Geometric Realization Of Curvature

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A central area of study in differential geometry is the examination of the relationship between purely algebraic properties of the curvature tensor and the underlying geometric properties of the manifold. Many authors have worked in this area in recent years. Nevertheless, many fundamental questions remain unanswered. When dealing with a geometric problem, it is frequently convenient to work first purely algebraically and pass later to the geometric setting. For this reason, many questions in differential geometry are often phrased as problems involving the geometric realization of curvature.

We suppose given a vector space V and a family of tensors $\{T_1, ..., T_k\}$ on V. The structure $(V, T_1, ..., T_k)$ is said to be *geometrically realizable* if there exists a manifold M, if there exists a point P of M, and if there exists an isomorphism $\phi : V \to T_P M$ such that $\phi^* L_i(P) = T_i$ where $\{L_1, ..., L_k\}$ is a corresponding geometric family of tensor fields on M. Thus, for example, if $k = 1$ and if $T_1 = \langle\cdot,\cdot\rangle$ is a non-degenerate inner product on V, then a geometric realization of $(V, \langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian manifold (M, g), a point P of M, and an isomorphism $\phi : V \to T_P M$ so that $\phi^* g_P = \langle\cdot,\cdot\rangle$; this is of course, a trivial problem.

Many people have worked in this area or in a closely related area including D. Alekseevsky, N. Blažić, M. Brozos-Vázquez, E. Calviño-Louzao, L. Cordero, A. Derdzinski, C. Dunn, I. Dotti, E. García-Río, P. Gilkey, T. Hervella, H. Kang, O. Kowalski, M. Fernandez, Y. Matushita, Y. Nikolayevsky, B. Opzoda, JH. Park, K. Sekigawa, I. Stavrov, S. Nikčević, U. Simon, E. Vázquez-Abel, R. Vázquez-Lorenzo, L. Vanhecke, V. Videv, D. Westerman, and G. Weingart; it is not possible to mention everyone working in this field!

We shall give a brief survey of the field and report on a few results studying geometric realizations of:

1. Riemannian algebraic curvature tensors by pseudo-Riemannian manifolds.
2. Affine curvature tensors by affine manifolds.
3. Weyl curvature tensors by Weyl manifolds.
4. Kähler affine curvature tensors by affine Kähler manifolds.
5. Kähler Riemannian curvature tensors by Kähler manifolds.
6. Hermitian Riemannian curvature tensors by Hermitian manifolds.
7. Covariant derivative Kähler tensors by almost pseudo-Hermitian manifolds.

In each instance, curvature decompositions of the appropriate space of tensors under a suitable structure group play a crucial role. And it is important to write down the appropriate symmetries of the curvature tensors involved; for example the Gray identity plays an important role in the study of Hermitian geometry. Furthermore, some problems are not geometrically realizable; for example, a Ricci-antisymmetric projectively flat affine curvature tensor R is geometrically realizable by a Ricci-antisymmetric projectively flat affine connection if and only if $R = 0$. We discuss not only the positive definite setting but also higher signature geometry and para-complex geometry.