Simple Lie groups and pseudoRiemannian manifolds

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Examples of pseudoRiemannian manifolds with "large" isometry group.

A semisimple Lie group H, by definition, has a bi-invariant pseudoRiemannian metric given by the Killing form of its Lie algebra. For this we have $\mathrm{Iso}_0(H) = H \times H$.

If $\Gamma \subset H$ is a cocompact lattice, then H/Γ is a compact pseudoRiemannian manifold for which $\mathrm{Iso}_0(H/\Gamma) = H$.

Let $G \subset H$ be a simple Lie group and $K \subset H$ a compact subgroup that centralizes G and such that $K \cap \Gamma = \{e\}$. Then the double coset:

$$K \backslash H / \Gamma$$

admits a left G-action preserving a pseudoRiemannian structure induced by the metric on H and such that $G \subset \mathrm{Iso}_0(K \backslash H/\Gamma)$.

We will be interested in the case where G is noncompact and H has no compact factors.

Some basic examples.

- A finite volume example is given by taking $H = \mathsf{SL}(n+m,\mathbb{R})$, $\Gamma = \mathsf{SL}(n+m,\mathbb{Z})$, and

$$G = \mathsf{SL}(n,\mathbb{R}) \subset \left\{ \begin{pmatrix} * & 0 \\ 0 & I_m \end{pmatrix} \right\} \quad K = \mathsf{SO}(m) \subset \left\{ \begin{pmatrix} I_n & 0 \\ 0 & * \end{pmatrix} \right\}$$

with the double coset:

$$\mathsf{SO}(m)\backslash \mathsf{SL}(n+m,\mathbb{R})/\,\mathsf{SL}(n+m,\mathbb{Z}).$$

- Let $\Gamma \subset H = \mathsf{SL}(n,\mathbb{R}) \times \mathsf{SL}(m,\mathbb{R})$ be a cocompact lattice (it can be choosen irreducible if $n = m$) and

$$\begin{aligned} G &= \mathsf{SL}(n,\mathbb{R}) \times I_m \\ K &= \mathsf{SO}(m) \subset I_n \times \mathsf{SL}(n,\mathbb{R}). \end{aligned}$$

Then the example is the double coset:

$$(\mathsf{SL}(n,\mathbb{R}) \times \mathsf{SO}(m)\backslash \mathsf{SL}(m,\mathbb{R}))/\Gamma.$$

A program on pseudoRiemannian manifolds with a "large" isometry group.

Suppose we are given:

- M a compact pseudoRiemannian manifold,
- G a noncompact simple Lie group, and
- a smooth action of G on M preserving the metric.

Problem: Determine to what extent M is related to the previous examples.

For such an M and an arbitrary pseudoRiemannian manifold N, the manifold:

$$M \times N$$

is a new example carrying an isometric G-action where we have no control on N. We want to avoid these examples.

We can avoid such examples by considering G-actions on M with either of the following hypothesis:

- There is a dense G-orbit.

- The manifold M is locally irreducible, i.e. its universal covering space is an irreducible pseudoRiemannian manifold.

Conjecture (on pseudoRiemannian manifolds with large isometry group):

Suppose we are given:

- M a compact pseudoRiemannian manifold,
- G a noncompact simple Lie group, and
- a smooth action of G on M preserving the metric and with a dense orbit.

Then up to "trivial" constructions M is a double coset example $K\backslash H/\Gamma$ as described before.

Here "trivial" means up to a finite covering.

A first related result is the following.

Theorem (Zimmer)**.** *Suppose we are given:*

- M *a compact pseudoRiemannian manifold,*
- G *a noncompact simple Lie group, and*
- *a smooth ergodic action of* G *on* M *preserving the metric.*

Then there is a Lie algebra embedding:

$$\mathfrak{g} \hookrightarrow \mathfrak{so}(m_1, m_2)$$

of the Lie algebra $\mathfrak{g}$ *of* G *into* $\mathfrak{so}(m_1, m_2)$*, where* (m_1, m_2) *is the signature of* M*.*

The above Zimmer's result provide a relation between the Lie algebra $\mathfrak{g}$ and the signature of M, for an isometric G-action on M as above.

Recall that G itself is a pseudoRiemannian manifold and so has a signature, say (n_1, n_2). A natural problem is to determine a relation between the signatures of G and M for our setup.

From now on, we will denote:

- (m_1, m_2) the signature of M,
- (n_1, n_2) the signature of G,
- $m_0 = \min(m_1, m_2)$ and $n_0 = \min(n_1, n_2)$, which we will call the null dimensions of M and G, respectively.

Theorem (Gromov). *For G and M as above:*

$$n_0 \leq m_0.$$

In other words, the dimension of the tangent null subspaces to G is bounded by the dimension of the maximal null tangent subspaces of M.

Problem: Study the nature of M for suitable relations between n_0 and m_0.

Theorem (Quiroga)**.** *Let G and M as before be such that G with* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. *If G acts faithfully on M and $n_0 = m_0$, then there exist:*

(1) *a finite covering $\widehat{M} \to M$,*

(2) *a connected semisimple Lie group of the form $H = G \times G'$,*

(3) *a lattice $\Gamma \subset H$ and $K \subset H$ a compact subgroup centralizing G,*

for which the G-action on M lifts to $\widehat{M}$ so that:

$$\widehat{M} \cong K \backslash H / \Gamma = K \backslash (G \times G') / \Gamma$$

are G-equivariantly diffeomorphic.

Techniques used in the proof.

- (Zimmer, Gromov, Szaro) The G-actions considered are known to be locally free, i.e. with discrete stabilizers.

- The orbit maps:

$$\begin{array}{ccc} G & \longrightarrow & M \\ g & \longmapsto & gx, \end{array}$$

  allow us to compare the metric on M with the natural metric on G.

- (Gromov, Candel-Quiroga) An essential tool is obtained by showing the existence of additional local isometries for M.

More precisely, in a the neighborhood of almost every point x there exists a Lie algebra $\mathfrak{g}(x)$ of Killing vector fields satisfying:

(1) $Z_x = 0$, for every $Z \in \mathfrak{g}(x)$,

(2) the local one-parameter subgroups of $\mathfrak{g}(x)$ preserve the G-orbits,

(3) $\mathfrak{g}(x)$ and $\mathfrak{g}$ are isomorphic Lie algebras, so that the canonical vector space isomorphism $T_xGx \cong \mathfrak{g}$ is an isomorphism of $\mathfrak{g}$-modules.

This are obtained using Gromov-Nomizu results on rigid geometric structures and Zimmer's algebraic hull for actions.

- If $T\mathcal{O}^\perp$ denotes the orthogonal complement to the tangent bundle to the G-orbits in M, then the Lie algebras $\mathfrak{g}(x)$ are used to apply representation theory and prove that $T\mathcal{O}^\perp$ is integrable. The leaves of this bundle provide the factor $K\backslash G'$.

Another example of a result that can be obtained with these techniques.

Theorem (Quiroga). *Let $G = \mathrm{SO}(n,n)$ and M be as before with the manifold and action assumed to be analytic. Suppose that M is complete and locally irreducible, that there is a dense orbit and that:*

$$\begin{aligned} \dim M \leq 2n^2 + n &= \dim \mathrm{SO}(n,n) + 2n \\ &= \dim \mathrm{SO}(n,n+1). \end{aligned}$$

Then, M is $\mathrm{SO}(n,n)$-equivariantly diffeomorphic to a manifold of the form:

$$\mathrm{SO}(n,n+1)/\Gamma,$$

where Γ is a cocompact lattice.