A Variational Approach to Robertson-Walker Spacetimes with Homogeneous Scalar Fields

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IMLG 2006, Santiago de Compostela

Outline

1. **History**
   - Scalar field spacetimes
2. **Homogeneous scalar field with a potential**
   - The model
   - Variational formulation
   - Main result
3. **The functional framework**
   - Abstract critical point theory
4. **Proof of the main result**
   - The approximation scheme
   - Convergence to a solution
   - Conclusions

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How many ways scalar fields' evolution study has been approached in?

Some - very unexhaustive! - literature

- qualitative [causal structure analysis](Christodoulou 1991–'98)
- late time behavior (Joshi et al 2004, R.G. 2005, Rendall, Miritzis 2006)
- numerical (Choptuik 1983, Garfinkle 2004, Alcubierre *et al* 2004)

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Assumptions and equations

Assumptions

1. metric: $k = 0$ FRW
2. matter: scalar field ϕ with potential $V(\phi)$

Equations

- line element:

$$g = -dt \otimes dt + a^2(t)\left[dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3\right]$$

- field equations (in the unknowns $a(t), \phi(t)$):

$$(G^0_0 = 8\pi T^0_0): \qquad -\frac{3\dot{a}^2}{a^2} = -(\dot{\phi}^2 + 2V(\phi))$$

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The Hilbert–Palatini action functional

$$\mathcal{L} = \int_M \sqrt{-\det g}\,(L_g + L_f)\,\mathrm{d}V$$

The Hilbert–Palatini action functional

$$\mathcal{L}(a,\phi) = \int_0^T 3a(t)\dot{a}^2(t) - a^3(t)\dot{\phi}^2(t) + 2a^3(t)V(\phi(t))\,\mathrm{d}t$$

Proposition

If $(a,\phi) \in C^2(\mathbb{R}^+,\mathbb{R})$ solves Euler–Lagrange equation for $\mathcal{L}$, and

$$3\dot{a}(0)^2 = a_0^2(\dot{\phi}(0)^2 + 2V(\phi_0)), \tag{1}$$

then it is a solution for homogeneous scalar field equations.

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Question

How to get rid of the arrival time T (unknown, in principle)?

Solution

Use a *modified* version of Euler–Maupertuis least action principle [van Groesen 1985, see also R.G, F Giannoni, P Piccione 2006]

$$F(a,\phi) = \left(\int_0^1 3a\dot{a}^2 - a^3\dot{\phi}^2\,\mathrm{d}t\right)\cdot\left(\int_0^1 2a^3V(\phi)\,\mathrm{d}t\right)$$

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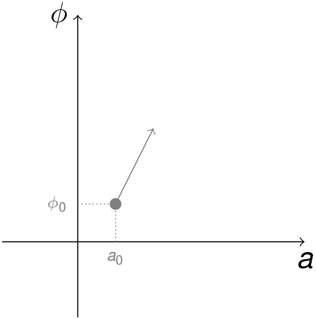
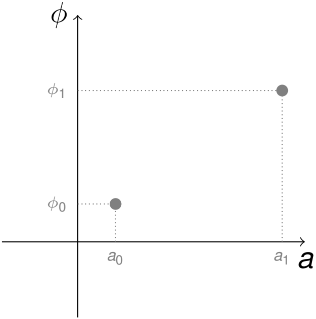
We look for critical points between *prescribed* configurations





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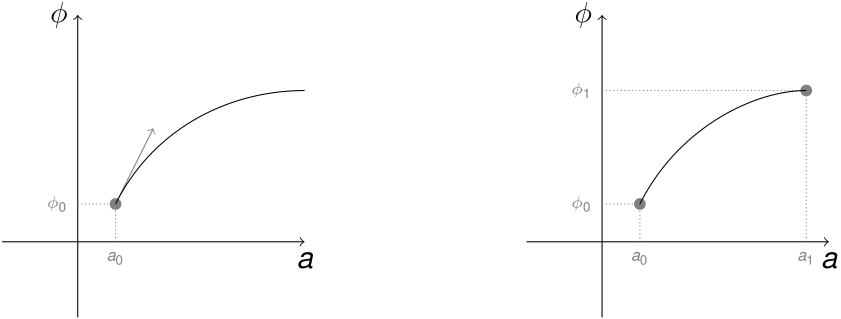

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Problem

Given

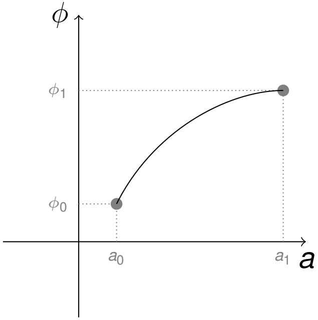
$a_0, a_1 \in \mathbb{R}^+, \phi_0, \phi_1 \in \mathbb{R}, V \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$,

find critical points of the functional

$F(a, \phi) = \left(\int_0^1 3a(t)\dot{a}^2(t) - a^3(t)\dot{\phi}^2(t)\, \mathrm{d}t \right) \cdot \left(\int_0^1 2a^3(t) V(\phi(t))\, \mathrm{d}t \right)$,

with positive critical value, in the space of $\mathcal{C}^2$ curves $(a, \phi) : [0, 1] \to \mathbb{R}^+ \times \mathbb{R}$ such that

$a(0) = a_0, a(1) = a_1, \phi(0) = \phi_0, \phi(1) = \phi_1$.



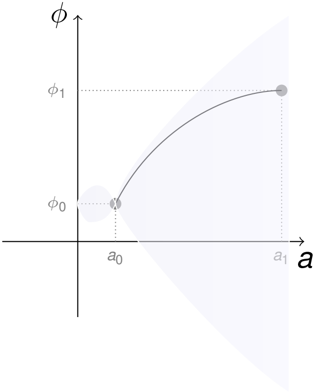
Existence of solutions

Theorem (R.G, F. Giannoni, G. Magli, *JMP* 2006)

Under the assumptions

- $3\min\{a_0, a_1\}(a_1 - a_0)^2 > \max\{a_0, a_1\}(\phi_1 - \phi_0)^2$
- $V \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$,

there exists $T > 0$ *and* $(a(t), \phi(t)) \in \mathcal{C}^2([0, T], \mathbb{R}^2)$ *solutions, with the boundary conditions* $a(0) = a_0, a(T) = a_1, \phi(0) = \phi_0, \phi(T) = \phi_1$.



Open issue (for future developments...)

How to find evolution leading to a singularity ($a(T) = 0$)?

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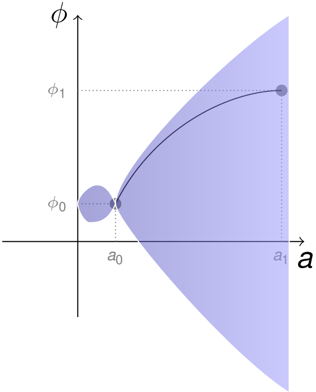

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What is the main obstruction to develop a variational theory in this case?

Problem

The functional is not positive definite on the velocities

Jump to the functional

Solution (Rabinowitz, 1986)

Saddle Point Theorem

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Saddle Point Theorem

Rabinowitz' Saddle Point Theorem

Theorem

- Ω *Hilbert manifold,* Y *finite dimensional affine space*
- $\mathfrak{X} = \Omega \times Y$, $f \in \mathcal{C}^1(\mathfrak{X}, \mathbb{R})$
- $\exists \omega_0 \in \Omega$, $e_0 \in Y$, $R > 0$ *such that, called* $B_R(e_0) = \{e \in Y \,:\, \|e - e_0\| \le R\}$, *it is*

$$b_0 \equiv \sup_{e \in \partial B_R(e_0)} f(\omega_0, e) < b_1 \equiv \inf_{\omega \in \Omega} f(\omega, e_0);$$

- *if* $b_2 = \sup_{e \in B_R(e_0)} f(\omega_0, e)$, *the strip* $\{x \in \mathfrak{X} \,:\, b_1 \le f(x) \le b_2\} \subset \mathfrak{X}$ *is complete;*
- f *satisfies* $(PS)_c$ *at any* $c \in [b_1, b_2]$.

$\Longrightarrow$ *there exists a critical value* c *for* f *in* $[b_1, b_2]$.

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Palais–Smale condition

Any sequence $\{x_n\}_{n \in} \subset \mathfrak{X}$ such that $f(x_n) \to c$, and $\nabla f(x_n) \to 0$ has a converging subsequence in $\mathfrak{X}$.

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Saddle Point Theorem cannot be applied *as is*...

...in our case:

$\Omega = \{a \in H^1([0,1],]m, M[) \ : \ a(0) = a_0, \ a(1) = a_1\}$,
$Y = \{\phi = \widehat{\phi} + \phi_* \ : \ \widehat{\phi} \in H^1_0([0,1], \mathbb{R})\}$, where $\phi_*(t) = (1-t)\phi_0 + t\phi_1$.

Obstructions

1. $\dim Y = +\infty$
2. Ω is not complete
3. $V(\phi)$ unbounded above

Solution

We need to approximate both the functional and the space.

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Initial vs approximating problems

Our problem

$\mathfrak{X} = \Omega \times Y$

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Approx problem

$\mathfrak{X}_k = \Omega \times Y_k$

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Proposition

The functional $F_{\epsilon,\lambda}$ satisfies Saddle Point Theorem hypotheses on the space $\mathfrak{X}_k = \Omega \times Y_k$.

Initial vs approximating problems

Our problem

$\mathfrak{X} = \Omega \times Y$

$Y = \{\phi = \widehat{\phi} + \phi_* : \widehat{\phi} \in H^1_0([0,1], \mathbb{R})\}$
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Proof of the main result

Proof.

- $\exists$ critical point (a_k, ϕ_k) for $F_{\epsilon,\lambda}$ on $\mathfrak{X}_k = \Omega \times Y_k$ such that $F_{\epsilon,\lambda}(a_k, \phi_k) \in [b_1, b_2]$
- b_1, b_2 are independent of k
- $(a_{k_j}, \phi_{k_j}) \xrightarrow{j\to\infty} (a, \phi)$ critical point for $F_{\epsilon,\lambda}$ on $\mathfrak{X}$
- for ϵ sufficiently small, $U_\epsilon(a) = 0 \Rightarrow (a, \phi)$ is a critical point for the functional $F_{0,\lambda}$ on $\mathfrak{X}$
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Summary

What we have done

- The problem of homogeneous scalar field in spherical symmetry has been tackled using a variational approach.
- Since the functional is not positive definite, Rabinowitz Saddle Point Theorem has been applied (to an approximation of the original problem, actually)
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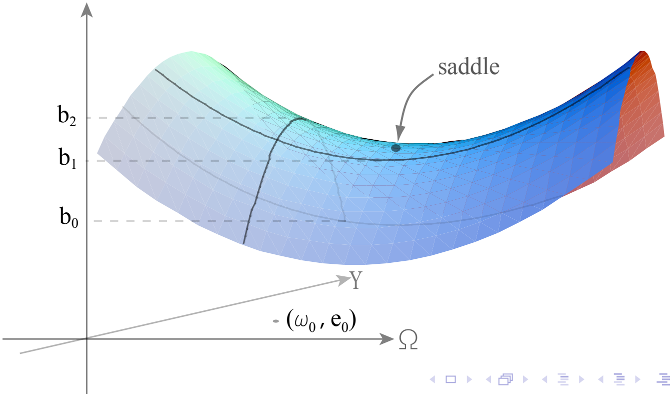
Rabinowitz' Saddle Point Theorem

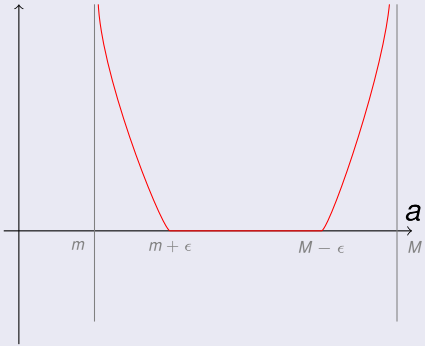
$b_0 \equiv \sup_{e \in \partial B_R(e_0)} f(\omega_0, e) < b_1 \equiv \inf_{\omega \in \Omega} f(\omega, e_0)$
$b_2 \equiv \sup_{e \in B_R(e_0)} f(\omega_0, e)$

Return



Initial vs approximating problems

$U_\epsilon(a)$



Close

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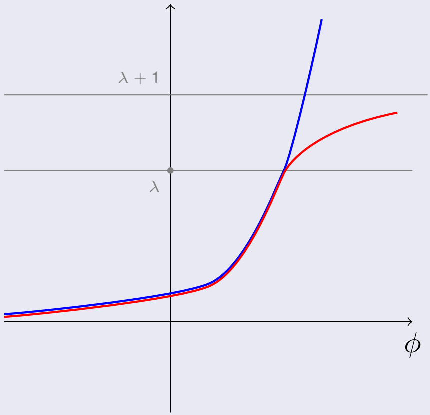

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