Conference in Geometry and Global Analysis Celebrating P. Gilkey's 65th Birthday

Principal curvatures of isoparametric hypersurfaces in complex hyperbolic spaces



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Contents

- Isoparametric hypersurfaces
 - Basic definitions and notation
 - Motivation: a problem in Geometric Optics
 - Homogeneous hypersurfaces
 - The problem in space forms
 - Generalization to spaces of nonconstant curvature
- The problem in complex space forms
 - Real hypersurfaces with constant principal curvatures
 - Isoparametric hypersurfaces in the complex hyperbolic space
 - New isoparametric hypersurfaces

 $(\bar{M},\langle\,,\,
angle)$ Riemannian manifold $ar{
abla}$ Levi-Civita connection of \bar{M}

 $M\subset \bar{M}$ hypersurface ξ unit normal vector field ∇ Levi-Civita connection of M

Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + II(X, Y)$$

Weingarten formula

$$\bar{\nabla}_X \xi = -\mathcal{S}X$$

$$\langle II(X,Y), \xi \rangle = \langle \mathcal{S}X, Y \rangle$$

 \mathcal{S} selfadjoint



 ${\cal S}$ diagonalizable



principal curvatures of M: eigenvalues of S $\lambda_1, \ldots, \lambda_{n-1}$

Mean curvature of M



$$H = \operatorname{tr}(\mathcal{S}) = \lambda_1 + \dots + \lambda_{n-1}$$

Number of distinct principal curvatures of M



 $\Phi(x,y,z,t)$ wave function in \mathbb{R}^3



wave equation $\Delta \Phi = \frac{\partial^2 \Phi}{\partial t^2}$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

 t_0 fixed instant

$$\{(x, y, z) \in \mathbb{R}^3 : \Phi(x, y, z, t_0) = c(t_0)\}$$

wavefront: points in \mathbb{R}^3 with the same phase or oscillating state at the instant t_0

Let us consider time-independent wavefronts

$$\Delta\Phi(x,y,z,t_0) = \frac{\partial^2\Phi}{\partial t^2}(x,y,z,t_0) = c''(t_0)$$

$$f(x, y, z) = \Phi(x, y, z, t_0)$$

 $\Delta f = \Delta \Phi(\cdot, \cdot, \cdot, t_0)$ is constant along the level sets of $f = \Phi(\cdot, \cdot, \cdot, t_0)$

We want to consider waves with parallel wavefronts

 $|\nabla f| = |\nabla \Phi(\cdot, \cdot, \cdot, t_0)|$ is constant along the level sets of $f = \Phi(\cdot, \cdot, \cdot, t_0)$



The possible wavefronts are level sets of functions f such that $|\nabla f|^2$ and Δf are constant along those level sets

 $(\bar{M},\langle\,,\,\rangle)$ Riemannian manifold

 $f \colon ar{M} o \mathbb{R}$ isoparametric function



 $|\nabla f|^2$ and Δf are constant along the level sets of f

 $M\subset \bar{M}$ isoparametric hypersurface



M is a codimension 1 level set of an isoparametric function

Theorem (Cartan)

 $(\bar{M},\langle\,,\,
angle)$ Riemannian manifold

 $M\subset \bar{M}$ isoparametric hypersurface



M and its nearby parallel hypersurfaces have constant mean curvature (CMC)

Theorem (Cartan)

 $(\bar{M},\langle\,,\,
angle)$ real space form: \mathbb{R}^n , $\mathbb{R}H^n$ ou \mathbb{S}^n

 $M\subset \bar{M}$ isoparametric hypersurface



 ${\it M}$ has constant principal curvatures

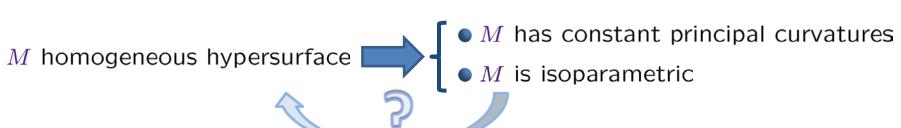
Homogeneous hypersurfaces

A hypersurface $M\subset \bar{M}$ is (extrinsically) homogeneous if $M=G\cdot o$ where $o\in M$ and G is a subgroup of $I(\bar{M})$ $G\times \bar{M}\to \bar{M} \text{ is a cohomogeneity one isometric action}$

- Objectives
 - classify homogeneous hypersurfaces of a manifold



characterize homogenous hypersurfaces using geometric data



lacktriangle The Euclidean space \mathbb{R}^n

Levi-Civita: \mathbb{R}^3

affine hyperplanes \mathbb{R}^{n-1} • spheres S^{n-1} products $S^k \times \mathbb{R}^{n-k-1}$

lacktriangle The hyperbolic space $\mathbb{R}H^n$

Cartan

geodesic hyperspheres horospheres

tot. geod. $\mathbb{R}H^{n-1}$ and equidistant hypersurfaces tubes around tot. geod. subspaces $\mathbb{R}H^k$

• The sphere S^n

Segre: \mathbb{R}^n

Cartan: classified g = 1, 2, 3

Hsiang, Lawson: classified homogeneous hypersurfaces

Münzner: proved $g \in \{1, 2, 3, 4, 6\}$ and $m_i = m_{i+1} \pmod{2}$

Ferus, Karcher, Münzner: inhomogeneous examples

Stolz: the possible triples (g, m_1, m_2) with g = 4 agree with those of the known homogeneous and inhomogeneous examples

Progress in cases g = 4,6: Abresch, Dorfmeister, Neher, Cecil, Chi, Jensen, Immervoll, Miyaoka...

Isoparametric hypersurface



Hypersurface with constant principal curvatures

 $\mathbb{R}^n, \mathbb{R}H^n, \mathbb{S}^n$

Isoparametric hypersurface



Hypersurface with constant principal curvatures

in general

Remark. Both conditions include the notion of a homogeneous hypersurface

Question. In which ambient spaces can homogeneous hypersurfaces be characterized by one of these concepts?

Remark. To our knowledge, the only known isoparametric hypersurfaces in Riemannian symmetric spaces which are not homogenous, or which do not have constant principal curvatures, are related to the FKM examples in spheres

D. Ferus, H. Karcher, H. F. Münzner: Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), no. 4, 479–502

The problem in CPⁿ and CHⁿ

Complex space forms

Complex hyperbolic space $\mathbb{C}H^n$ c<0 Bergman metric

Euclidean space \mathbb{C}^n c = 0 flat metric

Complex projective space $\mathbb{C}P^n$ c>0 Fubini-Study metric

Real hypersurfaces

M real hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$ ξ unit normal field $J\xi$ Hopf vector field tangent to M

h number of nontrivial projections of $J\xi$ onto the principal curvature spaces $(h \leq g)$



∄ hypersurfaces

with cpc

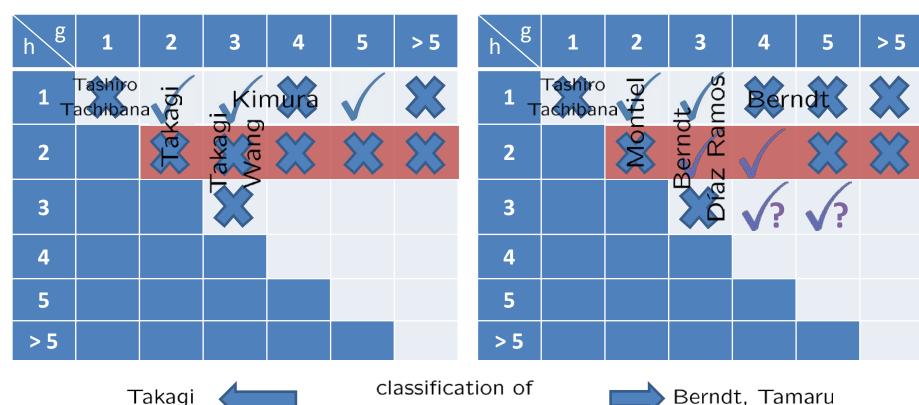
State of the problem in $\mathbb{C}P^n$

∃ hypersurfaces

been classified

with cpc and have

State of the problem in $\mathbb{C}H^n$



J. C. Díaz-Ramos, M. Domínguez-Vázquez: Non-Hopf real hypersurfaces with constant principal curvatures in complex space forms, arXiv:0911.3624v1 (to appear in *Indiana Univ. Math. J.*)

homogeneous hypersurfaces

∃ hypersurfaces

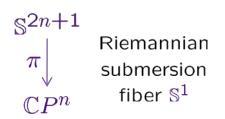
with cpc but have

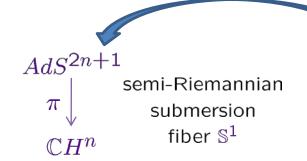
not been classified

The problem in CPⁿ and CHⁿ

Behaviour of isoparametric hypersurfaces with respect to **Hopf fibrations**

Hopf fibrations





anti De Sitter space Lorentzian space of constant negative curvature

Isoparametric hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$

M real hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$

 $\lambda_1,\ldots,\lambda_{2n-1}$ principal curvatures of M in the basis E_1,\ldots,E_{2n-1}

$$b_i = \langle J\xi, E_i \rangle$$
 shape operator of the lifted hypersurface $\pi^{-1}M$



$$\begin{pmatrix}
\lambda_1 & 0 & \pm b_1 \\
0 & \ddots & \vdots \\
0 & \lambda_{2n-1} & \pm b_{2n-1} \\
b_1 & \dots & b_{2n-1} & 0
\end{pmatrix}$$



M isoparametric $\pi^{-1}M$ isoparametric



 $\pi^{-1}M$ has constant principal curvatures

the mean curvatures of M and $\pi^{-1}M$ agree

 $\pi^{-1}M$ contained in a (Lorentzian) space form

The classification of homogeneous hypersurfaces in $\mathbb{C}H^n$ is known.

One can classify these examples according to the Jordan canonical form of the shape operator of its lift via the Hopf map:

I. Tube around a totally geodesic $\mathbb{C}H^k$

$$\left(egin{array}{ccc} \lambda_1 & & 0 \ & \ddots & \ 0 & & \lambda_{2n} \end{array}
ight)$$

II. Horosphere

$$\begin{pmatrix} \lambda_1 & 0 & & & \\ \varepsilon & \lambda_1 & & & \\ & & \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_{2n-1} \end{pmatrix}$$

Lohnherr hypersurface W^{2n-1}

III. Equidistant hypersurface to W^{2n-1} Tube around a Berndt-Brück submanifold W^{2n-k}_{ω}

$$\begin{pmatrix} \lambda_1 & 0 & 1 & & & & \\ 0 & \lambda_1 & 0 & & & & \\ 0 & 1 & \lambda_1 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_{2n-2} \end{pmatrix}$$

IV. Tube around a totally geodesic $\mathbb{R}H^n$

$$\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix}$$

$$\lambda_3$$

$$\vdots$$

$$\lambda_{2n}$$

Theorem. Let M be an isoparametric hypersurface in $\mathbb{C}H^n$ and $p \in M$. Then, the principal curvatures of M at p and their multiplicities coincide with those of the homogeneous hypersurfaces in $\mathbb{C}H^n$.

In particular, $h(p) \in \{1, 2, 3\}$ and $g(p) \in \{2, 3, 4, 5\}$ and:

•
$$h(p) = 1$$
 $g(p) \in \{2, 3\}$

•
$$h(p) = 2$$
 $g(p) \in \{2, 3, 4\}$

•
$$h(p) = 3$$
 $g(p) \in \{3, 4, 5\}$

Remark. This is not a local result, but a pointwise result. The principal curvatures and even h and g may vary from point to point.

Remark. The cases g(p) = h(p) = 2 and g(p) = h(p) = 3 do not arise in the known isoparametric hypersurfaces.

Question. Are isoparametric hypersurfaces in $\mathbb{C}H^n$ open parts of homogeneous hypersurfaces?

Question. Are isoparametric hypersurfaces in $\mathbb{C}H^n$ open parts of homogeneous hypersurfaces?

NO

There are inhomogeneous isoparametric hypersurfaces in $\mathbb{C}H^n$

Theorem. Let M be a connected isoparametric hypersurface in $\mathbb{C}H^n$ with $h \leq 2$ nontrivial projections of the Hopf vector field onto the principal curvature spaces. Then, h is constant and M has constant principal curvatures. Moreover:

- If h = 1, then M is an open part of:
 - ullet a tube around a totally geodesic ${\Bbb C}H^k$
 - ullet a tube around a totally geodesic $\mathbb{R}H^n$
 - a horosphere
- If h = 2, then M is an open part of:
 - a Lohnherr hypersurface W^{2n-1}
 - ullet an equidistant hypersurface to W^{2n-1}
 - ullet a tube around a Berndt-Brück submanifold W^{2n-k}
- J. Berndt: Real hypersurfaces with constant principal curvatures in complex hyperbolic space. *J. Reine Angew. Math.* **395** (1989), 132–141.
- J. C. Díaz-Ramos, M. Domínguez-Vázquez: Non-Hopf real hypersurfaces with constant principal curvatures in complex space forms, to appear in *Indiana Univ. Math. J.*

The complex hyperbolic space $\mathbb{C}H^n$ can be seen as a solvable Lie group AN with a left invariant metric

$$\mathbb{C}H^n \cong AN$$

$$A$$
 abelian dim $A = 1$

$$N$$
 nilpotent dim $N = 2n - 1$

$$\mathfrak{a} \oplus \mathfrak{n}$$

Lie algebra
of AN

$$\mathfrak{n}=\mathfrak{z}\oplus\mathfrak{v}$$
 \mathbb{C}^{n-1}

 $B \in \mathfrak{a}$ unit length $Z = JB \in \mathfrak{z}$

$$[B, Z + U] = Z + \frac{1}{2}U$$
$$[U, V] = \langle JU, V \rangle Z$$

$$Z = JB \in \mathfrak{z}$$

$$U, V \in \mathfrak{v}$$

w subspace of v

$$\mathfrak{s}_{\mathfrak{w}}=\mathfrak{a}\oplus\mathfrak{w}\oplus\mathfrak{z}$$

Lie subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$



 $S_{\mathfrak{w}}$ corresponding subgroup of AN

$$W_{\mathfrak{w}} = S_{\mathfrak{w}} \cdot o$$



homogeneous minimal submanifold

Theorem. The tubes around $W_{\mathfrak{w}}$ are isoparametric hypersurfaces. Moreover, these tubes are homogeneous hypersurfaces if and only if $\mathfrak{w}^{\perp} = \mathfrak{v} \ominus \mathfrak{w}$ has constant Kähler angle.

the angle between Jv and \mathfrak{w}^{\perp} is the same for all $v \in \mathfrak{w}^{\perp}$ If \mathfrak{w}^{\perp} has constant Kähler angle φ $W_{\mathfrak{w}} \text{ is a Berndt-Brück submanifold } W_{\varphi}^{2n-k}$ $(\text{if } \varphi = \pi/2 \text{ then we put } W^{2n-k} = W_{\pi/2}^{2n-k})$ $\text{If } \mathfrak{w}^{\perp} \text{ does not have constant Kähler angle } \varphi$ $\text{tubes around } W_{\mathfrak{w}} \text{ have nonconstant principal curvatures}$

Remark. For the inhomogeneous examples, the functions h and g may be nonconstant, and we can have $h \in \{1,2,3\}$ and $g \in \{3,4,5\}$.

- J. Berndt, M. Brück: Cohomogeneity one actions on hyperbolic spaces, *J. Reine Angew. Math.* **541** (2001), 209-235
- J. C. Díaz-Ramos, M. Domínguez-Vázquez: Inhomogeneous isoparametric hypersurfaces in complex hyperbolic spaces, preprint arXiv:1011.5160v1 [math.DG].

Thanks for your attention!