Holonomy groups of Lorentzian manifolds

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Outline

Parallel displacement

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$$\begin{array}{cccccc} \mathcal{P}_\gamma & : & T_{\gamma(0)}M & \stackrel{\sim}{\longrightarrow} & T_{\gamma(1)}M \\ & & X_0 & \longmapsto & X(1), \end{array}$$

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where $X(t)$ is the solution to the ODE

$$\nabla_{\dot\gamma(t)} X(t) \equiv 0 \text{ with initial condition } X(0) = X_0.$$

Holonomy group of a linear connection

For $p \in M^n$ we define the Holonomy group

$$Hol_p\,(M, \nabla) \quad := \quad \left\{ \mathcal{P}_\gamma | \gamma(0) = \gamma(1) = p, \qquad \right\}$$
$$\cap$$
$$Gl(T_pM)$$

and its Lie algebra $\mathfrak{hol}_p(M, \nabla)$.

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For $p \in M^n$ we define the (Connected) Holonomy group

$$Hol_p^0(M,\nabla) \quad := \quad \left\{\mathcal{P}_\gamma | \gamma(0) = \gamma(1) = p, \gamma \sim \{p\}\right\} \subset Gl(T_pM)$$

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Example

- ∇ flat $\Rightarrow$ $Hol_p(M,\nabla) = \Pi_1(M)$ and $\mathfrak{hol}_p(M,\nabla) = \{0\}$.

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- S^n the round sphere: $Hol_p(S^n) = SO(n)$.

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Theorem (Ambrose/Singer)

M connected $\Longrightarrow \mathfrak{hol}_p(M, \nabla)$ *is spanned by*

$$\left\{ \underbrace{\mathcal{P}_\gamma^{-1} \circ \mathcal{R}(X, Y) \circ \mathcal{P}_\gamma} \,\middle|\, \gamma(0) = p \text{ and } X, Y \in T_{\gamma(1)} M \right\}$$

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satisifies Bianchi identity if $T^\nabla = 0$
$\Longrightarrow$ *$\mathfrak{hol}_p(M, \nabla)$ is a Berger algebra.*

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- Schwachhöfer/Merkulov '99: $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$.

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Assume also $T^\nabla = 0$, then $\nabla = \nabla^g$ Levi-Civita connnection and set

$$Hol_p(M,g) := Hol_p(M,\nabla^g)$$

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$\rightsquigarrow$ *Decomposition* of a semi-Riemannian manifold (M,g):
If $V \subset T_pM$ hol-invariant, non-degenerate,
i.e. $T_pM = V \oplus V^\perp$ hol-invariant, then

$$(M,g) \overset{locally}{\simeq} (N,h) \times (N^\perp, h^\perp)$$

with $V^{(\perp)} \simeq T_pN^{(\perp)}$ as $Hol_p(M,g)$–module.

De Rham/Wu decomposition

Complete decomposition of T_pM into $Hol_p(M,g)$–modules:

$$T_pM = \oplus_{i=0}^{k} V_k, \text{ with } V_0 \text{ trivial and } V_i \underbrace{\text{indecomposable}} \text{ for } i > 0$$

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Then $(M,g) \simeq (M_1, g_1) \times \ldots \times (M_k, g_k)$

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Theorem (de Rham '52, Wu '64)

Let (M,g) be semi-Riemannian, complete and 1-connected.

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- *(M_i,g_i) flat or with indecomposable holonomy representation,*
- $Hol_p(M,g) \simeq Hol_{p_1}(M_1,g_1) \times \ldots \times Hol_{p_k}(M_k,g_k)$.

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Let (M, g) be 1-connected, irreducible, non locally symmetric. Then

$Hol_p(M, g) \overset{O(n)}{\sim}$

$$\Big| \; SO(n) \; \Big| \; U(\tfrac{n}{2}) \; \Big| \; SU(\tfrac{n}{2}) \; \Big| \quad Sp(\tfrac{n}{4}) \quad \Big| \; Sp(1) \cdot Sp(\tfrac{n}{4}) \; \Big| \; G_2 \; \Big| \; Spin(7)$$

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Ric	—	$\neq 0$	0	0	$c \cdot g$	0	0

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	generic	*Kähler*		*hyper Kähler*	*quat. Kähler*		
par. field	—	J		J_1, J_2, J_3	$\langle J_1, J_2, J_3 \rangle$	ω^3	ω^4
Ric	—	$\neq 0$	0	0	$c \cdot g$	0	0
$dim\{\nabla \varphi = 0\}$ ↑ *par.spinor*	*0*	*0*	*2*	*q+1*	*0*	*1*	*1*

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Classify holonomy for these!

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The orthogonal part is reductive:

$$\mathfrak{g} := pr_{\mathfrak{so}(n)}\mathfrak{h} = \underbrace{\mathfrak{z}}_{centre} \oplus \underbrace{\mathfrak{g}'}_{= [\mathfrak{g},\mathfrak{g}] \text{ semisimple}} \qquad (Levi - decomposition)$$

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Note: Groups of uncoupled type III and IV can be non-closed, first examples in Berard-Bergery/Ikemakhen '96

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Theorem (— '03)

If $\mathfrak{h}$ *is a Berger algebra (e.g. a Lorentzian holonomy algebra), then* $\mathfrak{g} := proj_{\mathfrak{so}(n)}\mathfrak{h}$ *is a Riemannian holonomy algebra.*

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Theorem (B-B/I '96, Boubel '00, — '03, Galaev '05)

If $\mathfrak{g} := proj_{\mathfrak{so}(n)}\mathfrak{h}$ is a Riemannian holonomy algebra, then there is a Lorentzian metric h with $\mathfrak{hol}_p(h) = \mathfrak{h}$.

Proof of the first Theorem — problem

Let $\mathfrak{h} \subset \mathfrak{so}(1, n+1)_L$ be a Berger algebra, $\mathfrak{g} := proj_{\mathfrak{so}(n)}(\mathfrak{h})$.

- Problem: $\mathfrak{g}$ has “nice” algebraic properties (reductive, acts completely reducible) but is no Berger algebra, *apriori*.

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Let $L = \mathbb{R} \cdot X$ be the invariant line, $Z \in T_pM$ transversal to $X^\perp$. Then

$$T_pM \;=\; X^\perp \oplus \mathbb{R} \cdot Z \;=\; \mathbb{R} \cdot X \oplus \underbrace{X^\perp \cap Z^\perp}_{:=E} \oplus \mathbb{R} \cdot Z,$$

E is non degenerate and $\mathfrak{g} \subset \mathfrak{so}(E, g_p) = \mathfrak{so}(n)$ is reductive and completely reducible, and generated by two types of curvature endomorphisms

$$R|_E \in \mathcal{K}(\mathfrak{g}) \quad \text{but also} \quad R(Z, .)|_E \in Hom(E, \mathfrak{g}) \quad \text{for } R \in \mathcal{K}(\mathfrak{h})$$

$\checkmark$ $\quad\rightsquigarrow$ weak Berger algebras

Proof of the first Theorem — weak Berger algebras I

For $\mathfrak{g} \subset \mathfrak{so}(n)$ define weak curvature endomorphisms:

$$\mathcal{B}(\mathfrak{g}) := \left\{ Q \in Hom(\mathbb{R}^n, \mathfrak{g}) \middle| \langle Q(x)y, z\rangle + \langle Q(y)z, x\rangle + \langle Q(z)x, y\rangle = 0 \right\}.$$

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Theorem (— '02)

If $\mathfrak{h} \subset \mathfrak{so}(n)(1, n+1)_L$ is an indecomposable Berger algebra, then $\mathfrak{g} := proj_{\mathfrak{so}(n)}(\mathfrak{h})$ is a weak-Berger algebra.

Weak Berger algebras II

Decomposition Property for (weak) Berger algebras

Let $\mathfrak{g} \subset \mathfrak{so}(n)$ *be a (weak) Berger algebra, and* $\mathbb{R}^n$ *decomposed as follows:*

$$\begin{array}{rcccccccl} & \mathbb{R}^n & = & E_0 & \oplus & E_1 & \oplus \ldots \oplus & E_k, & E_0 \text{ trivial, } E_i \text{ irreducible.} \\ \text{Then} & \mathfrak{g} & = & & & \mathfrak{g}_1 & \oplus \ldots \oplus & \mathfrak{g}_k, & \mathfrak{g}_i \text{ ideals, such that} \end{array}$$

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$\Longrightarrow$ in order to classify $\mathfrak{g} = pr_{\mathfrak{so}(n)}\mathfrak{hol}(M,h)$ we need to classify irreducible weak Berger algebras.

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Corollary

Lorentzian holonomy groups of uncoupled type I and II are closed.

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Let $\left(\Sigma, \nabla^{\Sigma}\right)$ be the spinor bundle over (M, g).

Assume: $\exists \varphi \in \Gamma(\Sigma)$ with $\nabla^{\Sigma}\varphi = 0$ a parallel spinor field.

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Theorem (— '03)

(M^{n+2}, g) indecomposable Lorentzian spin with parallel spinor. Then $Hol_p(M, g) = G \ltimes \mathbb{R}^n$ where G is a product of the following groups:

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Parallel spinors on a Lorentzian spin manifold (M, g)

Let $\left(\Sigma, \nabla^{\Sigma}\right)$ be the spinor bundle over (M, g).

Assume: $\exists \varphi \in \Gamma(\Sigma)$ with $\nabla^{\Sigma}\varphi = 0$ a parallel spinor field.
$\Longrightarrow$ $\exists$ causal vector field $X_{\varphi} \in \Gamma(TM)$: $\nabla X_{\varphi} = 0$. Two cases:

$$\begin{array}{lclcl} g(X_\varphi, X_\varphi) < 0 & : & (M,g) & = & (\mathbb{R}, -dt^2) \\ g(X_\varphi, X_\varphi) = 0 & : & (M,g) & = & (\overline{M}, \overline{g}) \end{array} \times \begin{array}{l}\text{Riemannian mf.}\\ \text{with parallel spinor}\end{array}$$

$\uparrow$ indecomposable with parallel spinor

Theorem (— '03)

(M^{n+2}, g) indecomposable Lorentzian spin with parallel spinor. Then $Hol_p(M, g) = G \ltimes \mathbb{R}^n$ where G is a product of the following groups:

	$\{1\}$,	$SU(p)$,	$Sp(q)$,	G_2,	$Spin(7)$
$dim\{\nabla\varphi = 0\}$:	$2^{[k/2]}$	2	$q+1$	1	1

This generalizes the result for $n \leq 9$ in [Bryant '99].

Lorentzian Einstein manifolds

Theorem (Galaev/— '06)

The holonomy of an indecomposable non-irreducible Lorentzian Einstein manifold is uncoupled, i.e.

$$Hol_p^0(M,g) = \begin{cases} (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n, & \textit{or} \\ G \ltimes \mathbb{R}^n & \end{cases}$$

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Corollary

A Lorentzian Einstein manifold with parallel light-like vector field is Ricci-flat.

The uncoupled types $G \ltimes \mathbb{R}^n$ and $(\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n$

Construction method for the uncoupled types

Let (N^n, g) be a Riemannian manifold, $f \in C^\infty(\mathbb{R} \times N)$ suffiently generic, and $\varphi \in C^\infty(\mathbb{R})$. Then $M = \mathbb{R} \times N \times \mathbb{R}$ with the Lorentzian metric

$$h_{(x,p,z)} = 2dxdz + f(x,z)dz^2 + e^{2\varphi(z)} g_p$$

is indecomposable, non irreducible with

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- Complete solution in Galaev '05 $\curvearrowright$

Coupled types — Proof of Theorem [Galaev '05]

For a Riemannian holonomy algebra $\mathfrak{g}$, fix $Q_1, \ldots, Q_N$, a basis of $\mathcal{B}(\mathfrak{g})$, and define polynomials on $\mathbb{R}^{n+1}$:

$$u_i(y_1, \ldots, y_n, z) \;:=\; \sum_{A=1}^{N} \sum_{k,l=1}^{n} \frac{1}{3(A-1)!} \Big\langle Q_A(e_k) e_l, e_i \Big\rangle y_k y_l z^A .$$

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For any indecomposable $\mathfrak{h} \subset \mathfrak{so}(1, n+1)_L$, for which $\mathfrak{g} = proj_{\mathfrak{so}(n)}(\mathfrak{h})$ is a Riemannian holonomy algebra, exists an analytic $f \in C^\infty(\mathbb{R}^{n+2})$ such that the following Lorentzian metric has holonomy $\mathfrak{h}$:

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Example: Coupled type III

If $\mathfrak{h}$ is of type III, such that $\mathfrak{g}$ acts trivial on $\mathbb{R}^{n_0}$, $n_0 < n-2$, and defined by $\varphi : \mathfrak{z} \twoheadrightarrow \mathbb{R}$ set

$$\varphi_{Ai} \quad := \quad \frac{1}{(A-1)!}\widetilde{\varphi}(proj_{\mathfrak{z}}(Q_A(e_i))),$$

for $A = 1, \ldots, N$ and $i = n_0 + 1, \ldots, n$.
Then f can be given by

$$f(x, y_1, \ldots, y_n, z) = 2x \sum_{A=1}^{N} \sum_{i=n_0+1}^{n} \varphi_{Ai} y_i z^{A-1} + \sum_{k=1}^{n_0} y_k^2.$$

Parallel distributions

Let (M, g) be a Lorentzian manifolds with $Hol_p(M, g) \subset SO_0(1, n+1)_L$.
This corresponds to filtrations

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Definition

A Lorentzian manifold with parallel light-like vector field is called Brinkmann wave.

The screen bundle

Definition

Let (M, g) be an Lorentzian manifold with parallel light like line distribution $\mathcal{L}$. The vector bundle

$$\left(\mathcal{S} = \bigcup_{p\in M} \mathcal{L}_p^{\perp}/_{\mathcal{L}_p}, \quad g^{\mathcal{S}}([U],[V]) := g(U,V), \quad \nabla_U^{\mathcal{S}}[V] := [\nabla_U V]\right)$$

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Geometric structures on $\mathcal{S}$ correspond to algebraic structures of the screen holonomy, e.g. parallel complex structure etc.

Coordinates for a Lorentzian manifold (M, h) with recurrent light-like vector field X

Theorem (Brinkmann'25, Walker'49)

$\exists$ *coordinates* $(x, y_1, \ldots, y_n, z)$: $\frac{\partial}{\partial x} = X$, $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\rangle = X^{\perp}$, *and*

- $$h = 2\,dxdz + \underbrace{\sum_{i=1}^{n} u_i dy_i}_{=\phi_z} \; dz + fdz^2 + \underbrace{\sum_{i,j=1}^{n} g_{ij} dy_i\, dy_j}_{=g_z},$$

  (ϕ_z: *family of 1-forms*; g_z: *family of Riem. metrics*)

  with $\frac{\partial g_{ij}}{\partial x} = \frac{\partial u_i}{\partial x} = 0$, $f \in C^{\infty}(M)$, *and* X *parallel* $\iff \frac{\partial f}{\partial x} = 0$.

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$\implies$ $Hol_p(g_z) \subset pr_{SO(n)} Hol_p(h)$, but in general $\neq$ (see Galaev's examples).

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- Symmetric spaces with solvable transvection group (Cahen-Wallach spaces) $\iff$ f is a quadratic polynomial in the y_i's.

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Examples

- Symmetric spaces with solvable transvection group (Cahen-Wallach spaces) $\iff$ f is a quadratic polynomial in the y_i's.
- Plane waves: f is a quadratic polynomial in the y_i's with coefficients depending on z (Important in supergravity theories. [Figueroa O'Farrill/Papadopoulos '02]

Generalisations

Better description:

$$\text{pp-wave} \iff \begin{cases} (P) & \exists \text{ parallel light-like vector field, and} \\ (1) & \mathcal{R}(U,V) : X^{\perp} \to \mathbb{R} \cdot X \quad \forall\, U, V \in TM \end{cases}$$

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Generalise (P) and (1):

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Obvious consequence (– '06)

An indecomposable Lorentzian manifold has solvable holonomy $\mathbb{R}^+ \ltimes \mathbb{R}^n$ $\iff$ (R) but not (P), and (1).

This means: (R) and (1) $\iff$ trivial screen holonomy.

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- *If (P), in Schimming coordinates ($h = 2dxdz + g_z$) the g_z is a z-dependent family of flat Riemannian metrics.*

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Remark

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- *All of Galaev's examples have light-like hypersurface curvature, i.e. all possible holonomy groups can be realised by such metrics.*

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Proposition (— '06)

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A Brinkmann wave has light-like hypersurface curvature $\iff ||\mathcal{R}||^2 = 0$.

Further properties for $h = 2dxdz + fdz^2 + \phi_z + \sum_{i=1}^{n} dy_i^2$:

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Manifolds with light-like hypersurface curvature II

(M, h) has light-like hypersurface curvature

$\Longleftrightarrow$ The curvature of the light-like hypersurfaces defined by $\mathcal{L}^\perp$ has a light-like image.

$\Longleftrightarrow$ The screen bundle $\mathcal{S}$ restricted to these hypersurfaces is flat.

Proposition (— '06)

A Brinkmann wave has light-like hypersurface curvature $\Longleftrightarrow$ $||\mathcal{R}||^2 = 0$.

Further properties for $h = 2dxdz + fdz^2 + \phi_z + \sum_{i=1}^n dy_i^2$:

1. h has trivial screen holonomy $\Longleftrightarrow$ $d\phi_z = 0\ \forall z$.
2. h is Ricci isotropic $\Longleftrightarrow$ $d^* d\phi_z = 0\ \forall z$, and Ricci flat if in addition $\Delta f = 0$.

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5. Study further spinor field equations for these manifolds.