{1-\cosh^3\beta}{\sinh^3\beta}$$

$$u(R) - u_0 < R\;\frac{\cosh(\beta)-1}{\sinh(\beta)}$$

Existence (volume)

Given κ, a contact angle β and a volume V, there exists a unique stationary compact surface in $\mathbb{L}^3$ resting in a spacelike plane whose enclosed volume is V.

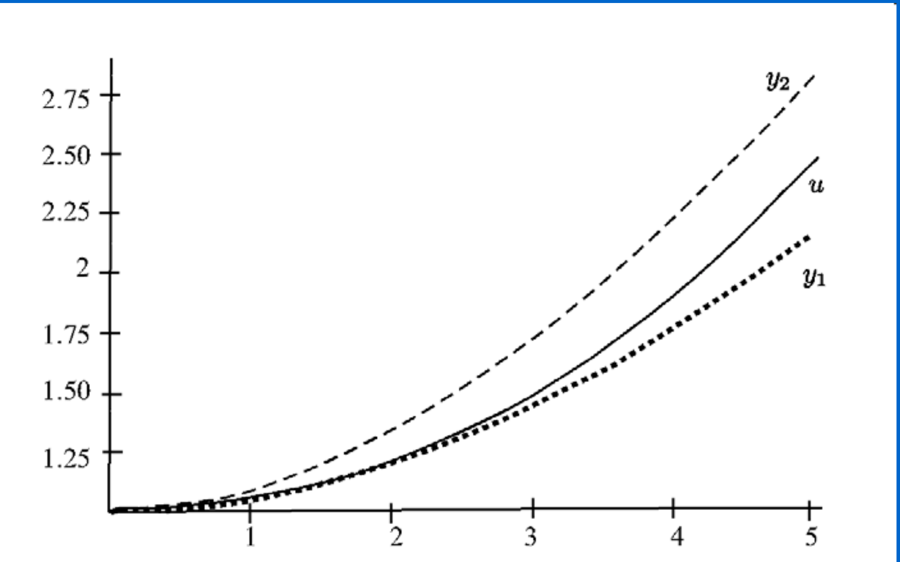
$$\mathcal{V} \leq \frac{\pi R^3}{3} \frac{\cosh^3\beta - 3\cosh\beta + 2}{\sinh^3\beta}$$

Σ_1 and Σ_2 the hyperbolic caps $y_i(r) = \sqrt{r^2 + \mu^2} + c$

Σ_1: $H_1 = \dfrac{\kappa u_0}{2}$ and $y_1(0) = u_0$

Σ_2: $y_2'(R) = u'(R)$ and $y_2(0) = u_0$

Claim: $y_1 \leq u \leq y_2$



CONCLUSIONS

- Stationary surfaces are solutions of a variational problem: cmc surfaces as particular case.
- Assumptions on the rotational / cylindrical hypothesis.
- Description of the shape of the surface.
- A priori estimates in terms of the initial conditions.