HARMONIC FIELDS ON MIXED

RIEMANNIAN-LORENTZIAN MANIFOLDS

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Example of a Riemannian-Lorentzian manifold:

The extended projective disc $\mathbb{P}^2$.

Equip the unit disc centered at the origin of coordinates in $\mathbb{R}^2$ with the distance function

$$ds^2 = \frac{\left(1 - y^2\right) dx^2 + 2xydxdy + \left(1 - x^2\right) dy^2}{\left(1 - x^2 - y^2\right)^2}.$$

Riemannian inside the disc, Lorentzian outside the disc, singular on the unit circle.

The unit circle is the curve at projective infinity, so the Lorentzian points are *ideal*: beyond the absolute.

Harmonic fields: Solutions of the Hodge equations

$$d\alpha = \delta\alpha = 0.$$

On extended $\mathbb{P}^2$ the Hodge equations are no longer uniformly elliptic; they are

elliptic on ordinary points inside the unit disc

hyperbolic on ideal points

parabolic on the unit circle

Why study such a peculiar system?

i) to learn what the geometry of the extended projective disc reveals about elliptic-hyperbolic PDEs, and

ii) to learn what elliptic-hyperbolic PDEs on the extended projective disc reveal about the geometry of space-time.

i) What the geometry of the extended projective disc tells us about elliptic-hyperbolic pde's:

There is no canonical way to decide what constitutes a natural boundary-value problem for an equation that changes from elliptic to hyperbolic type on a smooth curve. Historically, physical analogies have been the main tool, chiefly analogies to the physics of compressible flow. However, it is also possible to approach the problem using a geometric analogy.

Traditional classification (linear, second-order, dimension 2):

$$Lu = \alpha\,(x,y)\,u_{xx} + 2\beta(x,y)u_{xy} + \gamma(x,y)u_{yy},$$

A **class** of equations on **one** domain. Characterize the type of the equations by the sign of

$$\Delta\,(x,y) = \alpha\gamma - \beta^2$$

positive $\rightarrow$ elliptic

negative $\rightarrow$ hyperbolic

zero $\rightarrow$ parabolic

changes sign on a smooth curve $\rightarrow$ mixed type

Alternative approach:

$$\mathcal{L}_g u = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{|g|} \frac{\partial u}{\partial x^j} \right)$$

One equation on a **class** of domains. Characterize the domain by the signature of the metric tensor:

Riemannian $\rightarrow$ elliptic

Lorentzian $\rightarrow$ hyperbolic

Changes signature on a smooth curve $\rightarrow$ elliptic-hyperbolic

According to this point of view, in order to decide which boundary-value problems are natural for a second-order linear elliptic-hyperbolic equations on $\mathbb{R}^2$, one should study the geometry of the underlying pseudo-Riemannian metric.

Example: *Lavrent'ev-Bitsadze equation*

$$sgn(y)u_{xx} + u_{yy} = 0,$$

an elliptic-hyperbolic equation on $\mathbb{R}^2$;

also, the Laplace-Beltrami operator on a metric which is Euclidean above the x-axis and Minkowskian below the x-axis.

ii) What elliptic-hyperbolic PDEs on the extended projective disc reveal about the geometry of space-time:

Area functional for a smooth surface Σ in $\mathbb{M}^{2,1}$ having graph $z = f(x, y)$:

$$A = \int\int_{\Sigma} \sqrt{\left|1 - f_x^2 - f_y^2\right|}\, dx dy.$$

Σ is time-like $\leftarrow f_x^2 + f_y^2 > 1$

Σ is space-like $\leftarrow f_x^2 + f_y^2 < 1$

Lagrange's notation:

$$p = f_x,\ q = f_y,$$

Boundary between the space-like and time-like surfaces:

$$p^2 + q^2 = 1.$$

Necessary condition for Σ to be extremal on $\mathbb{M}^{2,1}$: Its graph $f(x,y)$ must satisfy

$$\left(1-p^2\right)q_y+2pqp_y+\left(1-q^2\right)p_x=0,$$

a quasilinear partial differential equation, elliptic for space-like surfaces and hyperbolic for time-like surfaces.

Linearize by *Legendre transformation*

$$z=px+qy-\varphi(p,q),\, x=\varphi_p,\, y=\varphi_q.$$

Obtain the linear equation (C-H. Gu, *LNM* **1255**)

$$\left(1-p^2\right)\varphi_{pp}-2pq\varphi_{pq}+\left(1-q^2\right)\varphi_{qq}=0.$$

In homogeneous coordinates: (u, v, w) for $w \neq 0$:

$$\left[\left(1-p^{2}\right)\psi_{p}\right]_{p}-2pq\psi_{pq}+\left[\left(1-q^{2}\right)\psi_{q}\right]_{q}=0,$$

where $p=-u/w$ and $q=-v/w$ (C-H. Gu, *Acta Math. Sinica n.s.* **1**)

This is the Laplace-Beltrami equation on extended $\mathbb{P}^2$

Now we consider a space-like surface Σ associated to a multi-valued 0-form $\mathbf{f}$ defined over a multi-connected domain of $\mathbb{M}^{2,1}$.

Variational equations of the area functional:

$$\left[\left(1-\omega_2^2\right)\omega_1\right]_x +$$

$$\omega_1\omega_2(\omega_{1y}+\omega_{2x}) + \left[(1-\omega_1)^2\,\omega_2\right]_y = 0,$$

$$\omega_{2x} - \omega_{1y} = 0,$$

for

$$Q = \delta_{\ell m} g^{ij} \frac{\partial f^\ell}{\partial x^i}\frac{\partial f^m}{\partial x^j}.$$

This is an equation for a multi-connected extremal surface which is space-like in some regions of $\mathbb{M}^{2,1}$ and time-like in others. However, the resulting system, and the surface that it describes, are singular on the circle $Q = 1$.

If the Gaussian curvature

$$G = \frac{\omega_{1x}\omega_{2y} - |\omega_{1y}|^2}{\left[1 - \left(\omega_1^2 + \omega_2^2\right)\right]^2}$$

is nonvanishing, apply a Legendre transformation to obtain

$$\left[\left(1 - x^2\right) u_1\right]_x -$$

$$(xyu_1)_y - (xyu_2)_x + \left[\left(1 - y^2\right) u_2\right]_y = 0,$$

$$u_{1y} - u_{2x} = 0$$

This system has a geometric interpretation as the Hodge equations on the extended projective disc $\mathbb{P}^2$.

From this point of view, doing Hodge theory on $\mathbb{P}^2$ is a way to approach generalized Plateau problems in Minkowski space. The first step in such a program would be a local existence theorem for the Dirichlet problem.

Problems with this approach:

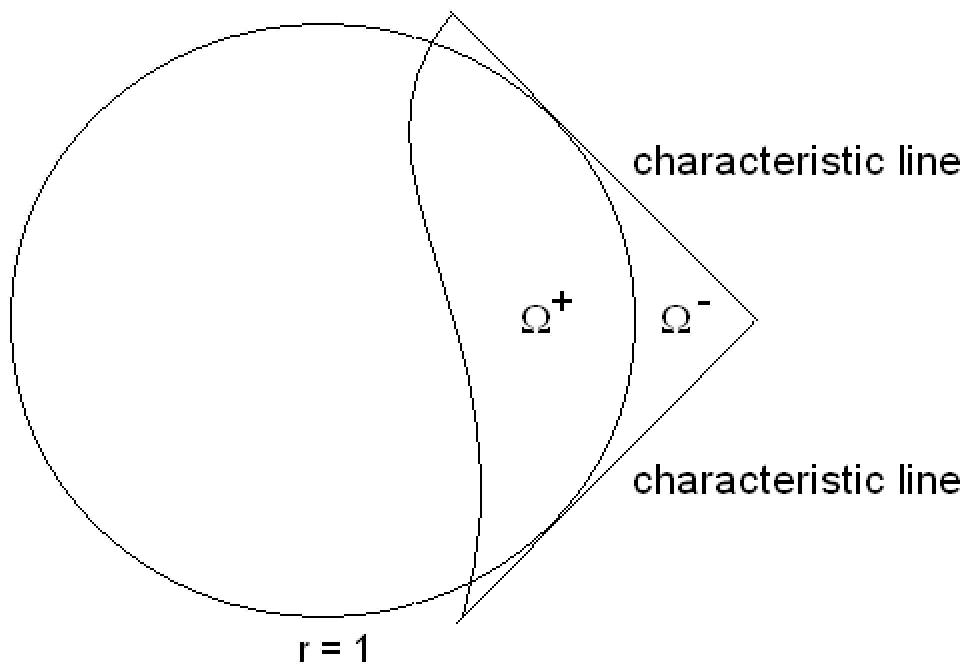
1. The Legendre transformation may itself introduce singularities. It is known that, in the elliptic region, such singularities can only occur on at most a point set. However, higher-order singularities in the parabolic and hyperbolic regions of the equations are possible.

2. Although the Legendre transformation makes the equation simpler, it makes boundary conditions more complicated, so the interpretation of results for harmonic 1-forms on extended $\mathbb{P}^2$ in terms of extremal hypersurfaces in $\mathbb{M}^{2,1}$ is not always straightforward.

Nevertheless, the Dirichlet problem for a quasilinear elliptic-hyperbolic system, having a line singularity on the parabolic curve, is sufficiently formidable that the approach via linearization remains the one with the most apparent promise.

Boundary-value problems for harmonic fields on extended $\mathbb{P}^2$:

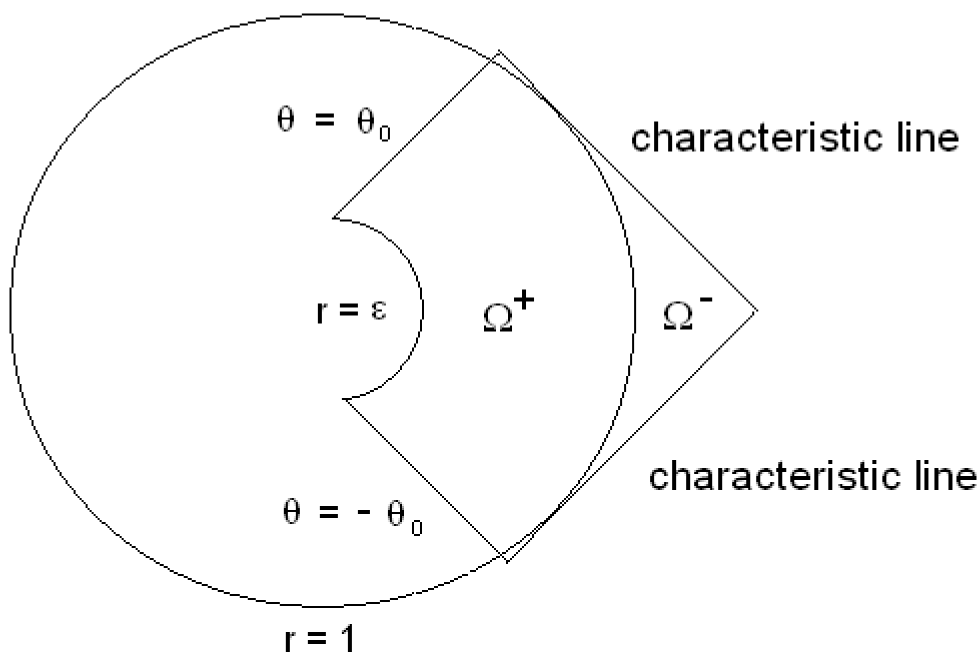
1. Let Ω be the domain formed by the polar lines of a chord of the unit disc in extended $\mathbb{P}^2$. Then there exists a weak solution on Ω with boundary values prescribed on the non-characteristic part of the boundary. In fact, we can deform the chord in such a way that there is also a non-characteristic hyperbolic boundary on which data are prescribed, providing a mild monotonicity condition is met. Solutions lie in a weighted function space. The weak solution is strong (and thus unique) if we round off the sharp points on the boundary and impose an arbitrarily small perturbation on the lower-order terms.


characteristic line
Ω^+
Ω^-
characteristic line
r = 1

2. Weak solutions to the Dirichlet problem – and to mixed Dirichlet-Neumann problems – in which data are prescribed on the entire boundary exist under a list of technical conditions on Ω which are roughly equivalent to the requirement that the boundary of Ω be star-like with respect to the flow of an appropriately defined vector field.

3. The existence of a strong solution to a boundary-value problem in an annulus about the unit circle can be proven under an arbitrarily small perturbation of the lower-order terms.

4. The closed Dirichlet problem for classical solutions is over-determined on the hyperbolic boundary.



θ = θ0
characteristic line
r = ε
Ω+
Ω-
characteristic line
θ = - θ0
r = 1

Other mixed Riemannian-Lorentzian manifolds:

1. *Special relativity*: The wave equation on Minkowski spacetime, in a reference frame rotating with constant angular velocity ω with respect to another reference frame, is expressible in cylindrical coordinates (ρ, φ, z) as the elliptic-hyperbolic equation

$$\frac{1}{\rho}\left(\rho u_\rho\right)_\rho + \left(\frac{1}{\rho^2} - \omega^2\right) u_{\varphi\varphi} + u_{zz} = 0.$$

($M\ Schönberg$, *Phys. Rev.* **69**)

2. *Quantum cosmology*: These examples arise from the (controversial) *Hartle-Hawking hypothesis*, that the universe might have originated as a manifold having Euclidean signature and subsequently undergone a transition to a model having Lorentzian signature across a hypersurface which was space-like as seen from the Lorentzian side.

Some 2-dimensional variants (J. M. Stewart, *Class. Quant. Grav.* **18**):

i) continuous change of signature:

$$ds^2 = -t dt^2 + dz^2;$$

ii) discontinuous change of signature:

$$ds^2 = -z^{-1} dt^2 + dz^2;$$

iii) continuous change of signature with a curvature singularity:

$$ds^2 = -z dt^2 + dz^2;$$

3. *Binary black hole spacetimes with a helical killing vector*, a generalization of example 1 (C. Klein, *Phys. Rev. D* **70**).

Elliptic-hyperbolic differential operators of *real principal type:* the major analytic properties of the operator depend only on the Hamiltonian system associated to the principal symbol – they do not depend on the form of the lower-order terms.

The physical examples tend to be of real principal type. The geometric examples tend not to be (exception: isometric embedding of Riemannian surfaces in $\mathbb{R}^3$, C-S. Lin, *CPAM* **39**)

As a result, what can or cannot be said about solutions in the geometric examples tends to depend delicately on the precise form of the lower-order terms. In particular, this dependence prevents the derivation of uniqueness theorems by the expected arguments.