A Variational Approach to Robertson-Walker Spacetimes with Homogeneous Scalar Fields

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IMLG 2006, Santiago de Compostela

History	Homogeneous	scalar	field	with	a potentia

Proof of the main result

Outline

History

Scalar field spacetimes

2 Homogeneous scalar field with a potential

- The model
- Variational formulation
- Main result
- 3 The functional framework
 - Abstract critical point theory

Proof of the main result

- The approximation scheme
- Convergence to a solution
- Conclusions

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The functional framework $\circ\circ$

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Scalar field spacetimes

How many ways scalar fields' evolution study has been approached in?

Some - very unexhaustive! - literature

- qualitative [causal structure analysis](Christodoulou 1991–'98)
- late time behavior (Joshi et al 2004, R.G. 2005, Rendall, Miritzis 2006)
- numerical (Choptuik 1983, Garfinkle 2004, Alcubierre *et al* 2004)

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The functional framework

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The model

Assumptions and equations

Assumptions

- metric: k = 0 FRW
-) matter: scalar field ϕ with potential $V(\phi)$

Equations

• line element:

 $g = -dt \otimes dt + a^{2}(t) \left[dx^{1} \otimes dx^{1} + dx^{2} \otimes dx^{2} + dx^{3} \otimes dx^{3} \right]$

field equations (in the unknowns a(t), φ(t)):

$$(G_0^0 = 8\pi T_0^0): -\frac{3\dot{a}^2}{a^2} = -(\dot{\phi}^2 + 2V(\phi))$$

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$$(G_0^0 = 8\pi T_0^0): \qquad -\frac{3a^2}{a^2} = -(\dot{\phi}^2 + 2V(\phi))$$
$$(G_1^1 = 8\pi T_1^1): \qquad -\frac{a^2}{a^2} + 2aa$$

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Homogeneous scalar field with a potential $\circ \bullet \circ \circ$

The functional framework

Proof of the main result

Variational formulation

The Hilbert–Palatini action functional

$$\mathcal{L} = \int_{M} \sqrt{-\det g} \left(L_g + L_f \right) \mathrm{d}V$$



History	Homogeneous
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scalar field with a potential The f

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$$\mathcal{L}(a,\phi) = \int_0^1 3a(t)\dot{a}^2(t) - a^3(t)\dot{\phi}^2(t) + 2a^3(t)V(\phi(t))\,\mathrm{d}t$$

Proposition

If $(a, \phi) \in C^2(\mathbb{R}^+, \mathbb{R})$ solves Euler–Lagrange equation for \mathcal{L} , and $3\dot{a}(0)^2 = a_0^2(\dot{\phi}(0)^2 + 2V(\phi_0)),$ (1)

then it is a solution for homogeneous scalar field equations.

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Homogeneous scalar field with a potential $\circ \bullet \circ \circ$

The functional framework

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Question

How to get rid of the arrival time T (unknown, in principle)?

Solution

Use a *modified* version of Euler–Maupertuis least action principle [van Groesen 1985, see also R.G, F Giannoni, P Piccione 2006]

$$F(\boldsymbol{a},\phi) = \left(\int_0^1 3\boldsymbol{a}\dot{\boldsymbol{a}}^2 - \boldsymbol{a}^3\dot{\phi}^2\,\mathrm{d}t\right)\cdot\left(\int_0^1 2\boldsymbol{a}^3V(\phi)\,\mathrm{d}t\right)$$

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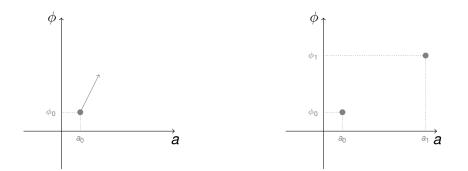
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Variational formulation

We look for critical points between *prescribed* configurations



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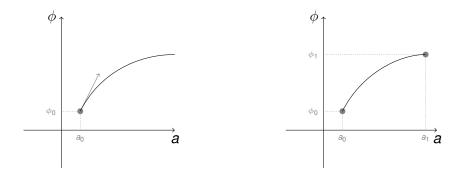
Homogeneous scalar field with a potential $\circ \circ \circ \circ$

The functional framework

Proof of the main result

Variational formulation

We look for critical points between *prescribed* configurations



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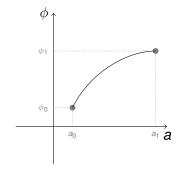
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Problem

Given

 $\begin{aligned} a_0, a_1 \in \mathbb{R}^+, \phi_0, \phi_1 \in \mathbb{R}, V \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}), \\ \text{find critical points of the functional} \\ F(a, \phi) &= \left(\int_0^1 3a(t)\dot{a}^2(t) - a^3(t)\dot{\phi}^2(t)\,\mathrm{d}t\right) \cdot \\ \left(\int_0^1 2a^3(t)V(\phi(t))\,\mathrm{d}t\right), \\ \text{with positive critical value, in the space of} \\ \mathcal{C}^2 \text{ curves } (a, \phi) : [0, 1] \to \mathbb{R}^+ \times \mathbb{R} \text{ such that} \\ a(0) &= a_0, a(1) = a_1, \phi(0) = \phi_0, \phi(1) = \phi_1. \end{aligned}$



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Homogeneous scalar field with a potential $\circ \circ \circ \bullet$

The functional framework

Proof of the main result

Main result

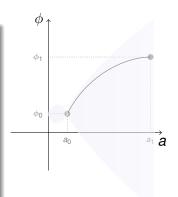
Existence of solutions

Theorem (R.G, F. Giannoni, G. Magli, JMP 2006)

Under the assumptions

- $3\min\{a_0, a_1\}(a_1 a_0)^2 > \max\{a_0, a_1\}(\phi_1 \phi_0)^2$
- $V \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$,

there exists T > 0 and $(a(t), \phi(t)) \in C^2([0, T], \mathbb{R}^2)$ solutions, with the boundary conditions $a(0) = a_0, a(T) = a_1, \phi(0) = \phi_0, \phi(T) = \phi_1.$



Open issue (for future developments...

How to find evolution leading to a singularity (a(T) = 0)?

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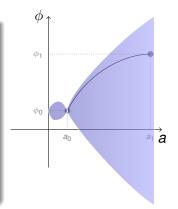
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there exists T > 0 and $(a(t), \phi(t)) \in C^2([0, T], \mathbb{R}^2)$ solutions, with the boundary conditions $a(0) = a_0, a(T) = a_1, \phi(0) = \phi_0, \phi(T) = \phi_1.$



Open issue (for future developments...)

How to find evolution leading to a singularity (a(T) = 0)?

History	Homogeneous scalar	field with a	poten

Proof of the main result

Outline

1 History

Scalar field spacetimes

Provide the second s

ntial

- The model
- Variational formulation
- Main result

3 The functional framework

Abstract critical point theory

Proof of the main result

- The approximation scheme
- Convergence to a solution
- Conclusions

The functional framework ●○ Proof of the main result

Abstract critical point theory

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Problem

The functional is not positive definite on the velocities

Jump to the functional

Solution (Rabinowitz, 1986)

Saddle Point Theorem

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The functional framework ●○ Proof of the main result

Abstract critical point theory

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Homogeneous scalar field with a potential 0000

The functional framework

Proof of the main result

Abstract critical point theory

Rabinowitz' Saddle Point Theorem

Theorem

- Ω Hilbert manifold, Y finite dimensional affine space
- ∃ω₀ ∈ Ω, e₀ ∈ Y, R > 0 such that, called B_R(e₀) = {e ∈ Y : ||e − e₀|| ≤ R}, it is

 $b_0 \equiv \sup_{e \in \partial B_R(e_0)} f(\omega_0, e) < b_1 \equiv \inf_{\omega \in \Omega} f(\omega, e_0);$

- if $b_2 = \sup_{e \in B_R(e_0)} f(\omega_0, e)$, the strip $\{x \in \mathfrak{X} : b_1 \leq f(x) \leq b_2\} \subset \mathfrak{X}$ is complete;
- f satisfies $(PS)_c$ at any $c \in [b_1, b_2]$

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listory	Homogeneous	scalar	field	with	potential

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Abstract critical point theory

Rabinowitz' Saddle Point Theorem

Theorem

Palais–Smale condition

- Any sequence $\{x_n\}_{n\in \mathbb{C}} \in \mathfrak{X}$ such that $f(x_n) \to c$, and $\nabla f(x_n) \to 0$ has a converging subsequence in \mathfrak{X} .
- if $b_2 = \sup_{e \in B_B(e_0)} f(\omega_0, e)$, are same
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Proof of the main result

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Proof of the main result

The approximation scheme

Saddle Point Theorem cannot be applied as is ...

...in our case:

$$\begin{split} \Omega &= \{ \pmb{a} \in H^1([0,1],]m, \pmb{M}[) : \ \pmb{a}(0) = \pmb{a}_0, \ \pmb{a}(1) = \pmb{a}_1 \}, \\ Y &= \{ \phi = \widehat{\phi} + \phi_* : \ \widehat{\phi} \in H^1_0([0,1], \mathbb{R}) \}, \text{ where } \phi_*(t) = (1-t)\phi_0 + t\phi_1. \end{split}$$

Obstructions

- dim $Y = +\infty$
- **2** Ω is not complete
- 3 $V(\phi)$ unbounded above

Solution

We need to approximate both the functional and the space.

History	Homogeneous	scalar	field	with a	a potentia

Proof of the main result

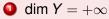
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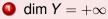
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History

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The approximation scheme

Initial vs approximating problems

Our problem

 $\mathfrak{X}=\Omega\times\,Y$

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 $\begin{aligned} F(\mathbf{a},\phi) &= \\ \left(\int_0^1 3\mathbf{a}\dot{\mathbf{a}}^2 - \mathbf{a}^3\dot{\phi}^2 \mathrm{d}t\right) \cdot \left(\int_0^1 2\mathbf{a}^3 V(\phi) \mathrm{d}t\right) \end{aligned}$

Approx problem $\begin{aligned} \widehat{\mathbf{X}}_{k} &= \Omega \times \mathbf{Y}_{k} \\ Y_{k} &= \{ \phi = \widehat{\phi_{k}} + \phi_{*} : \widehat{\phi_{k}} \in W_{k} \} \\ \text{where} \\ W_{k} &= \text{span}\{ \sin(\pi \ell t) : t \in [0, 1], \ell = 1, ..., k \} \\ F_{\epsilon, \mathbf{X}}(a, \phi) &= \left(\int_{0}^{1} (3a + U_{\epsilon}(a)) \dot{a}^{2} - a^{3} \dot{\phi}^{2} dt \right) \cdot \left(\int_{0}^{1} 2a^{3} V_{\mathbf{X}}(\phi) dt \right) \end{aligned}$

Proposition

The functional $F_{\epsilon,\lambda}$ satisfies Saddle Point Theorem hypotheses on the space $\mathfrak{X}_k = \Omega \times Y_k$.

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History

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History	Homogeneous s	calar field	with a	potential

Proof of the main result

Convergence to a solution

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- \exists critical point (a_k, ϕ_k) for $F_{\epsilon,\lambda}$ on $\mathfrak{X}_k = \Omega \times Y_k$ such that $F_{\epsilon,\lambda}(a_k, \phi_k) \in [b_1, b_2]$
- b₁, b₂ are independent of k
- $(a_{k_j}, \phi_{k_j}) \xrightarrow{f \to \infty} (a, \phi)$ critical point for $F_{\epsilon, \lambda}$ on \mathfrak{X}
- for ε sufficiently small, U_ε(a) = 0 ⇒ (a, φ) is a critical point for the functional F_{0,λ} on X
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History	Homogeneous scalar field with a potential

Proof of the main result

Conclusions

Summary

What we have done

- The problem of homogeneous scalar field in spherical symmetry has been tackled using a variational approach.
- Since the functional is not positive definite, Rabinowitz Saddle Point Theorem has been applied (to an approximation of the original problem, actually)
- Approximating solutions are shown to converge to the solution of the original problem

Open Problems

- How to modify this scheme for the case of singular solutions?
- What about non-homogeneous case?

History	Homogeneous scalar field with a potential

Proof of the main result

Conclusions

Summary

What we have done

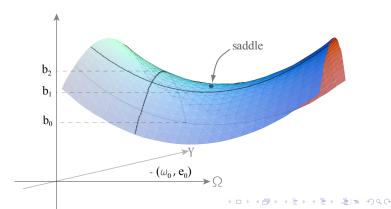
- The problem of homogeneous scalar field in spherical symmetry has been tackled using a variational approach.
- Since the functional is not positive definite, Rabinowitz Saddle Point Theorem has been applied (to an approximation of the original problem, actually)
- Approximating solutions are shown to converge to the solution of the original problem

Open Problems

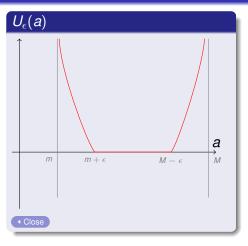
- How to modify this scheme for the case of singular solutions?
- What about non-homogeneous case?

Rabinowitz' Saddle Point Theorem

$$\begin{array}{l} b_0 \equiv \sup_{e \in \partial B_R(e_0)} f(\omega_0, e) < b_1 \equiv \inf_{\omega \in \Omega} f(\omega, e_0) \\ b_2 \equiv \sup_{e \in B_R(e_0)} f(\omega_0, e) \end{array}$$



Initial vs approximating problems



Approx problem

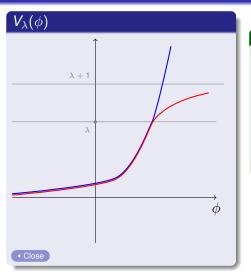
 $\mathfrak{X}_k = \Omega \times Y_k$

$$Y_k = \{ \phi = \widehat{\phi_k} + \phi_* : \widehat{\phi_k} \in W_k \}$$

where
$$W_k = \text{span}\{\sin(\pi \ell t) : t \in [0, 1], \ell = 1, ..., k\}$$

$$F_{\epsilon,\lambda}(\mathbf{a},\phi) = \left(\int_0^1 (3\mathbf{a} + U_{\epsilon}(\mathbf{a}))\dot{\mathbf{a}}^2 - \mathbf{a}^3\dot{\phi}^2 \mathrm{d}t \right) \cdot \left(\int_0^1 2\mathbf{a}^3 V_{\lambda}(\phi) \mathrm{d}t \right)$$

Initial vs approximating problems



Approx problem

 $\mathfrak{X}_k = \Omega \times Y_k$

$$\begin{array}{l} Y_k = \{ \phi = \widehat{\phi_k} + \phi_* : \widehat{\phi_k} \in W_k \} \\ \text{where} \\ W_k = \text{span}\{ \sin(\pi \ell t) : t \in [0, 1], \ell = 1, ..., k \} \end{array}$$

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