ZEROS OF CONFORMAL VECTOR FIELDS IN ANY METRIC SIGNATURE

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`http://www.math.ohio-state.edu/~andrzej/santiago.pdf`

CONFORMAL VECTOR FIELDS

A vector field v on a pseudo-Riemannian manifold (M, g) of dimension $n \geq 2$ is called *conformal* if its local flow consists of conformal diffeomorphisms.

Equivalently, for some $\phi : M \to \mathrm{I\!R}$,

$$\mathcal{L}_v g \;=\; \phi g\,, \quad \text{that is,} \quad v_{j,k} + v_{k,j} = \phi g_{jk}\,,$$

Thus, $\operatorname{div} v = n\phi/2$.

Example: *Killing fields* v, characterized by $\phi = 0$.

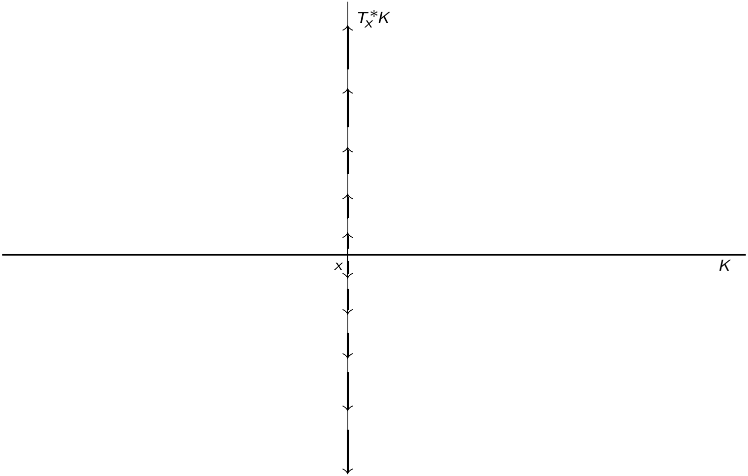
RIEMANN EXTENSIONS

E. Calviño-Louzao, E. García-Río, P. Gilkey, R. Vázquez-Lorenzo:
The geometry of modified Riemannian extensions
(Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 2009).

Let D be a connection on a manifold K (of any dimension).
We denote by $\pi : T^*K \to K$ the bundle projection of the cotangent bundle of K.
The Patterson-Walker *Riemann extension metric* on $M = T^*K$ is the neutral-signature metric g^{D} defined by requiring that

- all vertical and all D-horizontal vectors be g^{D}-null, while
- $g^{\mathrm{D}}_y(\xi, w) = \xi(d\pi_y w)$ for any $y \in M$, any vertical vector $\xi \in \mathrm{Ker}\, d\pi_y = T^*_x K$, with $x = \pi(y)$, and any $w \in T_y M$.

THE RADIAL VECTOR FIELD v ON T^*K



The radial field v is conformal for any Riemann extension metric.

SUMMARY

The connected components of the zero set of any conformal vector field, in a pseudo-Riemannian manifold of arbitrary signature, are totally umbilical conifold varieties, that is, smooth submanifolds except possibly for some quadric singularities.

The singularities occur only when the metric is indefinite, including the Lorentzian case.

Preprint: *Zeros of conformal fields in any metric signature*, at

`http://www.math.ohio-state.edu/~andrzej/preprints/zcf.pdf`

(more recent than the arXiv version: references updated).

BACKGROUND

- Kobayashi (1958): for a Killing field v on a Riemannian manifold (M, g), the connected components of the zero set of v are mutually isolated totally geodesic submanifolds of even codimensions.

- Blair (1974): if M is compact, this remains true for conformal vector fields, as long as one replaces the word ‘geodesic’ by ‘umbilical’ and the codimension clause is relaxed in the case of one-point connected components.

- Belgun, Moroianu and Ornea (2010, arXiv:1002:0482v3, to appear in J. Geom. Phys. 61, no. 3, 2011): Blair's conclusion holds without the compactness hypothesis.

LINEARIZABILITY

- The last result is also a direct consequence of the following theorem of Frances (2009, arXiv:0909:0044v2): at any zero z, a conformal field is linearizable unless z has a conformally flat neighborhood.

- Frances and Melnick (2010, arXiv:1008.3781): the above statement is true in real-analytic Lorentzian manifolds as well.

- Leitner (1999): in Lorentzian manifolds, zeros of a conformal field with certain additional properties lie, locally, in a null geodesic.

THE SIMULTANEOUS KERNEL

Z always denotes the *zero set* of a given conformal field v.

If $z \in Z$, we use the symbol

$$\mathcal{H}_z = \mathrm{Ker}\,\nabla v_z \cap \mathrm{Ker}\,d\phi_z$$

for the *simultaneous kernel*, at z, of the differential $d\phi$ and the bundle morphism $\nabla v : TM \to TM$ (which sends each vector field w to $\nabla_w v$).

When z is fixed, we also write H instead of $\mathcal{H}_z$.

NONESSENTIAL ZEROS

Given a conformal vector field v on a pseudo-Riemannian manifold (M, g), a point $z \in M$ is said to be *essential* for v if no conformal change of g on any neighborhood of z turns v into Killing field for the new metric.

If $z \in Z$ is a nonessential zero of v, we may assume that v is a Killing field (by changing the metric conformally near z). Thus, $z \in Z$ has a neighborhood U' in M such that, for some star-shaped neighborhood U of 0 in $T_z M$, the exponential mapping $\exp_z$ is a diffeomorphism $U \to U'$ and

$$Z \cap U' = \exp_z[H \cap U].$$

Here $H = \mathcal{H}_z = \mathrm{Ker}\, \nabla v_z$, since $\phi = 0$.

MAIN RESULT

THEOREM: *Let* Z *be the zero set of a conformal vector field* v *on a pseudo-Riemannian manifold* (M,g) *of dimension* $n \geq 3$.

If $z \in Z$ *is an essential zero of* v, *then*

$$Z \cap U' \;=\; \exp_z[C \cap H \cap U],$$

for any sufficiently small star-shaped neighborhood U *of* 0 *in* T_zM *mapped by* $\exp_z$ *diffeomorphically onto a neighborhood* U' *of* z *in* M, *where* $C = \{u \in T_zM : g_z(u,u) = 0\}$ *is the null cone.*

SINGULARITIES OF THE ZERO SET Z

In other words:

The zero set Z is, near any essential zero z, the $\exp_z$-image of the null cone in the simultaneous kernel H.

Consequently:

The singular subset Δ of $Z \cap U'$ equals $\exp_z[H \cap H^\perp \cap U]$, if the metric restricted to H is not semidefinite, and $\Delta = \emptyset$ otherwise.

THE CONNECTED COMPONENTS OF Z

The connected components of $(Z \cap U') \smallsetminus \Delta$ are totally umbilical submanifolds of (M, g), and their codimensions are even unless $\Delta = \emptyset$ and $Z \cap U'$ is a null totally geodesic submanifold.

In addition, ϕ is constant along each connected component of Z.

WHY TOTALLY UMBILICAL

Let b *be the second fundamental form of a submanifold* K *in a manifold* M *endowed with a torsionfree connection* ∇*.*

If $z \in M$*, a neighborhood* U *of* 0 *in* $T_z M$ *is mapped by* $\exp_z$ *diffeomorphically onto a neighborhood of* z *in* M*, and* $K = \exp_z[V \cap U]$ *for a vector subspace* V *of* $T_z M$*, then* $b_z = 0$*.*

THE CONFORMAL-FIELD CONDITION, REWRITTEN

We always denote by $t \mapsto x(t)$ a geodesic of (M, g), by $\dot{x} = \dot{x}(t)$ its velocity, and write $\dot{f} = d[f(x(t))]/dt$, $\ddot{f} = d^2[f(x(t))]/dt^2$ for vector-valued functions f on M.

The equality $v_{j,k} + v_{k,j} = \phi g_{jk}$ means that, along every geodesic,

$$\langle v, \dot{x}\rangle^{\cdot} = \phi\langle \dot{x}, \dot{x}\rangle,$$

and, rewritten as $\nabla v + [\nabla v]^* = \phi\,\mathrm{Id}$, is obviously equivalent to

$$2\nabla v \;=\; A \;+\; \phi\,\mathrm{Id} \quad \text{with} \quad A^* = -A.$$

Here $A = \nabla v - [\nabla v]^*$ is twice the skew-adjoint part of ∇v.

IDENTITIES RELATED TO THE CARTAN CONNECTION

If $t \mapsto u(t) \in T_{x(t)}M$ and $\nabla_{\dot{x}} u = 0$, one has

$$\begin{aligned} 2\nabla_{\dot{x}}\nabla_u v &= 2R(v \wedge \dot{x})u + [(d\phi)(u)]\dot{x} + \dot{\phi}u - \langle \dot{x}, u\rangle \nabla\phi, \\ (1 - n/2)[(d\phi)(u)]^{\boldsymbol{\cdot}} &= S(u, \nabla_{\dot{x}} v) + S(\dot{x}, \nabla_u v) + [\nabla_v S](u, \dot{x}), \end{aligned}$$

$S = \mathrm{Ric} - (2n-2)^{-1}\,\mathrm{Scal}\,g$ being the Schouten tensor. Thus,

$$\begin{aligned} \nabla_{\dot{x}}\nabla_{\dot{x}} v &= R(v \wedge \dot{x})\dot{x} + \dot{\phi}\dot{x} - \langle \dot{x}, \dot{x}\rangle \nabla\phi/2, \\ (1 - n/2)\ddot{\phi} &= 2S(\dot{x}, \nabla_{\dot{x}} v) + [\nabla_v S](\dot{x}, \dot{x}), \end{aligned}$$

Hence: *if the geodesic is null and* v, $\nabla_{\dot{x}} v$, $\dot{\phi}$ *vanish for some* t, *then they vanish for every* t.

ONE INCLUSION – FOR FREE

Once again: *if the geodesic is null and* v, $\nabla_{\dot{x}} v$, $\dot{\phi}$ *vanish for some* t, *then they vanish for every* t.

Therefore, for any zero z of v, essential or not,

$$\exp_z[C \cap H \cap U] \subseteq Z \cap U',$$

where $H = \mathrm{Ker}\,\nabla v_z \cap \mathrm{Ker}\, d\phi_z$, that is,

the $\exp_z$*-image of the null cone in the simultaneous kernel* H *always consists of zeros of* v.

The clause about constancy of ϕ will now follow immediately, once the above inclusion is shown to be an equality.

INTERMEDIATE SUBMANIFOLDS

Given a zero z of a section ψ of a vector bundle $\mathcal{E}$ over a manifold M, we denote by $\partial\psi_z$ the linear operator $T_zM \to \mathcal{E}_z$ with the components $\partial_j\psi^a$. (Thus, $\partial\psi_z = \nabla\psi_z$ if ∇ is a connection in $\mathcal{E}$.)

A trivial consequence of the rank theorem:
All zeros of ψ near z then lie in a submanifold $N \subseteq M$ such that $T_zN = \operatorname{Ker}\partial\psi_z$ and $\operatorname{Ker}\partial\psi_x \subseteq T_xN$ for all $x \in N$ with $\psi_x = 0$.

Note that the zero set Z of ψ can, in general, be any closed subset of M. An *intermediate submanifold* N chosen as above provides some measure of control over Z.

CONNECTING LIMITS

Whenever M is a manifold, $z \in M$, and $L \subseteq T_zM$ is a line through 0, while $x_j, y_j \in M$, $j = 1, 2, \ldots$, are sequences converging to z with $x_j \neq y_j$ whenever j is sufficiently large, let us call L a *connecting limit* for this pair of sequences if some norm $|\,|$ in T_zM and some diffeomorphism Ψ of a neighborhood of 0 in T_zM onto a neighborhood of z in M have the property that $\Psi(0) = z$ and $d\Psi_0 = \mathrm{Id}$, while the limit of the sequence $(w_j - u_j)/|w_j - u_j|$ exists and spans L, the vectors u_j, w_j being characterized by $\Psi(u_j) = x_j$, $\Psi(w_j) = y_j$ for large j.

RADIAL LIMIT DIRECTIONS

For M, z and x_j, y_j as above, neither L itself nor the fact of its existence depends on the choice of $|\,|$ and Ψ.

In the case where $N \subseteq M$ is a submanifold, both sequences x_j, y_j lie in N, and L is their connecting limit, one has $L \subseteq T_z N$.

By a *radial limit direction* of a subset $Z \subseteq M$ at a point $z \in M$ we mean a connecting limit of for a pair of sequences as above, of which one is constant and equal to z, and the other lies in Z.

BEIG'S THEOREM (1992)

$z \in Z$ is nonessential if and only if

$\phi(z) = 0$ and $\nabla\phi_z \in \nabla v_z(T_z M)$.

In other words: *$z \in Z$ is essential if and only if*

either $\phi(z) \neq 0$, or $\phi(z) = 0$ and $\nabla\phi_z \notin \nabla v_z(T_z M)$.

A proof can be found in a 1999 paper by Capocci.

CASE I: $\phi(z) \neq 0$

Choose U, U' so that $\phi \neq 0$ everywhere in U'. For $x \in (Z \cap U') \smallsetminus \{z\}$, let $L_x = T_z \Gamma_x$ be the initial tangent direction of of the geodesic segment Γ_x joining z to x in U'.

Recall that

$$\langle v, \dot{x} \rangle^{\boldsymbol{\cdot}} = \phi \langle \dot{x}, \dot{x} \rangle ,$$

and so Γ_x is null.

CASE I: $\phi(z) \neq 0$ (CONTINUED)

Next, for U, U' small enough, $L_x \subseteq \mathrm{Ker}\,\nabla v_z$.

In fact, Γ_x is rigid. Hence v is tangent to Γ_x, and $L_x \subseteq \mathrm{Ker}\,(\nabla v_z - \lambda_x \mathrm{Id})$ for some eigenvalue λ_x.

Now, if we had $\lambda_x \neq 0$ for some sequence $x \in (Z \cap U') \smallsetminus \{z\}$ converging to z, passing to a suitable subsequence such that $L_x \to L$ for some L we would get $\lambda_x = \lambda$ (independent of x), and a contradiction would ensue: $L \subseteq T_z N = \mathrm{Ker}\,\partial\psi_z = \mathrm{Ker}\,\nabla v_z$, where N is an intermediate submanifold for $\psi = v$ and z.

CASE I: $\phi(z) \neq 0$ (STILL)

Furthermore, as $2\nabla v = A + \phi\,\mathrm{Id}$ with $A^* = -A$, it follows that

both $\mathrm{Ker}\,\nabla v_z$ and $H \subseteq \mathrm{Ker}\,\nabla v_z$ are null subspaces of $T_z M$.

If $\mathrm{Ker}\,\nabla v_z \subseteq \mathrm{Ker}\,d\phi_z$, so that $H = \mathrm{Ker}\,\nabla v_z$, the one inclusion we already have completes the proof.

CASE I: $\phi(z) \neq 0$ (FINAL STEP)

Therefore, assume that $\mathrm{Ker}\,\nabla v_z$ is *not* contained in $\mathrm{Ker}\,d\phi_z$. Thus, $K = \exp_z[H \cap U]$ is a codimension-one submanifold of $N = \exp_z[\mathrm{Ker}\,\nabla v_z \cap U]$, while the restriction of ϕ to N has a nonzero differential at z, and $\phi = \phi(z)$ on K. Making U, U' smaller, we ensure that $\phi \neq \phi(z)$ everywhere in $N \smallsetminus K$. This shows that no zero x of v lies in $N \smallsetminus K$, for the existence of one would result in a contradiction: we have

$\nabla_{\dot{x}}\nabla_{\dot{x}}(v \wedge \dot{x}) = [R(v \wedge \dot{x})\dot{x}] \wedge \dot{x}$ (for null geodesics) and
$\nabla_{\dot{x}}\nabla_{\dot{x}} v = \dot{\phi}\dot{x}$ (for null geodesics to which v is tangent);

integrating the latter, one obtains $\nabla_{\dot{x}} v = [\phi - \phi(z)]\dot{x}$.

CASE II: $\phi(z) = 0$ **AND** $\nabla\phi_z \notin \nabla v_z(T_zM)$

SUBCASE II-a: in addition, $\mathrm{Ker}\,\nabla v_z$ is not null.

For $K = \exp_z[H \cap H^\perp \cap U]$ and any $x \in K$:

the parallel transport from z to x sends the simultaneous kernel $H = \mathrm{Ker}\,\nabla v_z \cap \mathrm{Ker}\,d\phi_z$ onto $\mathcal{H}_x = \mathrm{Ker}\,\nabla v_x \cap \mathrm{Ker}\,d\phi_x$,

while

$\dim \mathcal{H}_x$ *is independent of $x \in K$*, and

if $\phi(z) = 0$, both $\mathrm{rank}\,\nabla v_x$ *and* $\dim \mathrm{Ker}\,\nabla v_x$ *are constant as functions of $x \in K$.*

SUBCASE II-a, PROOF OF THE ABOVE CLAIMS

From the "inclusion for free" and the second order identities related to the Cartan connection:

$$2\nabla_{\dot{x}}\nabla_u v = [(d\phi)(u)]\,\dot{x}, \qquad (1 - n/2)[(d\phi)(u)]^{\cdot} = S(\dot{x}, \nabla_u v).$$

Uniqueness of solutions: the parallel transport sends $H = \mathcal{H}_z$ INTO $\mathcal{H}_x$. Now 'ONTO' follows as $\dim \mathcal{H}_x \leq \dim \mathcal{H}_z$ (semicontinuity). Thus, for $x \in K$ and $p_x = \dim \operatorname{Ker} \nabla v_x$,

$$p_z - 1 \leq p_x \leq p_z.$$

As $\phi(x) = 0$, the codimension $n - p_x$ is even (note that $2\nabla v = A + \phi\,\mathrm{Id}$ with $A^* = -A$). Hence $p_x = p_z$.

SETS OF CONNECTING LIMITS

Suppose that M is a manifold, $X, Y \subseteq M$, and $z \in M$.

We denote by $\mathbb{L}_z(X, Y)$ the set of all connecting limits for pairs x_j, y_j of sequences in X and, respectively, Y, converging to z, with $x_j \neq y_j$ for all j.

For instance:

$\mathbb{L}_z(\{z\}, Z)$ is the set of all radial limit directions of a subset $Z \subseteq M$ at a point $z \in M$.

INTERMEDIATE SUBMANIFOLDS REVISITED

As before: we are given a zero z of a section ψ of a vector bundle $\mathcal{E}$ over a manifold M.

For $r = \mathrm{rank}\, \partial\psi_z$, we choose an r-dimensional real vector space W and a base-preserving bundle morphism $G : \mathcal{E} \to M \times W$ such that $G_z : \partial\psi_z(T_zM) \to W$ is an isomorphism. Now we amy set $N = U \cap F^{-1}(0)$ for a suitable neighborhood U of z in M and $F : M \to W$ defined by $F(x) = G_x\psi_x$.

If ξ is a section of $\mathcal{E}^*$ and $\partial\psi_z(T_zM) \subseteq \mathrm{Ker}\,\xi_z$, then $Q = \xi(\psi) : N \to \mathrm{I\!R}$ has a critical point at z with the Hessian of Q characterized by $\partial dQ_z(u,u) = \xi\,([\nabla_u(\nabla\psi)]u)$.

SUBCASE II-a CONTINUED

Recall: this means that

$\phi(z) = 0, \quad \nabla\phi_z \notin \nabla v_z(T_z M), \quad \mathrm{Ker}\,\nabla v_z$ not null.

Fix a section w of the bundle $\mathrm{Ker}\,\nabla v$ over $K = \exp_z[H \cap H^\perp \cap U]$ lying outside the subbundle $\mathrm{Ker}\,\nabla v \cap \mathrm{Ker}\, d\phi$, and apply the intermediate submanifold construction to $\psi = v$, $\mathcal{E} = TM$ and $\xi = 2g(w,\cdot\,)$.

Then $Q = 2g(w,v) : N \to \mathbb{R}$ has, at z, the Hessian

$$\partial dQ \;=\; d\phi \otimes g(w,\cdot\,) + g(w,\cdot\,) \otimes d\phi \,-\, [d\phi(w)]g\,.$$

THE MORSE-BOTT LEMMA

Given a manifold N, *a submanifold* $K \subseteq N$, *a function* $Q : N \to \mathrm{I\!R}$, *and a point* $z \in K \cap Q^{-1}(0)$, *let* $dQ = 0$ *on* K, *and let* $\mathrm{rank}\, \partial dQ_z \geq \dim N - \dim K$.

Then, for some diffeomorphism Ψ *between neighborhoods* U *of* 0 *in* $T_z N$ *and* U' *of* z *in* N, *such that* $\Psi(0) = z$ *and* $d\Psi_0 = \mathrm{Id}$, *the composition* $Q \circ \Psi$ *equals the restriction to* U *of the quadratic function of* ∂dQ_z.

Consequently, $U' \cap Q^{-1}(0) = \Psi(C \cap U)$ *and* $K \cap U' = \Psi(V \cap U)$, *where* $C, V \subseteq T_z M$ *are the null cone and nullspace of* ∂dQ_z.

QUADRICS

Given a subset Z of a manifold N, and a point $z \in Z$, and a symmetric bilinear form $(\,,\,)$ in T_zM, we say that Z is a *quadric at* z *in* N *modelled on* $(\,,\,)$ if some diffeomorphism Ψ between neighborhoods of 0 in T_zN and of z in N, with $\Psi(0) = z$ and $d\Psi_0 = \mathrm{Id}$, makes Z, (near z) correspond to the null cone of $(\,,\,)$ (near 0). For instance:

- the conclusion of the Morse-Bott lemma states, in particular, that $Q^{-1}(0)$ is a quadric at z in N, modelled on ∂dQ_z,

- our Main Theorem implies that the zero set Z is a quadric at z in $\exp_z[H \cap U]$, modelled on the restriction of g_z to H.

CONSEQUENCES OF THE MORSE-BOTT LEMMA

In Subcase II-a, one has the equality

$$Z \cap \phi^{-1}(0) \cap U' \;=\; \exp_z[C \cap H \cap U].$$

Secondly, lying in H but not in $H \cap H^{\perp}$ is forbidden for connecting limit between $Z \smallsetminus \phi^{-1}(0)$ and K:

$$\mathbb{L}_z(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H) \;\subseteq\; \mathbb{P}(H \cap H^{\perp}),$$

where $\mathbb{P}(\;)$ is the projective-space functor. Note:

$H \cap H^{\perp} = T_z K$ and $T_z(N \cap \phi^{-1}(0)) = H$ is a codimension-one subspace of $\operatorname{Ker} \nabla v_z = T_z N$.

PROOF OF THE FIRST RELATION

For suitably chosen w, both $Q = 2g(w, v) : N \to \mathrm{I\!R}$ and the restriction of Q to $N \cap \phi^{-1}(0)$ satisfy, along with our z and $K = \exp_z[H \cap H^\perp \cap U]$, the hypotheses of the Morse-Bott lemma.

(FINALLY, the assumption "Ker ∇v_z not null" is used!)

So:

$$Z \cap \phi^{-1}(0) \cap U' \; = \; \exp_z[C \cap H \cap U],$$

since two quadrics modelled on the same symmetric bilinear form, such that one contains the other, must, essentially, coincide.

OUTLINE OF PROOF OF THE SECOND RELATION

The Morse-Bott lemma for Q, N and K allows us to identify Q with the quadratic function of a direct-sum symmetric bilinear form on $W \oplus V$, where the summand form on W is nondegenerate and that on V is zero.
If $L \in \mathbb{L}_z(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H)$, we have the convergence

$$\frac{s_j u_j + x_j - y_j}{|s_j u_j + x_j - y_j|} \to c\,u + x \in L \text{ as } j \to \infty,$$

for a fixed Euclidean sphere $\Sigma \subseteq W$, a neighborhood K of 0 in V, some $u_j, u \in \Sigma$, $s_j \in \mathrm{I\!R}$ and $x_j, y_j \in K$ with $u_j \to u$ and $|s_j| + |x_j| + |y_j| \to 0$. From the Hessian formula at the bottom of p. 28, $d\phi_z(u) \neq 0$. Hence $c = 0$, which proves that $L \in \mathbb{P}(H \cap H^\perp)$.

THE CRUCIAL IMPLICATION

In Subcase II-a, the inclusion

$$\mathbb{L}_z(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H) \;\subseteq\; \mathbb{P}(H \cap H^\perp)$$

implies, BY ITSELF, that

$$Z \cap U' \subseteq \phi^{-1}(0).$$

Here is why.

NONVANISHING OF ϕ

First, for a fixed positive-definite metric h, any $y \in U' \smallsetminus K$ is joined by a "rigid" g-geodesic segment $\varGamma_y$ to a point $p_y \in K$ is such a way that that $\varGamma_y$ is h-normal to K at p_y. Now:

if $y \in (Z \cap U') \smallsetminus \phi^{-1}(0)$, *then* $\phi \neq 0$ *everywhere in* $\varGamma_y \smallsetminus \{y, p_y\}$.

For, otherwise, a subsequence of a sequence of points y falsifying this claim and converging to z would produce, as the limit of their $T_y\varGamma_y$, an element L of $\mathbb{L}_z(Z \smallsetminus \phi^{-1}(0), K) \cap \mathbb{P}(H)$, and hence of $\mathbb{P}(H \cap H^\perp)$, which cannot happen as L would also be h-orthogonal to $H \cap H^\perp = T_z K$.

PROOF OF THE CRUCIAL IMPLICATION, CONTINUED

Next, whenever a conformal vector field v is tangent to a null geodesic segment $\mathit{\Gamma}$, so that $x(0) = y$ and $\nabla_{\dot{x}} v = \lambda \dot{x}$ at $t = 0$ for some $y \in M$ and $\lambda \in \mathrm{I\!R}$, we have

- $\nabla_{\dot{x}} v = [\lambda + \phi - \phi(y)]\dot{x}$ *along* $\mathit{\Gamma}$,

- ∇v *restricted to* $\mathit{\Gamma}$ *descends to a parallel section of* $\mathfrak{conf}\,[(T\mathit{\Gamma})^\perp/(T\mathit{\Gamma})]$ *and has the same characteristic polynomial at all points of* $\mathit{\Gamma}$, *if, in addition,* ϕ *is constant along* $\mathit{\Gamma}$.

To see this, it suffices to integrate the equality $\nabla_{\dot{x}} \nabla_{\dot{x}} v = \dot{\phi}\dot{x}$ (see the final step of Case I), and, respectively, use the first one of the second-order identities related to the Cartan connection.

PROOF OF THE CRUCIAL IMPLICATION, FINAL STEP

We prove that $Z \cap U' \subseteq \phi^{-1}(0)$ by contradiction. Suppose that some points $y \in Z \cap U'$ with $\phi(y) \neq 0$ form a sequence converging to z. Our Γ_y are tangent to v, so (see p. 36) $\nabla_{\dot{x}} v = [\lambda + \phi - \phi(y)]\dot{x}$, $x(0) = y$, $x(1) = p_y$, where λ may depend on y, but not on the curve parameter t. Thus, $\dot{x}(1)$ is an eigenvector of ∇v at p_y for the eigenvalue $\lambda_y = \lambda - \phi(y)$. Constancy of the spectrum of ∇v along Γ (see p. 36) implies that λ_y is an eigenvalue of ∇v_z and, as the limit L of any convergent subsequence of the directions $T_y \Gamma_y$ must lie in $T_z N = \mathrm{Ker}\, \nabla v_z$, we eventually have $\lambda_y = 0$, that is, $\lambda = \phi(y)$. The equality $\nabla_{\dot{x}} v = [\lambda + \phi - \phi(y)]\dot{x}$ now becomes $\nabla_{\dot{x}} v = \phi \dot{x}$, and Rolle's theorem contradicts the conclusion about nonvanishing of ϕ on p. 35.

SUBCASE II-a WRAPPED UP

The inclusion on p. 31 combined with the crucial implication (p. 34) shows that $\phi(x) = 0$ for every $x \in Z$, near z.

The equality on p. 31 now proves the assertion of the Main Theorem in Subcase II-a.

CASE II: $\phi(z) = 0$ AND $\nabla\phi_z \notin \nabla v_z(T_z M)$

SUBCASE II-b: in addition, $\mathrm{Ker}\,\nabla v_z$ is null.

Since $\mathrm{Ker}\,\nabla v_z$ is null, so is $H \subseteq \mathrm{Ker}\,\nabla v_z$. Hence $H = H \cap H^\perp$ and the inclusion on p. 31 is satisfied trivially. The crucial implication (p. 34) now gives $Z \cap U' \subseteq \phi^{-1}(0)$.

We choose an intermediate submanifold N containing $K = \exp_z[H \cap H^\perp \cap U]$ (that is, $K = \exp_z[H \cap U]$) as a codimension-one submanifold.

Since $T_z N \cap \mathrm{Ker}\, d\phi_z = T_z K$, it follows that $U' \cap \phi^{-1}(0) \subseteq K$, and so $Z \cap U' \subseteq K$.
This completes the proof of the Main Theorem.

EXAMPLE: COMPACTIFIED RIEMANN EXTENSIONS

Let E be a neutral-signature pseudo-Euclidean space of even dimension $n+2$. Any fixed endomorphism $B \in \mathfrak{so}(E)$ with eigenvalues $c, -c \neq 0$ and null $[(n+2)/2]$-dimensional eigenspaces constitutes a conformal vector field v on the projective null cone ("Einstein universe") of E, with its standard conformally flat conformal structure.

The two connected components of the zero set Z of v are copies of the projective space $\mathrm{I\!RP}^{n/2}$, and in a tubular neighborhood of each of them v is diffeomorphically equivalent to a radial vector field in a Riemann extension manifold.

A EUCLIDEAN DESCRIPTION OF THE SAME

Up to a finite covering, an equivalent description of the above is the product manifold $S^{n/2}_{\mathrm{can}} \times S^{n/2}_{-\mathrm{can}}$ (where can is the canonical metric of constant curvature 1), with the vector field v given by

$$v_{(x,y)} = (y - \langle y, x\rangle x, x - \langle x, y\rangle y),$$

$S^{n/2}_{\mathrm{can}}$ being treated here as the unit sphere in a Euclidean space with the inner product $\langle\,,\rangle$.