The Lorentzian conformal analogon of Calabi Yau manifolds

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The conformal analog of Calabi Yau manifolds

1. Motivation
2. Cartan connections and their holonomy groups
3. Holonomy groups of conformal manifolds
4. Conformal holonomy groups and Einstein metrics
5. Holonomy groups of Riemannian conformal structures (Classification)
6. Holonomy groups of Lorentzian conformal structures
7. Conformal Lorentzian manifolds with holonomy in $SU(1,m)$
   - CR geometry and Fefferman spaces
   - Conformal holomomy group of Fefferman spin manifolds

1. Motivation

Holonomy groups of Riemannian manifolds

(M^n, g) Riemannian manifold, complete, simply-connected.

DeRham Spitting Theorem: (G.DeRham 1952)

$$(M, g) \simeq \mathbb{R}^k \times (M_1, g_1) \times \cdots \times (M_k, g_k) \quad \text{where } (M_i, g_i) \text{ is irreducible}$$

Holonomy groups of symmetric spaces

Let (M, g) be a 1-connected symmetric space, $M = G/K$, where $G \subset Isom(M, g)$ is the transvection group of M and $K = G_x$ the stabilizer of a point $x \in M$. Then

1. $Hol_x(M, g) \simeq K$
2. The holonomy representation $Hol_x(M, g) \to SO(T_xM, g_x)$ is given by the isotropy representation of K.

Irreducible symmetric spaces are classified $\Rightarrow$ their holonomy groups are known

Berger's List: (M. Berger 1955)

Let (M^n, g) be an irreducible non-locally symmetric Riemannian manifold. Then the holonomy group $Hol^0(M, g)$ is (up to conjugation in $O(n)$) one of the following groups

$SO(n)$	generic type	0
$U(\frac{n}{2})$	Kähler	0
$SU(\frac{n}{2})$	Ricci-flat, Kähler	2
$Sp(\frac{n}{4})$	Hyperkähler	$\frac{n}{4} + 1$
$Sp(\frac{n}{4}) \cdot Sp(1)$	quaternionic Kähler	0
G_2	$n = 7,$ special parallel 3-form	1
$Spin(7)$	$n = 8,$ special parallel 4-form	1

special holonomy group $\Longleftrightarrow$ **special geometric structure**

$\Downarrow$

special spinor fields

Calabi-Yau manifold $:=$ Compact Riemannian manifold (M^{2m}, g)
with $Hol(M,g) = SU(m)$

$\Longrightarrow$
- (M,g) is Kähler ($\Rightarrow$ complex)
- (M,g) is Ricci-flat
- (M,g) is spin
- (M,g) has a 2-parameter family of parallel spinors $\nabla_X \varphi = 0$

Aim of the talk: Describe the analog situation in conformal geometry:

g metric $\rightsquigarrow$ $c = [g]$ conformal structure, $c := [g] = \{\tilde{g} \mid \tilde{g} = e^{2\sigma} g\}$

- What is the holonomy group of a *conformal* manifold (M,c)?
  $Hol(M^{p,q}, c) \subset O(p+1, q+1)$
- Describe Lorentzian conformal manifolds with
  $Hol(M,[g]) \subset SU(1,m) \subset O(2,2m)$.

Why conformal holonomy groups are interesting ?

- This would describe interesting special conformal geometries (described by special holonomy groups like in the Berger list)

- There is a link to conformally invariant spinor field equations:

$$\nabla_X \varphi + \frac{1}{n} X \cdot D\varphi = 0 \qquad \varphi \text{ conformal Killing spinor (CKS)}$$

For which manifolds (M, g) exist conformal Killing spinors ?
Answer is given by $Hol(M, [g])$:

$$\{\varphi \mid \varphi \text{ CKS}\} \stackrel{1:1}{\Longleftrightarrow} \{v \in \Delta_{p+1,q+1} \mid \widetilde{Hol(M, [g])}\, v = v\}.$$

What we have to do to define $Hol(M,[g])$?

- $(M^{p,q}, g)$ semi-Riemannian manifold, n=p+q

  $\rightsquigarrow$ (TM, ∇^g, g), ∇^g Levi-Civita-connection, unique

  $\rightsquigarrow$ $Hol(M,g) := Hol(TM, \nabla^g) \subset O(p,q)$

- $(M^{p,q}, [g])$ conformal manifold, n=p+q

  $\rightsquigarrow$ There is no unique connection ∇ on $(TM, [g])$.

  $\rightsquigarrow$ Find unique triple $(\mathcal{T}(M), \nabla^{nor}, \langle\cdot,\cdot\rangle)$ for $(M,[g])$

  $\rightsquigarrow$ $Hol(M,[g]) := Hol(\mathcal{T}(M), \nabla^{nor}) \subset O(p+1, q+1)$

$\mathcal{T}(M)$ vector bundle of rank n+2 on M
$\langle\cdot,\cdot\rangle$ bundle metric of sign (p+1,q+1)
∇^{nor} metric connection on $(\mathcal{T}(M), \langle\cdot,\cdot\rangle)$.

$\rightsquigarrow$ **Tool: Cartan geometry, Cartan connections**

2. Cartan connections and holonomy groups

$(P, \pi, M; B)$ B-principal fibre bundle, $B \subset G$ closed subgroup

$\omega \in \Omega^1(P; \mathfrak{g})$ is a Cartan connection $:\Longleftrightarrow$

- $R_b^* \omega = Ad(b^{-1}) \circ \omega \qquad \forall\, b \in B$
- $\omega(\tilde{X}) = X \qquad \forall\, X \in \mathfrak{b}$
- $\omega_u : T_u P \longrightarrow \mathfrak{g}$ linear isomorphism $\forall\, u \in P$

$\Longrightarrow$ no B-invariant horizontal splitting of TP

$T_x M \overset{\omega}{\simeq} \mathfrak{g}/\mathfrak{b} = T_e(G/B)$

P parallelizable

Example: Flat model: $G \xrightarrow{\pi} G/B = M$ homogeneous bundle

$\omega := \omega_G \in \Omega^1(G; \mathfrak{g})$ Maurer-Cartan form of G

$\Omega^\omega = d\omega + \frac{1}{2}[\omega, \omega] = 0$ curvature of $\omega = \omega_G$

Holonomy groups of Cartan connections

$\omega \in \Omega^1(N, \mathfrak{g})$ 1-form on N, $\quad \delta : [0,1] \longrightarrow N$ curve on N

$\Longrightarrow \ \exists\,! \ \delta_{[\omega]} : [0,1] \longrightarrow G$ (development of ω along δ):

- $\delta'_{[\omega]}(t) = dL_{\delta_{[\omega]}(t)}(\omega(\delta'(t)))$
- $\delta_{[\omega]}(0) = e \ \in G$

$(P, \pi, M; B)$ with Cartan connection $\omega \in \Omega^1(P, \mathfrak{g})$, $u \in P_x$

Holonomy group of ω wrto $u \in P_x$:

$Hol_u(P,\omega) := \left\{ \tilde{\gamma}_{[\omega]}(1) \in G \mid \gamma \text{ loop in } x \text{ with lift } \tilde{\gamma}\text{, } \textit{closed} \text{ in } u \in P \right\} \subset G$

Example: $G \xrightarrow{\pi} G/B$ with Maurer-Cartan form $\omega = \omega_G$.
If $\tilde{\gamma}(0) = u$, then $\tilde{\gamma}_{[\omega]}(t) = u^{-1} \cdot \tilde{\gamma}(t)$
$\Longrightarrow \quad Hol_u(G, \omega_G) = \{1\}$.

$(P, \pi, M; B)$ principal bundle with Cartan connection $\omega \in \Omega^1(P, \mathfrak{g})$
$\rho : G \longrightarrow GL(V)$ representation of G
$E := P \times_B V$ tractor bundle

$\rightsquigarrow$ $\nabla^\omega : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ covariant derivative (tractor connection)

$$\nabla^\omega_X \phi_{|_U} := \big[s, X(v) + \rho_*\big(\omega(ds(X))\big)\, v \big] \qquad \phi_{|_U} = [s, v]$$

Theorem: If B is connected or M is simply connected, then

$$\rho\big(Hol_u(P, \omega)\big) = [u]^{-1} \circ \underbrace{Hol_x(E, \nabla^\omega)}_{\subset GL(E_x)} \circ [u] \subset GL(V) \qquad ([u] : V \simeq E_x)$$

$$\rho \text{ faithful} \implies Hol(P, \omega) \simeq Hol(E, \nabla^\omega).$$

3. Holonomy groups of conformal structures

3.1. Flat model of conformal geometry

S^n with standard conformal structure $c = [g_{can}]$.
$\rightsquigarrow S^n \simeq Conf(S^n, c)/Conf(S^n, c)_{x_0} \simeq PO(1, n+1)/B$.

$L := \{x \in \mathbb{R}^{p+1,q+1} \setminus \{0\} \mid \langle x, x\rangle_{p+1,q+1}\} \subset \mathbb{R}^{p+1,q+1}$ lightcone

$\mathbb{P}L$ with conformal structure $c := [\mu^* \langle \cdot, \cdot \rangle_{p+1,q+1}]$ of signature (p, q),

$\mu : \mathbb{P}L \to L^+$ section

Riemannian: $(\mathbb{P}L, c) = (S^n, [g_{can}]) \simeq \partial H^{n+1}$
Lorentzian: $(\mathbb{P}L, c) = Ein_n \simeq \partial AdS^{n+1}$

$G := O(p+1, q+1)$.
$\hat{G} := PO(p+1, q+1) \simeq Conf(\mathbb{P}L, c)$ acts transitively and conformally on $\mathbb{P}L$

$B := \hat{G}_{p_\infty} \simeq CO(p, q) \ltimes \mathbb{R}^n \subset G$ stabilizer of an isotropic line p_∞

$\rightsquigarrow$ $\mathbb{P}L \simeq G/B$ flat model of conformal geometry

3.2. Curved conformal manifolds

(M,c) conformal manifold of signature (p,q), $n = p+q$.

$(\mathcal{P}^0, \pi^0, M; CO(p,q))$ bundle of conformal frames of (M,c)

$$\begin{array}{rcl} & \mathcal{P}^1 & \text{first prolongation of } \mathcal{P}^0 \\ \mathbb{R}^n & \downarrow & \\ & \mathcal{P}^0 & \text{conformal frames,} \quad \theta \text{ displacement form} \\ CO(p,q) & \downarrow & \\ & (M,c) & \end{array}$$

$\mathcal{P}^1 := \{ H \subset T_u\mathcal{P}^0 \mid u \in \mathcal{P}^0,\ H$ horizonal and torsion free$\}$ B- bundle over M

$B = CO(p,q) \ltimes \mathbb{R}^n \subset G$

torsion of a horizontal space H:

$$t(H) \in \Lambda^2(\mathbb{R}^n)^*) \otimes \mathbb{R}^n$$
$$t(H)(v,w) := d\theta_u(\theta|_H^{-1}(v), \theta|_H^{-1}(w))$$

Theorem: There exists a unique Cartan connection $\omega^{nor} : T\mathcal{P}^1 \longrightarrow \mathfrak{g}$, which ”lifts” θ and satisfies the curvature condition $\partial^* \Omega^{\omega^{nor}} = 0$.

ω^{nor} := normal conformal Cartan connection

- $\mathcal{T}(M) := \mathcal{P}^1 \times_B \mathbb{R}^{p+1,q+1}$ standard tractor bundle
- $\langle \cdot, \cdot \rangle$ bundle metric on $\mathcal{T}(M)$ defined by $\langle \cdot, \cdot \rangle_{p+1,q+1}$ on $\mathbb{R}^{p+1,q+1}$
- $\nabla^{nor} := \nabla^{\omega^{nor}}$ metric connection on $\mathcal{T}(M)$.

(M, c) conformal manifold $\Longrightarrow$ $(\mathcal{T}(M), \nabla^{nor}, \langle, \cdot, \cdot \rangle)$
conformal invariant data by definition.

Holonomy group of (M, c):

$$\boxed{Hol(M, c) := Hol(\mathcal{T}(M), \nabla^{nor}) \subset O(p+1, q+1)}$$

3.3. Metric description of $(\mathcal{T}(M), \nabla^{nor}, \langle \cdot, \cdot \rangle)$

$g \in c \quad \rightsquigarrow \quad$ reduction $\quad i : \mathcal{P}^g \subset \mathcal{P}^0 \longrightarrow \mathcal{P}^1$

$$u \longmapsto Ker A^g_u$$

$$\Longrightarrow \quad \mathcal{T}(M) \overset{g}{\simeq} \mathcal{P}^g \times_{0(p,q)} \mathbb{R} \oplus \mathbb{R}^{p,q} \oplus \mathbb{R} \simeq \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}}$$

$$\phi \in \Gamma(\mathcal{T}(M)) \quad \Longrightarrow \quad \phi \overset{g}{\simeq} \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} \begin{matrix} \text{function on } M \\ \text{vector field on } M \\ \text{function on } M \end{matrix}$$

$$\langle \phi, \hat{\phi} \rangle \overset{g}{\simeq} \alpha\hat{\beta} + \beta\hat{\alpha} + g(Y, \hat{Y})$$

Theorem: Fix a metric $g \in c$. Then for the tractor connection ∇^{nor} hold:

$$\nabla^{nor}_X \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} \overset{g}{\simeq} \begin{pmatrix} X(\alpha) - K^g(X,Y) \\ \nabla^g_X Y + \alpha X + \beta K^g(X) \\ X(\beta) - g(X,Y) \end{pmatrix}$$

$K^g(X,Y) := \frac{1}{n-2}(Ric^g - \frac{1}{2(n-2)} scal^g g)$ Schouten tensor

4. Reducible conformal holonomy groups and Einstein metrics

$g \in c$ Einstein metric $\Longrightarrow$ $\exists\, \phi \in \Gamma(\mathcal{T}(M))$ with $\nabla^{nor}\phi = 0$
hence $Hol(M,c)$ is non-irreducible.

$\exists\, g \in c|_U$ Einstein $\Longleftarrow$ $\exists\, \phi \in \Gamma(\mathcal{T}(M))$ with $\nabla^{nor}\phi = 0$
U open and dense in M

- $g \in c$ Einstein $\Longrightarrow$ $K^g = \frac{1}{2n(n-1)} scal^g g$

$$\Longrightarrow \nabla^{nor}\Big(-\frac{scal^g}{2n(n-1)}, 0, 1\Big)^{\perp} = 0$$

- Let $g \in c$ arbitrary and $\nabla^{nor}\phi = 0$. Then $\phi \overset{g}{\simeq} (\alpha, Y, \beta)^{\perp}$.

$\Longrightarrow$ $U = \{x \in M \mid \beta(x) \neq 0\} \subset M$ open and dense

$\tilde{g} = \beta^{-2} g$ is Einstein

$scal^{\tilde{g}} > 0 \quad \Leftrightarrow \quad \phi$ timelike
$scal^{\tilde{g}} < 0 \quad \Leftrightarrow \quad \phi$ spacelike
$scal^{\tilde{g}} = 0 \quad \Leftrightarrow \quad \phi$ lightlike

- $V^1 \subset \mathbb{R}^{p+1,q+1}$ 1-dim Hol-invariant subspace $\Longrightarrow \exists\, g \in c|_U$ Einstein

- Local Splitting Theorem (St. Armstrong '05, F. Leitner '04):

  $V^k \subset \mathbb{R}^{p+1,q+1}$ k-dim Hol-invariant *non-degenerate* subspace, $k \geq 2$

  $$\Longrightarrow \quad (M,c) \overset{locally}{\simeq} (M_1 \times M_2, [g_1 \times g_2]), \qquad (M_i, g_i) \text{ Einstein}$$
  $$Hol(M,c) = Hol(M_1,[g_1]) \times Hol(M_2,g_2).$$

- Holonomy of Einstein spaces (St. Armstrong '05, Th. Leistner '05):

  $g \in c$ Einstein. $\quad (M,[g]) \rightsquigarrow (\tilde{M},\tilde{g})$ Ricci-flat space of sign $(p+1,q+1)$

  $$\tilde{M} := \mathbb{R} \times M \times \mathbb{R}^+$$

  $$\tilde{g} \ := \begin{cases} \frac{n(n-1)}{scal^g}(d^2t - d^2s) + t^2 g & \text{if } scal^g \neq 0 \\ 2dsdt + t^2 g & \text{if } scal^g = 0 \end{cases}$$

  $$\Longrightarrow \quad Hol_x(M,c) = Hol_{(1,x,1)}(\tilde{M},\tilde{g}) =$$

  $$= \begin{cases} Hol_{(1,x)}(cone(M), \frac{n(n-1)}{scal^g} dt^2 + t^2 g) & \text{if } scal^g \neq 0 \\ Hol_x(M,g) \ltimes \mathbb{R}^n & \text{if } scal^g = 0 \end{cases}$$

5. Riemannian conformal structures (St. Armstrong 2005)

$$\boxed{Hol(M^n,[g]) \subset O(1,n+1)}$$

a) Irreducible case: $Hol^0(M,[g]) \simeq SO^0(1,n+1)$ (Di Scala, Olmos 2001)

b) Non-irreducible case:

- 1-dim. invariant subspace $\Longrightarrow$ $g \overset{loc}{\sim} \tilde{g}$ Einstein metric
- k-dim. non-degenerate invariant subspace $\Longrightarrow$ $g \overset{loc}{\sim}$ product of Einstein metrics

c) Conformally indecomposable Einstein spaces:

- $scal^g < 0 \quad \Longrightarrow \quad Hol(M,[g]) = SO_0(1,n)$.
- $scal^g > 0 \quad \Longrightarrow \quad Hol(M,[g]) = SO(n+1), SU(\frac{n+1}{2}), Sp(\frac{n+1}{4}),$ G_2 ($n=6$), $Spin(7)$ ($n=7$)
- $scal^g = 0 \quad \Longrightarrow \quad Hol(M,[g]) = SO(n) \ltimes \mathbb{R}^n, SU(\frac{n}{2}) \ltimes \mathbb{R}^n, Sp(\frac{n}{4}) \ltimes \mathbb{R}^n,$ $G_2 \ltimes \mathbb{R}^7$ ($n=7$), $Spin(7) \ltimes \mathbb{R}^8$ ($n=8$).

6. Lorentzian conformal structures:

$$\boxed{Hol(M^{1,n-1},[g]) \subset O(2,n)}$$

a) Non-irreducible case:

- 1-dim. invariant subspace $\Longrightarrow\; g \overset{loc}{\sim} \tilde{g}$ Einstein metric
- k-dim. non-degenerate invariant subspace $\Longrightarrow\; g \overset{loc}{\sim}$ product of Einstein metrics
- 2-dim. totally isotropic invariant subspace (Th. Leistner '05):
  $\Longrightarrow\; g \overset{loc}{\sim} \tilde{g},\quad \tilde{g}$ Lorentzian metric with light-like recurrent vector field
  and totally isotropic Ricci tensor

b) Irreducible case:

Thm: (Di Scala, Leistner, 2008)

$H \subset O(2,n)$ connected, irreducibly acting $\Longrightarrow$

$H \simeq$
- $SO^0(2,n)$
- $U(1,\frac{n}{2})\,,\, SU(1,\frac{n}{2})$, if n is even
- $S^1 \cdot SO^0(1,\frac{n}{2}) \subset U(1,\frac{n}{2})$, if n is even and $n \geq 4$
- $SO^0(1,2) \subset SO^0(2,3)$, if $n = 3$.

Thm: (Leitner 2007)

$$Hol^0(M^{2m},c) \subset U(1,m) \quad \Rightarrow \quad Hol^0(M,c) \subset SU(1,m).$$

Thm: (M^n,c) conformal Lorentzian manifold, $n \geq 4$. $Hol^0(M,c)$ acts irreducibly

$\Rightarrow \quad Hol^0(M,c) \simeq SO^0(2,n)$ or

$Hol^0(M,c) \simeq SU(1,\frac{n}{2})$.

7. Lorentzian conformal manifolds (M,c) with $Hol(M,c) \subset SU(1,m)$

(N^{2m-1}, H, J) CR manifold $:\Longleftrightarrow$

- $H \subset TN$ subbundle of codim 1
- $J : H \to H$, $J^2 = -id_H$, J integrable
  $N_J(X,Y) := J\Big([JX,Y] + [X,JY]\Big) - [JX,JY] + [X,Y] = 0$

Examples: Real hypersurfaces of complex manifolds,
Sasaki manifolds, Heisenberg manifolds

(N^{2m-1}, H, J, θ) strictly pseudo convex manifold $:\Longleftrightarrow$

- N oriented, θ contact 1-form on N with Reeb vector field T_θ
- positive definite Levi form: $L_\theta(X,Y) = d\theta(X,JY)$, $X, Y \in H$

$g_\theta := L_\theta + \theta \circ \theta$ Riemannian metric on N.

$\nabla^W : \Gamma(TN) \to \Gamma(TN^* \otimes TN)$ Tanaka-Webster connection $:\Longleftrightarrow$

- $\nabla^W g_\theta = 0$
- $Tor^W(X,Y) = L_\theta(JX,Y) \cdot T_\theta$
  $Tor^W(T_\theta, X) = -\frac{1}{2}\Big\{[T_\theta, X] + J([T_\theta, JT_\theta])\Big\}$

(N^{2m-1}, H, J, θ) strictly pseudoconvex and spin

$\Longrightarrow \quad \sqrt{K} := \sqrt{\Lambda^{m,0} N}$ square root of the canonical line bundle K

$\Longrightarrow \quad M^{2m} := \sqrt{K}^{*} / \mathbb{R}^{+} \xrightarrow{\pi} N \qquad S^1$-bundle 0n N with connection form

$$\boxed{A_\theta := A^W - \frac{i}{4m} scal^W \circ \theta}$$ Fefferman connection

$$TM \;=\; \underbrace{H^* \oplus \mathbb{R}T_\theta^*}_{horizontal\ wrto\ A_\theta} \;\oplus\; TvM$$

$\Longrightarrow$ Lorentzian metric on M

$$\boxed{h_\theta := \pi^* L_\theta - \frac{8i}{m} \pi^* \theta \circ A_\theta}$$ Fefferman metric on M

$$\underset{stricly\ pseudoconvex}{(N, H, \underset{\uparrow}{J}, \theta, spin)} \;\rightsquigarrow\; \underset{independent\ on\ \theta}{(M, \underset{\uparrow}{[h_\theta]}, spin)}$$ Fefferman spin space

Theorem: $(M^{1,2m-1}, [h_\theta])$ Fefferman spin space

$$\Longrightarrow \; Hol(M, [h_\theta]) \subset SU(1,m)$$

- hermitian almost complex structure $\mathcal{J}$ on $\mathcal{T}(M) = \mathcal{P}^1 \times_B \mathbb{R}^{2,2m}$:

$$\mathcal{J}\begin{pmatrix} \alpha \\ \gamma\, X^* + T^*_\theta + \delta\, V \\ \beta \end{pmatrix} \overset{h_\theta}{:=} \begin{pmatrix} -\frac{\delta}{2} \\ \nabla^{h_\theta}_{X^*} V + \frac{\beta}{2}\, T^*_\theta + 2\alpha\, V \\ -2\gamma \end{pmatrix}$$

$$\nabla^{nor}\mathcal{J} = 0 \;\Longrightarrow\; Hol(M, [h_\theta]) \subset U(1,m)$$

- $\exists$ 2-parameter family of conformal Killing spinors on (M, h_θ) (Baum '97)

$\Longrightarrow$ 2-dimensional space of $\widetilde{Hol(M, [h_\theta])}$-invariant spinors in $\Delta_{2,2m}$

$\Longrightarrow$ $Hol(M, [h_\theta]) \subset SU(1,m)$

Local characterizations of Fefferman spaces

Theorem: (Graham/Sparling '87)

(M,h) Lorentzian manifold with a light-like Killing field V such that

- $V \lrcorner W^h = 0$ $\qquad W^h$ Weyl tensor
- $V \lrcorner C^h = 0$ $\qquad C^h$ Cotton York tensor
- $Ric^h(V,V) = const > 0$

$\Longrightarrow$ (M,h) is locally isometric to a Fefferman space (M,h_θ) of a strictly pseudo-convex manifold.

Theorem: (F. Leitner '06)

(M,c) conformal Lorentzian manifold.

$Hol^0(M,c) \subset SU(1,m) \Longrightarrow (M,c)$ is locally a Fefferman space.