Conference in Geometry and Global Analysis
Celebrating P. Gilkey's 65th Birthday

Principal curvatures of isoparametric hypersurfaces in complex hyperbolic spaces

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Isoparametric hypersurfaces

$(\bar{M}, \langle\,,\,\rangle)$ Riemannian manifold
$\bar{\nabla}$ Levi-Civita connection of $\bar{M}$

$M \subset \bar{M}$ hypersurface
ξ unit normal vector field
∇ Levi-Civita connection of M

Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + II(X,Y)$$

Weingarten formula

$$\bar{\nabla}_X \xi = -\mathcal{S}X$$

$\langle II(X,Y), \xi \rangle = \langle \mathcal{S}X, Y \rangle$

$\mathcal{S}$ selfadjoint ⇒ $\mathcal{S}$ diagonalizable ⇒ principal curvatures of M: eigenvalues of $\mathcal{S}$
$\lambda_1, \ldots, \lambda_{n-1}$

Mean curvature of M ⇒ $H = \mathrm{tr}(\mathcal{S}) = \lambda_1 + \cdots + \lambda_{n-1}$

Number of *distinct* principal curvatures of M ⇒ g

Isoparametric hypersurfaces

$\Phi(x,y,z,t)$ wave function in $\mathbb{R}^3$ → wave equation $\Delta\Phi = \frac{\partial^2\Phi}{\partial t^2}$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

t_0 fixed instant → $\left\{(x,y,z) \in \mathbb{R}^3 : \Phi(x,y,z,t_0) = c(t_0)\right\}$

wavefront: points in $\mathbb{R}^3$ with the same phase or oscillating state at the instant t_0

Let us consider time-independent wavefronts

$$\Delta\Phi(x,y,z,t_0) = \frac{\partial^2\Phi}{\partial t^2}(x,y,z,t_0) = c''(t_0) \qquad f(x,y,z) = \Phi(x,y,z,t_0)$$

$\Delta f = \Delta\Phi(\cdot,\cdot,\cdot,t_0)$ is constant along the level sets of $f = \Phi(\cdot,\cdot,\cdot,t_0)$

We want to consider waves with parallel wavefronts

$|\nabla f| = |\nabla\Phi(\cdot,\cdot,\cdot,t_0)|$ is constant along the level sets of $f = \Phi(\cdot,\cdot,\cdot,t_0)$

↓

The possible wavefronts are level sets of functions f such that $|\nabla f|^2$ and Δf are constant along those level sets

Isoparametric hypersurfaces

$(\bar{M}, \langle\, ,\, \rangle)$ Riemannian manifold

$f : \bar{M} \to \mathbb{R}$ isoparametric function $\Leftrightarrow$ $|\nabla f|^2$ and Δf are constant along the level sets of f

$M \subset \bar{M}$ isoparametric hypersurface $\Leftrightarrow$ M is a codimension 1 level set of an isoparametric function

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Theorem (Cartan)

$(\bar{M}, \langle\, ,\, \rangle)$ Riemannian manifold

$M \subset \bar{M}$ isoparametric hypersurface $\Leftrightarrow$ M and its nearby parallel hypersurfaces have constant mean curvature (CMC)

Theorem (Cartan)

$(\bar{M}, \langle\, ,\, \rangle)$ real space form: $\mathbb{R}^n$, $\mathbb{R}H^n$ ou $\mathbb{S}^n$

$M \subset \bar{M}$ isoparametric hypersurface $\Leftrightarrow$ M has constant principal curvatures

Isoparametric hypersurfaces

- Homogeneous hypersurfaces

  A hypersurface $M \subset \bar{M}$ is (extrinsically) homogeneous if

  $M = G \cdot o$ where $o \in M$ and G is a subgroup of $I(\bar{M})$

  $G \times \bar{M} \to \bar{M}$ is a cohomogeneity one isometric action

- Objectives
  - classify homogeneous hypersurfaces of a manifold
    - $\mathbb{R}^n$ ⟹ Segre
    - $\mathbb{R}H^n$ ⟹ Cartan
    - S^n ⟹ Hsiang, Lawson
    - $\mathbb{C}P^n$ ⟹ Takagi
    - $\mathbb{C}H^n$ ⟹ Berndt, Tamaru
  - characterize homogenous hypersurfaces using geometric data

M homogeneous hypersurface ⟹
- M has constant principal curvatures
- M is isoparametric

?

Isoparametric hypersurfaces

- The Euclidean space $\mathbb{R}^n$

  Levi-Civita: $\mathbb{R}^3$
  Segre: $\mathbb{R}^n$
  → affine hyperplanes $\mathbb{R}^{n-1}$
  spheres S^{n-1}
  products $S^k \times \mathbb{R}^{n-k-1}$

- The hyperbolic space $\mathbb{R}H^n$

  Cartan →
  geodesic hyperspheres
  horospheres
  tot. geod. $\mathbb{R}H^{n-1}$ and equidistant hypersurfaces
  tubes around tot. geod. subspaces $\mathbb{R}H^k$

- The sphere S^n

  Cartan: classified $g = 1, 2, 3$
  Hsiang, Lawson: classified homogeneous hypersurfaces
  Münzner: proved $g \in \{1, 2, 3, 4, 6\}$ and $m_i = m_{i+1}(\text{mod } 2)$
  Ferus, Karcher, Münzner: inhomogeneous examples
  Stolz: the possible triples (g, m_1, m_2) with $g = 4$ agree with those of the known homogeneous and inhomogeneous examples
  Progress in cases $g = 4, 6$: Abresch, Dorfmeister, Neher, Cecil, Chi, Jensen, Immervoll, Miyaoka...

Isoparametric hypersurfaces

Isoparametric hypersurface	=	Hypersurface with constant principal curvatures	$\mathbb{R}^n, \mathbb{R}H^n, \mathbb{S}^n$
Isoparametric hypersurface	≠	Hypersurface with constant principal curvatures	in general

Remark. Both conditions include the notion of a homogeneous hypersurface

Question. In which ambient spaces can homogeneous hypersurfaces be characterized by one of these concepts?

Remark. To our knowledge, the only known isoparametric hypersurfaces in Riemannian symmetric spaces which are not homogenous, or which do not have constant principal curvatures, are related to the FKM examples in spheres

D. Ferus, H. Karcher, H. F. Münzner: Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), no. 4, 479–502

The problem in CP^n and CH^n

- Complex space forms

Complex hyperbolic space $\mathbb{C}H^n$	Euclidean space $\mathbb{C}^n$	Complex projective space $\mathbb{C}P^n$
$c < 0$	$c = 0$	$c > 0$
Bergman metric	flat metric	Fubini-Study metric

- Real hypersurfaces

  M real hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$

  ξ unit normal field

  $J\xi$ Hopf vector field ⟹ tangent to M

h number of nontrivial projections of $J\xi$ onto the principal curvature spaces ($h \leq g$)

M Hopf ⟺ $J\xi$ eigenvector of $\mathcal{S}$ ⟺ $h = 1$

The problem in CP^n and CH^n

State of the problem in $\mathbb{C}P^n$

h \ g	1	2	3	4	5	> 5
1	✗ Tashiro Tachibana	✓ Takagi	✓ Kimura	✗	✓	✗
2		✗	✗ Takagi Wang	✗	✗	✗
3			✗			
4						
5						
> 5						

State of the problem in $\mathbb{C}H^n$

h \ g	1	2	3	4	5	> 5
1	✗ Tashiro Tachibana	✓ Montiel	✓ Berndt Díaz Ramos	✗ Berndt	✗	✗
2		✗	✓	✓	✗	✗
3			✗	✓?	✓?	
4						
5						
> 5						

Takagi ⟵ classification of homogeneous hypersurfaces ⟶ Berndt, Tamaru

✓ ∃ hypersurfaces with cpc and have been classified

✓? ∃ hypersurfaces with cpc but have not been classified

✗ ∄ hypersurfaces with cpc

J. C. Díaz-Ramos, M. Domínguez-Vázquez: Non-Hopf real hypersurfaces with constant principal curvatures in complex space forms, arXiv:0911.3624v1 (to appear in *Indiana Univ. Math. J.*)

The problem in CP^n and CH^n

- Hopf fibrations

$\mathbb{S}^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ Riemannian submersion, fiber $\mathbb{S}^1$

$AdS^{2n+1} \xrightarrow{\pi} \mathbb{C}H^n$ semi-Riemannian submersion, fiber $\mathbb{S}^1$

anti De Sitter space

Lorentzian space of constant negative curvature

- Isoparametric hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$

M real hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$

$\lambda_1, \dots, \lambda_{2n-1}$ principal curvatures of M in the basis $E_1, \dots, E_{2n-1}$

$b_i = \langle J\xi, E_i \rangle$

shape operator of the lifted hypersurface $\pi^{-1}M$ ⟹

$$\begin{pmatrix} \lambda_1 & & 0 & \pm b_1 \\ 0 & \ddots & & \vdots \\ & & \lambda_{2n-1} & \pm b_{2n-1} \\ b_1 & \dots & b_{2n-1} & 0 \end{pmatrix}$$

M isoparametric ⟺ $\pi^{-1}M$ isoparametric ⟺ $\pi^{-1}M$ has constant principal curvatures

the mean curvatures of M and $\pi^{-1}M$ agree

$\pi^{-1}M$ contained in a (Lorentzian) space form

The problem in $\mathbb{C}H^n$

The classification of homogeneous hypersurfaces in $\mathbb{C}H^n$ is known.

One can classify these examples according to the Jordan canonical form of the shape operator of its lift via the Hopf map:

I. Tube around a totally geodesic $\mathbb{C}H^k$

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2n} \end{pmatrix}$$

II. Horosphere

$$\begin{pmatrix} \lambda_1 & 0 & & & \\ \varepsilon & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{2n-1} \end{pmatrix}$$

III. Lohnherr hypersurface W^{2n-1}

Equidistant hypersurface to W^{2n-1}

Tube around a Berndt-Brück submanifold W_{φ}^{2n-k}

$$\begin{pmatrix} \lambda_1 & 0 & 1 & & & \\ 0 & \lambda_1 & 0 & & & \\ 0 & 1 & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{2n-2} \end{pmatrix}$$

IV. Tube around a totally geodesic $\mathbb{R}H^n$

$$\begin{pmatrix} a & b & & & \\ -b & a & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_{2n} \end{pmatrix}$$

The problem in CH^n

Theorem. Let M be an isoparametric hypersurface in $\mathbb{C}H^n$ and $p \in M$. Then, the principal curvatures of M at p and their multiplicities coincide with those of the homogeneous hypersurfaces in $\mathbb{C}H^n$.

In particular, $h(p) \in \{1, 2, 3\}$ and $g(p) \in \{2, 3, 4, 5\}$ and:

- $h(p) = 1 \Rightarrow g(p) \in \{2, 3\}$
- $h(p) = 2 \Rightarrow g(p) \in \{2, 3, 4\}$
- $h(p) = 3 \Rightarrow g(p) \in \{3, 4, 5\}$

Remark. This is not a local result, but a pointwise result. The principal curvatures and even h and g may vary from point to point.

Remark. The cases $g(p) = h(p) = 2$ and $g(p) = h(p) = 3$ do not arise in the known isoparametric hypersurfaces.

Question. Are isoparametric hypersurfaces in $\mathbb{C}H^n$ open parts of homogeneous hypersurfaces?

The problem in CH^n

Question. Are isoparametric hypersurfaces in $\mathbb{C}H^n$ open parts of homogeneous hypersurfaces?

NO There are inhomogeneous isoparametric hypersurfaces in $\mathbb{C}H^n$

Theorem. Let M be a connected isoparametric hypersurface in $\mathbb{C}H^n$ with $h \leq 2$ nontrivial projections of the Hopf vector field onto the principal curvature spaces. Then, h is constant and M has constant principal curvatures. Moreover:

- If $h = 1$, then M is an open part of:
  - a tube around a totally geodesic $\mathbb{C}H^k$
  - a tube around a totally geodesic $\mathbb{R}H^n$
  - a horosphere
- If $h = 2$, then M is an open part of:
  - a Lohnherr hypersurface W^{2n-1}
  - an equidistant hypersurface to W^{2n-1}
  - a tube around a Berndt-Brück submanifold W^{2n-k}

J. Berndt: Real hypersurfaces with constant principal curvatures in complex hyperbolic space. *J. Reine Angew. Math.* **395** (1989), 132–141.

J. C. Díaz-Ramos, M. Domínguez-Vázquez: Non-Hopf real hypersurfaces with constant principal curvatures in complex space forms, to appear in *Indiana Univ. Math. J.*

The problem in CH^n

The complex hyperbolic space $\mathbb{C}H^n$ can be seen as a solvable Lie group AN with a left invariant metric

$$\mathbb{C}H^n \cong AN$$

A abelian, $\dim A = 1$

N nilpotent, $\dim N = 2n-1$

$\mathfrak{a} \oplus \mathfrak{n}$ Lie algebra of AN

$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, with $\mathfrak{z} = \mathbb{R}$, $\mathfrak{v} = \mathbb{C}^{n-1}$: Heisenberg algebra

$B \in \mathfrak{a}$ unit length $\quad Z = JB \in \mathfrak{z}$

$$[B, Z+U] = Z + \frac{1}{2}U$$

$$[U,V] = \langle JU, V\rangle Z \qquad U, V \in \mathfrak{v}$$

$\mathfrak{w}$ subspace of $\mathfrak{v}$

$\mathfrak{s}_{\mathfrak{w}} = \mathfrak{a} \oplus \mathfrak{w} \oplus \mathfrak{z}$ Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$ → $S_{\mathfrak{w}}$ corresponding subgroup of AN

$$W_{\mathfrak{w}} = S_{\mathfrak{w}} \cdot o$$

→ homogeneous minimal submanifold

The problem in CH^n

Theorem. The tubes around $W_{\mathfrak{w}}$ are isoparametric hypersurfaces. Moreover, these tubes are homogeneous hypersurfaces if and only if $\mathfrak{w}^\perp = \mathfrak{v} \ominus \mathfrak{w}$ has constant Kähler angle.

$\mathfrak{w}^\perp$ has constant Kähler angle ⟺ the angle between Jv and $\mathfrak{w}^\perp$ is the same for all $v \in \mathfrak{w}^\perp$

If $\mathfrak{w}^\perp$ has constant Kähler angle φ ⟹ $W_{\mathfrak{w}}$ is a Berndt-Brück submanifold W_φ^{2n-k} (if $\varphi = \pi/2$ then we put $W^{2n-k} = W_{\pi/2}^{2n-k}$)

If $\mathfrak{w}^\perp$ does not have constant Kähler angle φ ⟹ tubes around $W_{\mathfrak{w}}$ have nonconstant principal curvatures

Remark. For the inhomogeneous examples, the functions h and g may be nonconstant, and we can have $h \in \{1,2,3\}$ and $g \in \{3,4,5\}$.

J. Berndt, M. Brück: Cohomogeneity one actions on hyperbolic spaces, *J. Reine Angew. Math.* **541** (2001), 209-235

J. C. Díaz-Ramos, M. Domínguez-Vázquez: Inhomogeneous isoparametric hypersurfaces in complex hyperbolic spaces, preprint arXiv:1011.5160v1 [math.DG].

Thanks for your attention!