Actions of Cohomogeneity One on Symmetric Spaces of Noncompact Type

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Real hyperbolic plane (I)

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- (H, ds^2) = upper half plane model for real hyperbolic plane
- $H = G/K$ homogeneous space with

$$G = SL_2(\mathbb{R})\ ,\ \ K = \left\{ \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \middle|\ s \in \mathbb{R} \right\} = SO_2$$

Real hyperbolic plane (II)

- There is a unique decomposition

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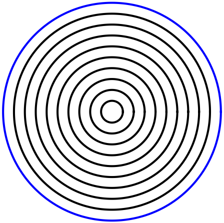
- $H = G/K \overset{\text{iso}}{=} AN$ solvable Lie group with left-invariant Riemannian metric

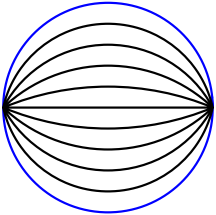
Real hyperbolic plane (III)

Orbit structures (Poincaré model)

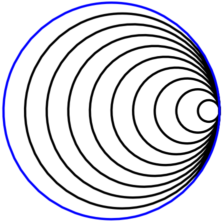
K-orbits



A-orbits



N-orbits



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Joint work with Hiroshi Tamaru (Hiroshima University)

The setup

- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
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- $M = G/K$ connected irreducible Riemannian symmetric space of noncompact type
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  $o \in M$ with $K \cdot o = o$
- H connected closed subgroup of G acting on M with cohomogeneity one

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3. *The action of H is orbit equivalent to the canonical extension of a cohomogeneity one action on a boundary component of M;*
4. *The action of H is orbit equivalent to one which is obtained by the "nilpotent construction".*

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- $G = KAN$ Iwasawa decomposition
- $M = G/K = AN$ solvable Lie group with left-invariant Riemannian metric
- Observation: Each codimension one subgroup of AN induces a cohomogeneity one action on M whose orbits form a Riemannian foliation on M
- There are two types of foliations arising in this way

The case of foliations (II)

- $\mathfrak{a} \oplus \mathfrak{n}$ Lie algebra of AN
- $\ell \subset \mathfrak{a}$ one-dimensional subspace
- $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ codimension one subalgebra

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- All orbits of $\mathcal{F}_\ell$ are isometrically congruent to each other
- Special case: Horosphere foliations

The case of foliations (III)

- $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$
- $\alpha_1, \ldots, \alpha_r$ simple roots
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- S_i acts on M with cohomogeneity one, induces foliation $\mathcal{F}_i$
- $\mathcal{F}_i$ has exactly one minimal orbit

The case of foliations (IV)

Berndt-Tamaru 2003: *Let M be a connected irreducible Riemannian symmetric space of noncompact type. Assume that H acts on M with cohomogeneity one such that the orbits form a Riemannian foliation $\mathcal{F}$ on M. Then $\mathcal{F}$ is congruent to one of the foliations $\mathcal{F}_\ell$ or $\mathcal{F}_i$.*

Totally geodesic singular orbit

Berndt-Tamaru 2004: *F is a totally geodesic singular orbit of a cohomogeneity one action on M* $\Longleftrightarrow$

- *F reflective and rank $F^\perp = 1$, or*
- *F is one of the following totally geodesic non-reflective submanifolds:*

F	M	$\dim F$	$\dim M$
$\mathbb{C}H^2$	G_2^2/SO_4	4	8
$SL_3(\mathbb{R})/SO_3$	G_2^2/SO_4	5	8
G_2^2/SO_4	$SO^o_{3,4}/SO_3SO_4$	8	12
$SL_3(\mathbb{C})/SU_3$	$G_2^{\mathbb{C}}/G_2$	8	14
$G_2^{\mathbb{C}}/G_2$	$SO_7^{\mathbb{C}}/SO_7$	14	21

Parabolic subalgebras (I)

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition
- $\mathfrak{a}$ maximal abelian subspace of $\mathfrak{p}$
- restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

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- $\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right)$, $\mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\alpha$
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- Every parabolic subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{q}_\Phi$ for some subset $\Phi \subset \Lambda$

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- $\mathfrak{a}_\Phi = \cap_{\alpha\in\Phi} \ker\alpha$, $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$
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- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ (**Langlands decomposition**)
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- $B_\Phi = M_\Phi \cdot o$ semisimple symmetric space with rank equal to $|\Phi|$, totally geodesic in M, boundary component of M w.r.t. maximal Satake compactification
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- $M = B_\Phi \times \mathbb{E}^{r-|\Phi|} \times N_\Phi$ (**horospherical decomposition**)

Canonical extension

Basic example: Extension of S^1-action on $\mathbb{R}^2$ to $S^1 \times \mathbb{R}$-action on $\mathbb{R}^3$

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Rank reduction - Such a cohomogeneity one action can be constructed by a CANONICAL EXTENSION OF A COHOMOGENEITY ONE ACTION ON A BOUNDARY COMPONENT

Nilpotent construction

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of Λ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

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Assume that

- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$
- $N_{L_j}^o(\mathfrak{n}_{j,\mathfrak{v}}) = \theta N_{L_j}^o(\mathfrak{v})$ acts transitively on $F_j = B_j \times \mathbb{E}$
- $N_{L_j \cap K}^o(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$ if $\dim \mathfrak{v} \geq 2$

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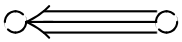
Then

$$H_{j,\mathfrak{v}} = N_{L_j}^o(\mathfrak{n}_{j,\mathfrak{v}}) N_{j,\mathfrak{v}} \text{ acts on } M \text{ with cohomogeneity one}$$

Nilpotent construction - An example (I)

- $M = G_2^2/SO_4$, dim $M = 8$, $\operatorname{rank} M = 2$
- root system Σ is of type G_2:



α_1 α_2

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- $\mathfrak{n}_1^1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^2$
- $\mathfrak{n}_1^2 = \mathfrak{g}_{2\alpha_1+\alpha_2} \cong \mathbb{R}$
- $\mathfrak{n}_1^3 = \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \cong \mathbb{R}^2$
- $\mathfrak{n}_1 = \mathfrak{n}_1^1 \oplus \mathfrak{n}_1^2 \oplus \mathfrak{n}_1^3$

Nilpotent construction - An example (II)

- $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$
- $\mathfrak{n}_1^1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^2$
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- $H_{1,\mathfrak{n}_1^1}$ acts on G_2^2/SO_4 with cohomogeneity one and singular orbit $H_{1,\mathfrak{n}_1^1} \cdot o$ with codimension 2

Nilpotent construction - rank $M = 1$

M	G	K	K_0	$\mathfrak{g}_\alpha$	$\mathfrak{n}$
$\mathbb{R}H^n$	$SO^o_{1,n}$	SO_n	SO_{n-1}	$\mathbb{R}^{n-1}$	$\mathbb{R}^{n-1}$
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	$\mathbb{C}^{n-1}$	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	$\mathbb{H}^{n-1}$	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
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<u>PROBLEM:</u> Find all k-dimensional ($k \geq 2$) linear subspaces $\mathfrak{v}$ of $\mathfrak{g}_\alpha$ for which there exists a subgroup of K_0 acting transitively on the unit sphere in $\mathfrak{v}$

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- $\mathbb{H}$: some linear subspaces $\mathfrak{v} \subset \mathbb{H}^{n-1}$ with constant quaternionic Kähler angle (no complete classification)

Open problems

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2. Two cohomogeneity one actions are known which can be constructed with the "nilpotent" method but not with any other method: one in G_2^2/SO_4, the other one in $G_2^{\mathbb{C}}/G_2$. Are there other examples?