Holonomy groups of Lorentzian manifolds

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Parallel displacement

Let \((M, \nabla)\) be an affine manifold, i.e. \(\nabla\) a linear connection.
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\[\sim\] Parallel displacement along \(\gamma : [0, 1] \to M\), piecewise smooth,
Parallel displacement

Let \((M, \nabla)\) be an affine manifold, i.e. \(\nabla\) a linear connection.

\(\leadsto\) Parallel displacement along \(\gamma : [0, 1] \to M,\) piecewise smooth,

\[
\mathcal{P}_\gamma : T_{\gamma(0)}M \xrightarrow{\sim} T_{\gamma(1)}M
\]

\[
X_0 \quad \mapsto \quad X(1),
\]
Parallel displacement

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\]

\[
X_0 \mapsto X(1),
\]

where \(X(t)\) is the solution to the ODE

\[
\nabla_{\dot{\gamma}(t)} X(t) \equiv 0 \text{ with initial condition } X(0) = X_0.
\]
Holonomy group of a linear connection

For $p \in M^n$ we define the Holonomy group

$$\text{Hol}_p (M, \nabla) := \left\{ \mathcal{P}_\gamma | \gamma(0) = \gamma(1) = p, \mathcal{P}_\gamma \right\} \cap \text{Gl}(T_p M)$$

and its Lie algebra $\text{hol}_p (M, \nabla)$. 

Example

$\nabla$ flat $\Rightarrow \text{Hol}_p (M, \nabla) = \Pi_1 (M)$ and $\text{hol}_p (M, \nabla) = \{0\}$. 

Sn the round sphere: $\text{Hol}_p (S^n) = \text{SO}(n)$. 

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Holonomy group of a linear connection

For $p \in M^n$ we define the (Connected) Holonomy group

$$\text{Hol}_p^0(M, \nabla) := \left\{ P_\gamma | \gamma(0) = \gamma(1) = p, \gamma \sim \{p\} \right\} \cap \text{Gl}(T_p M)$$

and its Lie algebra $\mathfrak{hol}_p(M, \nabla)$. 

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Example $\nabla$ flat $\Rightarrow \text{Hol}_p^0(M, \nabla) = \pi_1(M)$ and $\text{hol}_p(M, \nabla) = \{0\}$.

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holonomy representation $\cap \quad \text{Gl}(n, \mathbb{R}) \cong \text{Gl}(T_pM)$ (fixing a basis)

and its Lie algebra $\mathfrak{hol}_p(M, \nabla)$.

For $p, q \in M$:

$$\text{Hol}_p(M, \nabla) \sim \text{Hol}_q(M, \nabla)$$

conjugated in $\text{Gl}(n, \mathbb{R})$
Holonomy group of a linear connection

For $p \in M^n$ we define the (Connected) Holonomy group

$$\text{Hol}^0_p(M,\nabla) := \left\{ \gamma \in \text{GL}(M) \right\}$$

holonomy representation $\downarrow$

$$\text{GL}(n, \mathbb{R}) \cong \text{GL}(T_p M) \ (\text{fixing a basis})$$

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$$\text{Hol}_p(M,\nabla) \downarrow \text{conjugated in } \text{GL}(n, \mathbb{R}) \downarrow \text{Hol}_q(M,\nabla)$$

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For \( p \in M^n \) we define the (Connected) Holonomy group

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Conjugated in \( \text{Gl}(n, \mathbb{R}) \)

Example

- \( \nabla \text{ flat } \Rightarrow \text{Hol}_p(M, \nabla) = \Pi_1(M) \) and \( \mathfrak{h} \text{ol}_p(M, \nabla) = \{0\} \).
- \( S^n \) the round sphere: \( \text{Hol}_p(S^n) = SO(n) \).
Classification problem

Which groups may occur as holonomy groups?
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- Hano/Ozeki ’56: Any closed $G \subset GL(n, \mathbb{R})$. But $\nabla$ might have torsion.
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**Theorem (Ambrose/Singer)**

$M$ connected $\implies \text{hol}(M, \nabla)$ is spanned by

\[
\left\{ P_\gamma^{-1} \circ R(X, Y) \circ P_\gamma \, | \, \gamma(0) = p \text{ and } X, Y \in T_{\gamma(1)} M \right\}
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satisfies Bianchi identity if $T^\nabla = 0$
Classification problem

Which groups may occur as holonomy groups?

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Theorem (Ambrose/Singer)

\( M \) connected \( \implies \) \( h\text{ol}_p(M, \nabla) \) is spanned by

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satisfies Bianchi identity if \( T^\nabla = 0 \)

\( \implies \) \( h\text{ol}_p(M, \nabla) \) is a Berger algebra.
Berger algebras

Let $g \subset \mathfrak{gl}(n, \mathbb{R})$ be a subalgebra.
Berger algebras

Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be a subalgebra. The $\mathfrak{g}$ module of formal curvature endomorphisms is defined as

$$\mathcal{K}(\mathfrak{g}) := \left\{ R \in \Lambda^2 \mathbb{R}^n \otimes \mathfrak{g} \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \right\}$$
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$\mathfrak{g}$ is a Berger algebra $\iff \mathfrak{g} = \left\langle R(x, y) \mid R \in \mathcal{K}(\mathfrak{g}), x, y \in \mathbb{R}^n \right\rangle$
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$T\nabla = 0$: Ambrose-Singer $\implies \mathfrak{ho}_p(M, \nabla)$ is a Berger algebra.
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$T^\nabla = 0$: Ambrose-Singer $\implies$ $\mathfrak{hol}(M, \nabla)$ is a Berger algebra.

Classification of Berger algebras:

~ Classification of holonomy algebras of torsion free connections.
Berger algebras

Let $g \subset \mathfrak{gl}(n, \mathbb{R})$ be a subalgebra. The $g$ module of formal curvature endomorphisms is defined as

$$\mathcal{K}(g) := \{ R \in \Lambda^2 \mathbb{R}^n \otimes g \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \}$$

**$g$ is a Berger algebra $\iff g = \langle R(x, y) \mid R \in \mathcal{K}(g), x, y \in \mathbb{R}^n \rangle$**

$T^\nabla = 0$: Ambrose-Singer $\implies$ holp($M, \nabla$) is a Berger algebra.

Classification of irreducible Berger algebras:

- Berger ’55: $g \subset \mathfrak{so}(p, q)$,

$\leadsto$ Classification of irreducible holonomy algebras of torsion free connections.
Berger algebras

Let $g \subset \mathfrak{gl}(n, \mathbb{R})$ be a subalgebra. The $g$ module of formal curvature endomorphisms is defined as

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\[ g \text{ is a Berger algebra } \iff g = \left\langle R(x, y) \middle| R \in \mathcal{K}(g), x, y \in \mathbb{R}^n \right\rangle \]

$T^\nabla = 0$: Ambrose-Singer $\implies \mathfrak{hol}(M, \nabla)$ is a Berger algebra.

Classification of irreducible Berger algebras:

- Berger '55: $g \subset \mathfrak{so}(p, q)$,
- Schwachhöfer/Merkulov '99: $g \subset \mathfrak{gl}(n, \mathbb{R})$.

$\leadsto$ Classification of irreducible holonomy algebras of torsion free connections.
Holonomy and geometric structure I

\[ \otimes^r T_p M \otimes \otimes^s T^*_p M \]

//

\[
\{ F \in \otimes^r T_p M : \\
\quad \text{Hol}_p (M, \nabla) \cdot F = F \}
\]
Holonomy and geometric structure

\[ \bigotimes^r T_p M \otimes \bigotimes^s T^*_p M \]

\[ \{ F \in \bigotimes^r T_p M : \text{Hol}_p(M, \nabla) \cdot F = F \} \cong \{ \varphi \in \Gamma(\bigotimes^r T M) : \nabla \varphi = 0 \} \]
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//

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\left\{ F \in \otimes^r T_p M : \right. \\
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\]

$F \mapsto \varphi := \mathcal{P}_\gamma(F)$

independent of $\gamma$ with $\gamma(0) = p$
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independent of \( \gamma \) with \( \gamma(0) = p \)

- \( \text{Hol}_p(M, \nabla) \subset \text{SL}(n, \mathbb{R}) \iff \omega \in \Omega^n M : \nabla \omega = 0. \)
Holonomy and geometric structure

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\[
\begin{aligned}
\{ F \in \otimes^r T_p M : \\
\text{Hol}_p(M, \nabla) \cdot F = F \}
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\]

\[
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\]

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independent of \( \gamma \) with \( \gamma(0) = p \)

- \( \text{Hol}_p(M, \nabla) \subset \text{Sl}(n, \mathbb{R}) \iff \omega \in \Omega^n M : \nabla \omega = 0 \).
- \( \text{Hol}_p(M^{2k}, \nabla) \subset \text{Gl}(k, \mathbb{C}) \iff J \in \text{End}(TM) \) with \( J^2 = -\text{id} : \nabla J = 0 \).
Holonomy and geometric structure I

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\[ \text{//} \]

\[ \begin{cases} F \in \otimes^r T_p M : & \\text{Hol}_p(M, \nabla) \cdot F = F \\ \text{Hol}_p(M, \nabla) \cdot F = F \end{cases} \]

\[ \cong \]

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Independent of \( \gamma \) with \( \gamma(0) = p \)

- \( \text{Hol}_p(M, \nabla) \subset \text{Sl}(n, \mathbb{R}) \Leftrightarrow \omega \in \Omega^n M : \nabla \omega = 0. \)
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- \( \text{Hol}_p(M, \nabla) \subset \text{O}(p, q) \Leftrightarrow \text{metric } g \in \Gamma(\otimes^2 TM) : \nabla g = 0. \)
Holonomy and geometric structure

\[ \otimes^r T_p M \otimes \otimes^s T^*_p M \]

\[ \{ F \in \otimes^r_s T_p M \mid Hol_p(M, \nabla) \cdot F = F \} \cong \{ \varphi \in \Gamma(\otimes^r_s TM) \mid \nabla \varphi = 0 \} \]

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independent of \( \gamma \) with \( \gamma(0) = p \)

- \( Hol_p(M, \nabla) \subset Sl(n, \mathbb{R}) \iff \omega \in \Omega^n M : \nabla \omega = 0. \)
- \( Hol_p(M^{2k}, \nabla) \subset Gl(k, \mathbb{C}) \iff J \in \text{End}(TM) \text{ with } J^2 = -id : \nabla J = 0. \)
- \( Hol_p(M, \nabla) \subset O(p, q) \iff \text{metric } g \in \Gamma(\otimes^2 TM) : \nabla g = 0. \)

Assume also \( T^\nabla = 0 \), then \( \nabla = \nabla^g \) Levi-Civita connection and set

\[ Hol_p(M, g) := Hol_p(M, \nabla^g) \]
Geometric structure II

\[
\left\{ V \subset T_pM : \right. \\
\left. Hol_p(M, \nabla) \cdot V \subset V \right\}
\]
Geometric structure II

\[ \left\{ V \subset T_pM : Hol_p(M, \nabla) \cdot V \subset V \right\} \cong \left\{ \text{distribution } \mathcal{V} \subset TM \right\} \]

\[ \mathcal{V} \mapsto \mathcal{V} := \mathcal{P}_\gamma(\mathcal{V}) \]
Geometric structure II

\[
\left\{ V \subset T_pM : \begin{array}{c} \text{hol} \times \text{V} \subset V \\ \text{Hol}_p(M, \nabla) \times \text{V} \subset V \end{array} \right\} \cong \left\{ \text{distribution} \ V \subset TM \right\} \\
V \mapsto V \mapsto V := \mathcal{P}_\gamma(V)
\]

\[\mathcal{P}_\gamma(V) \subset V \iff \nabla_X : V \to V, \text{ in particular } V \text{ is integrable.}\]
Geometric structure II

\[
\left\{ \begin{array}{l}
V \subset T_p M : \\
Hol_p(M, \nabla) \cdot V \subset V
\end{array} \right\} \cong \left\{ \begin{array}{l}
distribution V \subset TM \\
\mathcal{P}_\gamma(V) \subset V
\end{array} \right\}
\]

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V \mapsto V := \mathcal{P}_\gamma(V)
\]

\(\mathcal{P}_\gamma(V) \subset V \iff \nabla_X : V \to V,\) in particular \(V\) is integrable.

\(\leadsto\) Decomposition of a semi-Riemannian manifold \((M, g)\):

If \(V \subset T_p M\) hol-invariant, non-degenerate,
i.e. \(T_p M = V \oplus V^\perp\) hol-invariant, then

\[
(M, g) \overset{locally}{\cong} (N, h) \times (N^\perp, h^\perp)
\]

with \(V^{(\perp)} \cong T_p N^{(\perp)}\) as \(Hol_p(M, g)\)-module.
De Rham/Wu decomposition

Complete decomposition of $T_pM$ into $Hol_p(M, g)$–modules:

$$T_pM = \bigoplus_{i=0}^{k} V_k,$$

with $V_0$ trivial and $V_i$ indecomposable for $i > 0$
De Rham/Wu decomposition

Complete decomposition of $T_p M$ into $Hol_p(M, g)$–modules:

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non-degenerate and only degenerate invariant subspaces
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Then

$$(M, g) \cong (M_1, g_1) \times \ldots \times (M_k, g_k)$$
De Rham/Wu decomposition

Complete decomposition of $T_p M$ into $Hol_p(M, g)$–modules:

$$T_p M = \bigoplus_{i=0}^{k} V_k,$$

with $V_0$ trivial and $V_i$ \underline{indecomposable} for $i > 0$

non-degenerate and only degenerate invariant subspaces

Theorem (de Rham ’52, Wu ’64)

Let $(M, g)$ be semi-Riemannian, complete and 1-connected.

Then there is a $k > 0$: $(M, g) \overset{\text{globally}}{\cong} (M_1, g_1) \times \ldots \times (M_k, g_k)$
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- $(M_i, g_i)$ complete and 1-connected,
- $(M_i, g_i)$ flat or with indecomposable holonomy representation,
- $\text{Hol}_p(M, g) \simeq \text{Hol}_{p_1}(M_1, g_1) \times \ldots \times \text{Hol}_{p_k}(M_k, g_k)$. 
Holonomy of Riemannian manifolds \((M, g)\)

Positive definite metric \(\implies\) indecomposable = irreducible
\(\implies\) \(\text{Hol}_p(M, g) \simeq\) product of irreducible holonomy groups.
Holonomy of Riemannian manifolds \((M, g)\)

Positive definite metric \(\implies\) indecomposable = irreducible
\(\implies\) \(\text{Hol}_p(M, g) \cong\) product of irreducible holonomy groups.

**Berger’s list (’55)**

Let \((M, g)\) be 1-connected, irreducible, non locally symmetric. Then

\[
\text{Hol}_p(M, g) \cong \begin{array}{c}
\text{SO}(n) \\
\text{U}(\frac{n}{2}) \\
\text{SU}(\frac{n}{2}) \\
\text{Sp}(\frac{n}{4}) \\
\text{Sp}(1) \cdot \text{Sp}(\frac{n}{4}) \\
G_2 \\
\text{Spin}(7)
\end{array}
\]
Holonomy of Riemannian manifolds \((M, g)\)

Positive definite metric \(\implies\) indecomposable \(=\) irreducible
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Thomas Leistner  (HU Berlin)  Holonomy groups of Lorentzian manifolds  Santiago de Compostela  10 / 30
Holonomy of Riemannian manifolds \((M, g)\)

Positive definite metric \(\implies\) indecomposable = irreducible
\(\implies\) \(\text{Hol}_p(M, g) \simeq\) product of irreducible holonomy groups.

Berger’s list (’55)
Let \((M, g)\) be 1-connected, irreducible, non locally symmetric. Then
\(\text{Hol}_p(M, g) \overset{O(n)}{\sim}\)

\[
\begin{array}{c|c|c|c|c|c|c}
 & SO(n) & U\left(\frac{n}{2}\right) & SU\left(\frac{n}{2}\right) & Sp\left(\frac{n}{4}\right) & Sp(1) \cdot Sp\left(\frac{n}{4}\right) & G_2 & \text{Spin}(7) \\
generic & \text{Kähler} & \text{hyper Kähler} & \text{quat. Kähler} & & & & \\
par. field & --- & J & J_1, J_2, J_3 & \langle J_1, J_2, J_3 \rangle & \omega^3 & \omega^4 \\
Ric & --- & \neq 0 & 0 & 0 & c \cdot g & 0 & 0 \\
\end{array}
\]
Holonomy of Riemannian manifolds \((M, g)\)

Positive definite metric \(\implies\) indecomposable = irreducible
\(\implies\) \(\text{Hol}_p(M, g) \cong\) product of irreducible holonomy groups.

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<th>(SO(n))</th>
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<td>(\uparrow) par. spinor</td>
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Wu–Decomposition for a Lorentz manifold \((M, g)\)

Let \((M, g)\) be a complete, 1-connected Lorentzian manifold.

\[
(M, g) \cong (\overline{M, \overline{g}}) \times (N_1, g_1) \times \ldots \times (N_k, g_k)
\]

which is either

1. Riemannian, irreducible or flat

2. Lorentzian manifold which is either

   - irreducible, i.e. \(\text{Hol}_p(M, g) = \text{SO}_0(1, n)\)

   - indecomposable, non-irreducible

   Classify holonomy for these!
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Classify holonomy for these!
We have to consider $H \subset SO_0(1, n + 1)$ indecomposable, non-irreducible, i.e. $\exists V \subset \mathbb{R}^{n+2} : H \cdot V \subset V$ such that
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i.e. $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L = \left\{ \begin{pmatrix} a & v^t & 0 \\ 0 & A & -v \\ 0 & 0^t & -a \end{pmatrix} \middle| \begin{array}{l} a \in \mathbb{R}, \\ v \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\} $
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The orthogonal part is reductive:

\[ \mathfrak{g} := \text{pr}_{\mathfrak{so}(n)} \mathfrak{h} = \begin{cases} \mathfrak{z} \oplus \mathfrak{g}' & (\text{Levi – decomposition}) \\ \text{centre} \end{cases} = [\mathfrak{g}, \mathfrak{g}] \text{ semisimple} \]
Classification I: $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L$ indecomposable

Theorem (Berard-Bergery/Ikemakhen '96)

For $\mathfrak{h}$ there are the following cases:

- **Type I:** $\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \rtimes \mathbb{R}^n$.
- **Type II:** $\mathfrak{h} = \mathfrak{g} \rtimes \mathbb{R}^n$.
- **Type III:** $\exists \phi: \mathfrak{z} \twoheadrightarrow \mathbb{R}: \mathfrak{h} = \begin{cases} \begin{bmatrix} \phi(A) & v t_0 & 0 \\ 0 & 0 & 0 \\ 0 & v 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\phi(A) & 0 \\ A + B - v & 0 & 0 \end{bmatrix} & | \begin{bmatrix} A \in \mathfrak{z} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^n \end{bmatrix} \end{cases}$
- **Type IV:** $\exists \phi: \mathfrak{z} \twoheadrightarrow \mathbb{R}^k$, for $0 < k < n$:
  $\mathfrak{h} = \begin{cases} \begin{bmatrix} 0 \psi(A) & vt & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A + B - v & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & = \psi(A) & 0 \\ 0 & v 0 & 0 \end{bmatrix} & | \begin{bmatrix} A \in \mathfrak{z} \\ B \in \mathfrak{g}' \\ v \in \mathbb{R}^n \end{bmatrix} \end{cases}$
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Theorem (Berard-Bergery/Ikemakhen '96)

For $\mathfrak{h}$ there are the following cases:

$\mathbb{R}^n \subset \mathfrak{h}$ –

$\mathbb{R}^n \not\subset \mathfrak{h}$ –
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For \( \mathfrak{h} \) there are the following cases:

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\( \mathbb{R}^n \not\subset \mathfrak{h} \) –
Lorentzian holonomy

Classification

Classification I: $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L$ indecomposable

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Type II: \[ \mathfrak{h} = g \ltimes \mathbb{R}^n. \]

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*Type III:* $\exists \varphi : \mathfrak{z} \to \mathbb{R}$.

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Classification I: \( \mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L \) indecomposable

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  \[ \mathfrak{h} = \begin{cases} \begin{pmatrix} 0 & \psi(A)^t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A + B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & A \in \mathfrak{z} \\ B \in \mathfrak{g}' \end{cases} \quad \psi(A) \end{cases} \quad \varphi \in \mathbb{R} \]

Note: Groups of uncoupled type III and IV can be non-closed, first examples in Berard-Bergery/Ikemakhen '96.
Classification I: $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L$ indecomposable

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\[ \mathbb{R}^n \not\subset \mathfrak{h} \]

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Classification II: $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L$ indecomposable
Classification II: \( \mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L \) indecomposable

**Theorem ( — ’03)**

If \( \mathfrak{h} \) is a Berger algebra (e.g. a Lorentzian holonomy algebra), then 
\( g := \text{proj}_{\mathfrak{so}(n)} \mathfrak{h} \) is a Riemannian holonomy algebra.
Classification II: $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L$ indecomposable

Theorem (— ’03)

If $\mathfrak{h}$ is a Berger algebra (e.g. a Lorentzian holonomy algebra), then

$$\mathfrak{g} := \text{proj}_{\mathfrak{so}(n)} \mathfrak{h}$$

is a Riemannian holonomy algebra.

Theorem (B-B/I ’96, Boubel ’00, — ’03, Galaev ’05)

If $\mathfrak{g} := \text{proj}_{\mathfrak{so}(n)} \mathfrak{h}$ is a Riemannian holonomy algebra, then there is a Lorentzian metric $h$ with $\text{hol}_p(h) = \mathfrak{h}$.
Proof of the first Theorem — problem

Let $\mathfrak{h} \subset \mathfrak{so}(1, n + 1)_L$ be a Berger algebra, $\mathfrak{g} := \text{proj}_{\mathfrak{so}(n)}(\mathfrak{h})$.

Problem: $\mathfrak{g}$ has “nice” algebraic properties (reductive, acts completely reducible) but is no Berger algebra, apriori.
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- **Idea:** Find algebraic restrictions on $\mathfrak{g}$ based on Bianchi identity, replacing the Berger condition.
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- **Idea:** Find algebraic restrictions on \( \mathfrak{g} \) based on Bianchi identity, replacing the Berger condition.

Let \( L = \mathbb{R} \cdot X \) be the invariant line, \( Z \in T_p M \) transversal to \( X^\perp \). Then

\[
T_p M = X^\perp \oplus \mathbb{R} \cdot Z = \mathbb{R} \cdot X \oplus X^\perp \cap Z^\perp \oplus \mathbb{R} \cdot Z, \]

\( E \) is non degenerate and \( \mathfrak{g} \subset \mathfrak{so}(E, g_p) = \mathfrak{so}(n) \) is reductive and completely reducible, and generated by two types of curvature endomorphisms

\[
R|_E \in \mathcal{K}(\mathfrak{g}) \quad \text{but also} \quad R(Z, .)|_E \in \text{Hom}(E, \mathfrak{g}) \quad \text{for} \ R \in \mathcal{K}(\mathfrak{h})
\]

\( \checkmark \quad \sim \) weak Berger algebras
For \( g \subset \mathfrak{so}(n) \) define weak curvature endomorphisms:

\[
\mathcal{B}(g) := \left\{ Q \in \text{Hom}(\mathbb{R}^n, g) \mid \langle Q(x)y, z \rangle + \langle Q(y)z, x \rangle + \langle Q(z)x, y \rangle = 0 \right\}.
\]
Proof of the first Theorem — weak Berger algebras

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\]

\( g \) is a weak Berger algebra \( \iff g = \left\langle Q(x) \mid Q \in \mathcal{B}(g), x \in \mathbb{R}^n \right\rangle \)
Proof of the first Theorem — weak Berger algebras I

For $g \subset \mathfrak{so}(n)$ define weak curvature endomorphisms:

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$$

$g$ is a weak Berger algebra $\iff$ $g = \langle Q(x) | Q \in \mathcal{B}(g), x \in \mathbb{R}^n \rangle$

Note: Berger $\implies$ weak Berger.
Proof of the first Theorem — weak Berger algebras I

For $g \subset \mathfrak{so}(n)$ define weak curvature endomorphisms:

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$g$ is a weak Berger algebra $\iff g = \langle Q(x) \mid Q \in \mathcal{B}(g), x \in \mathbb{R}^n \rangle$

Note: Berger $\implies$ weak Berger.

Theorem (— ’02)

If $\mathfrak{h} \subset \mathfrak{so}(n)(1, n + 1)_L$ is an indecomposable Berger algebra, then $g := \text{proj}_{\mathfrak{so}(n)}(\mathfrak{h})$ is a weak-Berger algebra.
Weak Berger algebras II

Decomposition Property for (weak) Berger algebras

Let $g \subset \mathfrak{so}(n)$ be a (weak) Berger algebra, and $\mathbb{R}^n$ decomposed as follows:

$$\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_k, \quad E_0 \text{ trivial, } E_i \text{ irreducible.}$$

Then $g = g_1 \oplus \ldots \oplus g_k$, $g_i$ ideals, such that $g_i$ acts irreducibly on $E_i$ and trivial on $E_j$, and is a (weak) Berger algebra.
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Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, $\mathfrak{g}_i$ ideals, such that $\mathfrak{g}_i$ acts irreducibly on $E_i$ and trivial on $E_j$, and is a (weak) Berger algebra.

$\implies$ in order to classify $\mathfrak{g} = \text{pr}_{\mathfrak{so}(n)}^\mathfrak{hol}(M, h)$ we need to classify irreducible weak Berger algebras.
Let $g \subset \mathfrak{so}(n)$ be a (weak) Berger algebra, and $\mathbb{R}^n$ decomposed as follows:

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where $E_0$ is trivial, and $E_i$ are irreducible.

Then $g = g_1 \oplus \ldots \oplus g_k$, where $g_i$ are ideals, such that $g_i$ acts irreducibly on $E_i$ and trivial on $E_j$, and is a (weak) Berger algebra.

\[
\implies \quad \text{in order to classify } g = \text{pr}_{\mathfrak{so}(n)} \mathfrak{hol}(M, h) \text{ we need to classify irreducible weak Berger algebras.}
\]

**Method:** Representation theory for (complex) semisimple Lie algebras.
Weak Berger algebras II

Decomposition Property for (weak) Berger algebras

Let $g \subset \mathfrak{so}(n)$ be a (weak) Berger algebra, and $\mathbb{R}^n$ decomposed as follows:

$$\mathbb{R}^n = E_0 \oplus E_1 \oplus \ldots \oplus E_k, \quad E_0 \text{ trivial, } E_i \text{ irreducible.}$$

Then

$$g = g_1 \oplus \ldots \oplus g_k, \quad g_i \text{ ideals, such that }$$

$g_i$ acts irreducibly on $E_i$ and trivial on $E_j$, and is a (weak) Berger algebra.

Corollary

Lorentzian holonomy groups of uncoupled type I and II are closed.

$\implies$ in order to classify $g = \text{pr}_{\mathfrak{so}(n)} \mathfrak{hol}(M, h)$ we need to classify irreducible weak Berger algebras.

Method: Representation theory for (complex) semisimple Lie algebras.
Parallel spinors on a Lorentzian spin manifold \((M, g)\)

Let \((\Sigma, \nabla^\Sigma)\) be the spinor bundle over \((M, g)\).

Assume: \(\exists \varphi \in \Gamma(\Sigma)\) with \(\nabla^\Sigma \varphi = 0\) a parallel spinor field.
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\[\implies \exists\ \text{causal vector field } X_\varphi \in \Gamma(TM) : \nabla X_\varphi = 0.\]
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\[\implies \exists \text{ causal vector field } X_\varphi \in \Gamma(TM): \nabla X_\varphi = 0.\] Two cases:

- \(g(X_\varphi, X_\varphi) < 0\) : \((M, g) = (\mathbb{R}, -dt^2)\) Riemannian mf.
- \(g(X_\varphi, X_\varphi) = 0\) : \((M, g) = (\overline{M}, \overline{g})\) with parallel spinor indecomposable with parallel spinor
Parallel spinors on a Lorentzian spin manifold \((M, g)\)

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\(\implies\) \(\exists\) causal vector field \(X_\varphi \in \Gamma(TM) : \nabla X_\varphi = 0\). Two cases:

\[
\begin{align*}
g(X_\varphi, X_\varphi) &< 0 & : & (M, g) & = & (\mathbb{R}, -dt^2) & \text{Riemannian mf.} \\
g(X_\varphi, X_\varphi) &= 0 & : & (M, g) & = & (\overline{M}, \overline{g}) & \uparrow \text{indecomposable with parallel spinor}
\end{align*}
\]

Theorem (\(\,\!-\,\!03\) )

\((M^{n+2}, g)\) indecomposable Lorentzian spin with parallel spinor. Then \(\text{Hol}_p(M, g) = G \ltimes \mathbb{R}^n\) where \(G\) is a product of the following groups:

\[
\{1\}, \quad SU(p), \quad Sp(q), \quad G_2, \quad \text{Spin}(7)
\]
Parallel spinors on a Lorentzian spin manifold \((M, g)\)

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\[ \Rightarrow \exists \text{ causal vector field } X_\varphi \in \Gamma(TM) : \nabla X_\varphi = 0. \]

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\[ g(X_\varphi, X_\varphi) < 0 : (M, g) = (\mathbb{R}, -dt^2) \]

\[ g(X_\varphi, X_\varphi) = 0 : (M, g) = (\bar{M}, \bar{g}) \]

\[ \text{indecomposable with parallel spinor} \]

Theorem (— ’03)

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\[ \{1\}, \quad SU(p), \quad Sp(q), \quad G_2, \quad Spin(7) \]

\[ \text{dim}\{\nabla \varphi = 0\} : \quad 2^{[k/2]} \quad 2 \quad q + 1 \quad 1 \quad 1 \]
Parallel spinors on a Lorentzian spin manifold \((M, g)\)

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Two cases:

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This generalizes the result for \(n \leq 9\) in [Bryant ’99].
Lorentzian Einstein manifolds

Theorem (Galaev—’06)

The holonomy of an indecomposable non-irreducible Lorentzian Einstein manifold is uncoupled, i.e.

\[ \text{Hol}^0_p(M, g) = \begin{cases} (\mathbb{R}^+ \times G) \rtimes \mathbb{R}^n, & \text{or} \\ G \rtimes \mathbb{R}^n \end{cases} \]

with a Riemannian holonomy group G. In the 2nd case the manifold is Ricci flat.
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In the second case $G$ is a product of $\{1\}, SU(p), Sp(q), G_2, Spin(7)$, or the holonomy of a non-Kählerian Riemannian symmetric space.
Lorentzian Einstein manifolds

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with a Riemannian holonomy group G. In the 2\textsuperscript{nd} case the manifold is Ricci flat.

In the second case G is a product of \{1\}, SU(p), Sp(q), G\textsubscript{2}, Spin(7), or the holonomy of a non-Kählerian Riemannian symmetric space.

Corollary

A Lorentzian Einstein manifold with parallel light-like vector field is Ricci-flat.
The uncoupled types $G \ltimes \mathbb{R}^n$ and $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$

Construction method for the uncoupled types

Let $(N^n, g)$ be a Riemannian manifold, $f \in C^\infty(\mathbb{R} \times N)$ sufficiently generic, and $\varphi \in C^\infty(\mathbb{R})$. Then $M = \mathbb{R} \times N \times \mathbb{R}$ with the Lorentzian metric

$$h(x,p,z) = 2dxdz + f(x, z)dz^2 + e^{2\varphi(z)} g_p$$

is indecomposable, non irreducible with

$$\text{Hol}_{(x,p,z)}(M, h) = \begin{cases} 
\text{Hol}_p(N, g) \ltimes \mathbb{R}^n, & \text{if } \frac{\partial f}{\partial x} = 0, \\
(\mathbb{R}^+ \times \text{Hol}_p(N, g)) \ltimes \mathbb{R}^n, & \text{otherwise.}
\end{cases}$$
Applications and Examples

Metrics realising all possible groups

The uncoupled types $G \ltimes \mathbb{R}^n$ and $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$

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Metrics for the coupled types:

- First examples in Berard-Bergery/Ikemakhen ’96
The uncoupled types $G \ltimes \mathbb{R}^n$ and $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$

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\end{cases}$$

Metrics for the coupled types:

- First examples in *Berard-Bergery/Ikemakhen ’96*
- Systematic study in *Boubel ’00*
- Complete solution in *Galaev ’05*
Coupled types — Proof of Theorem [Galaev ’05]

For a Riemannian holonomy algebra \( \mathfrak{g} \), fix \( Q_1, \ldots, Q_N \), a basis of \( \mathcal{B}(\mathfrak{g}) \), and define polynomials on \( \mathbb{R}^{n+1} \):

\[
    u_i(y_1, \ldots, y_n, z) := \sum_{A=1}^{N} \sum_{k,l=1}^{n} \frac{1}{3(A-1)!} \left\langle Q_A(e_k)e_l, e_i \right\rangle y_k y_l z^A.
\]
Coupled types — Proof of Theorem [Galaev ’05]

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Theorem (Galaev ’05)

For any indecomposable $\mathfrak{h} \subset \mathfrak{so}(1, n+1)_L$, for which $\mathfrak{g} = \text{proj}_{\mathfrak{so}(n)}(\mathfrak{h})$ is a Riemannian holonomy algebra, exists an analytic $f \in C^\infty(\mathbb{R}^{n+2})$ such that the following Lorentzian metric has holonomy $\mathfrak{h}$:

$$h = 2dxdz + f dz^2 + 2 \sum_{i=1}^{n} u_i dy_i \ dz + \sum_{k=1}^{n} dy_k^2.$$
Coupled types — Proof of Theorem [Galaev ’05]

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\[
h = 2dxdz + f dz^2 + 2 \sum_{i=1}^{n} u_i dy_i \quad dz + \sum_{k=1}^{n} dy_k^2,
\]

**family of 1-forms on \( \mathbb{R}^n \) flat metric**
Example: Coupled type III

If $\mathfrak{h}$ is of type III, such that $\mathfrak{g}$ acts trivial on $\mathbb{R}^{n_0}$, $n_0 < n - 2$, and defined by

$$\varphi : \mathfrak{h} \to \mathbb{R} \text{ set}$$

$$\varphi_{Ai} := \frac{1}{(A - 1)!} \widetilde{\varphi}(\text{proj}_\mathfrak{h}(Q_A(e_i))),$$

for $A = 1, \ldots, N$ and $i = n_0 + 1, \ldots, n$.

Then $f$ can be given by

$$f(x, y_1, \ldots, y_n, z) = 2x \sum_{A=1}^{N} \sum_{i=n_0+1}^{n} \varphi_{Ai} y_i z^{A-1} + \sum_{k=1}^{n_0} y_k^2.$$
Let \((M, g)\) be a Lorentzian manifold with \(\text{Hol}_p(M, g) \subset SO_0(1, n + 1)_L\). This corresponds to filtrations

\[
L \subset L^\perp \subset T_pM
\]

into holonomy invariant subspaces locally \(\leftrightarrow\) recurrent light-like vector field \(X\), i.e.

\[
\nabla X = \theta \otimes X
\]

with 1-form \(\theta\) \(\leftrightarrow\) foliation into totally geodesic light-like hypersurfaces.

If \(\text{Hol}_p^0(M, g) \subset SO(n) \rtimes \mathbb{R}^n\), i.e. \(L\) is spanned by an invariant vector, then the recurrent vector field is parallel.

Definition

A Lorentzian manifold with parallel light-like vector field is called Brinkmann wave.
Parallel distributions

Let \((M, g)\) be a Lorentzian manifolds with \(\text{Hol}_p(M, g) \subset SO_0(1, n + 1)_L\). This corresponds to filtrations

\[
\begin{align*}
L & \subset L^\perp \subset T_pM & \text{into holonomy invariant subspaces} \\
\mathcal{L} & \subset \mathcal{L}^\perp \subset TM & \text{into parallel distributions, } \mathcal{L}_p^{(\perp)} = L^{(\perp)}
\end{align*}
\]
Parallel distributions

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\[
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\[
\mathcal{L} \subset \mathcal{L}^\perp \subset TM \quad \text{into parallel distributions, } \mathcal{L}_p^{(\perp)} = L^{(\perp)}
\]

\(\mathcal{L}\) \(\xleftrightarrow{\text{locally}}\) recurrent light-like vector field \(X\), i.e. \(\nabla X = \theta \otimes X\) with 1-form \(\theta\)
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into holonomy invariant subspaces

\[ \mathcal{L} \subset \mathcal{L}^\perp \subset TM \]

into parallel distributions, \(\mathcal{L}^p(\perp) = L(\perp)\)

\(\mathcal{L}\) locally \(\longleftrightarrow\) recurrent light-like vector field \(X\), i.e. \(\nabla X = \theta \otimes X\) with 1-form \(\theta\)

\(\mathcal{L}^\perp\) \(\longleftrightarrow\) foliation into totally geodesic light-like hypersurfaces
Parallel distributions

Let \((M, g)\) be a Lorentzian manifolds with \(\text{Hol}_p(M, g) \subset SO_0(1, n + 1)_L\). This corresponds to filtrations

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\mathcal{L} \subset \mathcal{L}^\perp \subset TM \quad \text{into parallel distributions, } \mathcal{L}_p^{(\perp)} = L^{(\perp)}
\]

\(\mathcal{L} \xrightarrow{\text{locally}} \text{recurrent} \) light-like vector field \(X\), i.e. \(\nabla X = \theta \otimes X\) with 1-form \(\theta\)

\(\mathcal{L}^\perp \xrightarrow{\text{foliation}} \text{foliation into totally geodesic light-like hypersurfaces}\)

If \(\text{Hol}_p^0(M, g) \subset SO(n) \ltimes \mathbb{R}^n\), i.e. \(L\) is spanned by an invariant vector, then the recurrent vector field is parallel.
Applications and Examples

Geometric structures

Parallel distributions

Let \((M, g)\) be a Lorentzian manifolds with \(\text{Hol}_p(M, g) \subset SO_0(1, n + 1)_L\).
This corresponds to filtrations
\[
L \subset L^\perp \subset T_pM \quad \text{into holonomy invariant subspaces}
\]
\[
\mathcal{L} \subset \mathcal{L}^\perp \subset TM \quad \text{into parallel distributions, } \mathcal{L}_p^{(\perp)} = L^{(\perp)}
\]

\(\mathcal{L}\) \(\text{locally} \leftrightarrow \) \text{recurrent light-like vector field } X, \text{ i.e. } \nabla X = \theta \otimes X \text{ with 1-form } \theta

\(\mathcal{L}^\perp \leftrightarrow \) \text{foliation into totally geodesic light-like hypersurfaces}

If \(\text{Hol}^0_p(M, g) \subset SO(n) \ltimes \mathbb{R}^n\), i.e. \(L\) is spanned by an invariant vector, then the recurrent vector field is parallel.

Definition

A Lorentzian manifold with parallel light-like vector field is called \text{Brinkmann wave}.
The screen bundle

Definition

Let \((M, g)\) be an Lorentzian manifold with parallel light like line distribution \(\mathcal{L}\). The vector bundle

\[
\mathcal{S} = \bigcup_{p \in M} \mathcal{L}^\perp_p / \mathcal{L}_p, \quad g^S([U], [V]) := g(U, V), \quad \nabla^S_U[V] := [\nabla_U V]
\]

is called screen bundle. \(\text{Hol}_p(\mathcal{S}, \nabla^S)\) is called screen holonomy.
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is called \textit{screen bundle}. \(\text{Hol}_p(\mathcal{S}, \nabla^S)\) is called \textit{screen holonomy}.

\[
\Rightarrow \quad \text{proj}_{SO(n)} \text{Hol}_p(M, g) = \text{Hol}_p(\mathcal{S}, \nabla^S) [\text{''03}].
\]
Definition

Let \((M, g)\) be an Lorentzian manifold with parallel light like line distribution \(\mathcal{L}\). The vector bundle

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\mathcal{S} = \bigcup_{p \in M} \mathcal{L}_p^\perp / \mathcal{L}_p, \quad g^\mathcal{S}([U], [V]) := g(U, V), \quad \nabla^\mathcal{S}_U[V] := [\nabla_U V]
\]

is called **screen bundle**. \(Hol_p(\mathcal{S}, \nabla^\mathcal{S})\) is called **screen holonomy**.

\[
\Rightarrow \quad proj_{SO(n)} Hol_p(M, g) = Hol_p(\mathcal{S}, \nabla^\mathcal{S}) \quad [\text{— '03}].
\]

Geometric structures on \(\mathcal{S}\) correspond to algebraic structures of the screen holonomy, e.g. parallel complex structure etc.
Coordinates for a Lorentzian manifold \((M, h)\) with recurrent light-like vector field \(X\)

**Theorem (Brinkmann’25, Walker’49)**

\[ \exists \text{ coordinates } (x, y_1, \ldots, y_n, z): \frac{\partial}{\partial x} = X, \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\rangle = X^\perp, \text{ and} \]

\[ h = 2 \, dx dz + \sum_{i=1}^{n} u_i dy_i \, dz + fdz^2 + \sum_{i,j=1}^{n} g_{ij} dy_i \, dy_j, \]

with \(\frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0, f \in C^\infty(M), \text{ and } X \text{ parallel} \iff \frac{\partial f}{\partial x} = 0.\]
Coordinates for a Lorentzian manifold \((M, h)\) with recurrent light-like vector field \(X\)

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- \[ h = 2 \, dx \, dz + \sum_{i=1}^{n} u_i \, dy_i \, dz + fdz^2 + \sum_{i,j=1}^{n} g_{ij} \, dy_i \, dy_j, \]
  - \[ = \phi_z \text{ family of 1-forms} \]
  - \[ = g_z \text{ family of Riem. metrics} \]

with \( \frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0 \), \( f \in C^\infty(M) \), and \( X \) parallel \( \iff \frac{\partial f}{\partial x} = 0 \).

- \[ g = 2 \, dx \, dz + \sum_{i,j=1}^{n} g_{ij} \, dy_i \, dy_j, \text{ if } X \text{ is parallel } [\text{Schimming’78}]. \]
Coordinates for a Lorentzian manifold \((M, h)\) with recurrent light-like vector field \(X\)

**Theorem (Brinkmann’25, Walker’49)**

\[ \exists \text{ coordinates } (x, y_1, \ldots, y_n, z): \frac{\partial}{\partial x} = X, \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right) = X^\perp, \text{ and } \]

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with \( \frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0, f \in C^\infty(M), \) and \(X\) parallel \(\iff\) \(\frac{\partial f}{\partial x} = 0.\)

\[ g = 2 \, dx \, dz + \sum_{i,j=1}^{n} g_{ij} \, dy_i \, dy_j, \text{ if } X \text{ is parallel } [\text{Schimming’78}]. \]

\[ \implies \text{Hol}_p(g_z) \subset \text{pr}_{SO(n)}\text{Hol}_p(h), \text{ but in general } \neq \text{ (see Galaev’s examples).} \]
**pp-waves**

**Definition**

A Brinkmann wave is a **pp-wave** if

\[ \operatorname{tr}_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0. \]
pp-waves

Definition

A Brinkmann wave is a pp-wave: \[ \iff \text{tr}_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0. \]
\[ \iff h = dx dz + f dz^2 + \sum_{i=1}^{n} dy_i^2 : \frac{\partial f}{\partial x} = 0. \]
Definition

A Brinkmann wave is a pp-wave:

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\[ \iff \quad h = dx dz + f dz^2 + \sum_{i=1}^{n} dy_i^2 : \partial f / \partial x = 0. \]

Theorem (— ’01)

Let \((M^{n+2}, h)\) be an indecomposable Lorentzian manifold. \((M^{n+2}, h)\) has Abelian holonomy \(\mathbb{R}^n \iff \) it is a pp-wave.
pp-waves

Definition
A Brinkmann wave is a \( \text{pp-wave} \) :
\[
\iff \quad \text{tr}_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0.
\]
\[
\iff \quad h = dx dz + f dz^2 + \sum_{i=1}^{n} dy_i^2 : \quad \frac{\partial f}{\partial x} = 0.
\]

Theorem (— ’01)
Let \((M^{n+2}, h)\) be an indecomposable Lorentzian manifold. \((M^{n+2}, h)\) has \textit{Abelian} holonomy \(\mathbb{R}^n\) \iff it is a pp-wave.

Examples
- Symmetric spaces with solvable transvection group (Cahen-Wallach spaces) \iff \( f \) is a quadratic polynomial in the \(y_i\)’s.
**pp-waves**

**Definition**
A Brinkmann wave is a pp-wave: \( \iff \text{tr}_{(3,5)(4,6)}(\mathcal{R} \otimes \mathcal{R}) = 0. \)

\[
\iff h = dx dz + f dz^2 + \sum_{i=1}^{n} dy_i^2 : \frac{\partial f}{\partial x} = 0.
\]

**Theorem (— ’01)**

Let \((M^{n+2}, h)\) be an indecomposable Lorentzian manifold. \((M^{n+2}, h)\) has Abelian holonomy \(\mathbb{R}^n\) \(\iff\) it is a pp-wave.

**Examples**
- Symmetric spaces with solvable transvection group (Cahen-Wallach spaces) \(\iff\) \(f\) is a quadratic polynomial in the \(y_i\)'s.
- Plane waves: \(f\) is a quadratic polynomial in the \(y_i\)'s with coefficients depending on \(z\) (Important in supergravity theories. [Figueroa O’Farrill/Papadopoulos ’02])
Better description:

$$\text{pp-wave} \iff \begin{cases} 
(P) & \exists \text{parallel light-like vector field, and} \\
(1) & R(U, V) : X^\perp \rightarrow \mathbb{R} \cdot X \quad \forall \ U, V \in TM 
\end{cases}$$
Generalisations

Better description:

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\text{pp-wave} \iff \begin{cases} 
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Generalise (P) and (1):

\[
(R) \, \exists \text{ recurrent light-like vector field}
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Obvious consequence (– ’06)

An indecomposable Lorentzian manifold has solvable holonomy \( \mathbb{R}^+ \rtimes \mathbb{R}^n \) \iff (R) but not (P), and (1).

This means: (R) and (1) \iff trivial screen holonomy.
Manifolds with light-like hypersurface curvature I

Definition

A Lorentzian mf. has light-like hypersurface curvature $\iff$ (R) and (2).

Remark

If (P), in Schimming coordinates ($h = 2dxdz + g_z$) the $g_z$ is a $z$-dependent family of flat Riemannian metrics. All of Galaev's examples have light-like hypersurface curvature, i.e. all possible holonomy groups can be realised by such metrics.
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$\iff \exists$ coordinates $(x, y_1, \ldots, y_n, z)$:

$$h = 2 \, dx \, dz + f \, dz^2 + \sum_{i=1}^{n} u_i \, dy_i \, dz + \sum_{i=1}^{n} dy_i^2,$$

$$\underbrace{\partial u_i}_{\phi_z} = 0, \ f \in C^\infty(M).$$
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**Remark**

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Manifolds with light-like hypersurface curvature II

$(M, h)$ has light-like hypersurface curvature

$\iff$ The curvature of the light-like hypersurfaces defined by $\mathcal{L}^\perp$ has a light-like image.
Manifolds with light-like hypersurface curvature II

\((M, h)\) has light-like hypersurface curvature

\[\iff\] The curvature of the light-like hypersurfaces defined by \(L^\perp\) has a light-like image.

\[\iff\] The screen bundle \(S\) restricted to these hypersurfaces is flat.
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Proposition (— ’06)

A Brinkmann wave has light-like hypersurface curvature $\iff \|R\|^2 = 0$. 
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Proposition (— ’06)

A Brinkmann wave has light-like hypersurface curvature \( \iff \|\mathcal{R}\|^2 = 0. \)

Further properties for \( h = 2dxdz + fdz^2 + \phi_z + \sum_{i=1}^{n} dy_i^2 \):

1. \( h \) has trivial screen holonomy \( \iff d\phi_z = 0 \ \forall z. \)
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Proposition (— ’06)

A Brinkmann wave has light-like hypersurface curvature \(\iff\) \(||R||^2 = 0\).

Further properties for \(h = 2dxdz + fdz^2 + \phi_z + \sum_{i=1}^{n} dy_i^2\):

1. \(h\) has trivial screen holonomy \(\iff\) \(d\phi_z = 0 \ \forall z\).
2. \(h\) is Ricci isotropic \(\iff\) \(d^*d\phi_z = 0 \ \forall z\), and Ricci flat if in addition \(\Delta f = 0\).
Open Problems

**Special geometries** = not products but do not have full holonomy.
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Special geometries $\Rightarrow$ not products but do not have full holonomy.

Riemannian $\leadsto$ irreducible manifolds $\leadsto$ Berger list and subsequent results [Alekseevski, Bryant, Salomon, Joyce, ...]
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Riemannian \leadsto \text{irreducible manifolds} \leadsto \text{Berger list and subsequent results} [Alekseevski, Bryant, Salomon, Joyce, ...]

Lorentzian (irreducible \implies \text{SO}(1,n)) \leadsto \text{indecomposable, non-irreducible manifolds: groups are known, but many questions are open:}

1. Find global examples of metrics with prescribed holonomy, which are globally hyperbolic with complete or compact Cauchy surface (cylinder constructions in [Baum/Müller '06])
2. Describe the geometric structures corresponding to the coupled types III and IV.
3. Describe indecomposable, non-irreducible Lorentzian homogeneous spaces and their holonomy.
4. Find generalisations of Lorentzian symmetric spaces, e.g. screen holonomy is holonomy of Riemannian symmetric space.
5. Study further spinor field equations for these manifolds.
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